Kernel density in the use of the strong stability method to evaluate the proximity of G/M/1 and M/M/1 systems

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ABSTRACT

Bouallouche [3] has applied the strong stability method to study the proximity of the G/M/1 and M/M/1 systems when the general distribution of arrivals G is assumed to be hyper-exponantial.
In this paper, we show the applicability of the strong stability method to evaluate an approximation error of the G/M/1 and M/M/1 systems when the general distribution of arrivals G is unknown and must be estimated by the means of the kernel density estimation method. The boundary effects are taken in consideration. A simulation study is effectuated.

Keywords

Queueing systems, strnog stability, approximation, kernel density, bandwidth, boundary effects.

1. INTRODUCTION

When modelling practical problems, one may often replace a real system by another one which is close to it in some sense but simpler in structure and/or components. This approximation is necessary because real systems are generally very complicated, so their analysis can not lead to analytical results or it leads to complicated results which are not useful in practice.

The strong stability method elaborated in the beginning of the 1980s is applicable to all operation research models which can be represented by a Markov chain [1, 7]. It has been applied to queueing systems (see for example [2]). According to this approach, we suppose that the perturbation is small with respect to a certain norm. Such a strict condition allow us to obtain better estimations on the characteristics of the perturbed chain.

When the distribution of arrivals is general but close to Poisson distribution, it is possible to approximate the characteristics of the $G/M/1$ system by those of the $M/M/1$ system, if we prove the fact of stability (see [2]). In this case, is it possible to precise the error of the proximity? Results are known when the general distribution G is well fixed and close to the exponential one. For example when the distribution G is hyper-exponential, it's possible to obtain numerically the proximity of the stationary distribution of an $Hyp/M/1$ system by one of an $M/M/1$ system (see [3]). In this work, we are interested by the case where the distribution G is unknown so must be estimated by the means of estimating its density function. The most popular and attractive nonparametric method of estimating an unknown density is the kernel density method (see [9]). If $X_1, X_2, ..., X_n$ is a sample coming from a distribution F with an unknown density function f, the Rosenblatt kernel estimator (see [9]) is given by:

$$f_n(x) = \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right), \quad \forall x \in \mathbb{R}. \qquad (1)$$

Where K is a symmetric density function called kernel and h_n is the smoothing parameter (or bandwidth).
Several results are known in the literature when the density function is defined on the real line $\mathbb{R}$ [5, 6, 8, 9]. In the case of a density function defined on a bounded support, the boundary effects are present. To resolve this problem, many recent methods have been elaborated. We can cite: the "mirror image" of Schuster [10] and the asymmetric kernel estimators [4].

The aim of this paper is to show the applicability of the strong stability method to evaluate an approximation error on the stationary distributions of the G/M/1 and M/M/1 systems when the general law of arrivals G is unknown so its density function must be estimated by the means of the kernel density estimation method. The boundary effects are taken in consideration.

The paper is organized as follows: First, the basics of the $G/M/1$ and $M/M/1$ models and the strong stability approach are briefly reviewed in section 2. In section 3, we give some results and discussions concerning the kernel density method and the correction of boundary effects. In the last section, we apply the kernel density method in the study of the strong stability of the $M/M/1$ system. We are interested by the approximation of the stationary distribution of the $G/M/1$ system by one of the $M/M/1$ system, when the distribution G is general and unknown.

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2. STRONG STABILITY OF THE $M/M/1$ SYSTEM AFTER PERTURBATION OF THE ARRIVAL FLOW

2.1 Description of $M/M/1$ and $G/M/1$ models

Let us consider $G/M/1(FIFO, \infty)$ system where inter-arrival times are independently distributed with general distribution $H(t)$ and service times are distributed with $E_\gamma(t)$ (exponential with parameter γ).

Let X_n^* be the number of customers left behind in the system by the n^{th} departure. It's easy to prove that X_n^* forms a Markov chain (see [7]) with a transition operator $P^* = \|P_{ij}^*\|_{i,j\geq 0}$ where:

$$P_{ij}^* = \begin{cases} d_{i+1-j}^* = \int_0^{+\infty} \frac{1}{(i+1-j)!} e^{-\gamma t} (\gamma t)^{i+1-j} dH(t), & \text{if } 1 \leq j \leq i+1 \\ 1 - \sum_{k=0}^{i} d_k^*, & \text{if } j = 0 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Consider also an $M/M/1(FIFO, \infty)$ system, which has Poisson arrivals with parameter λ and the same distribution of the service time as the precedent system. It's known that X_n (the number of customers left behind in the system by the n^{th} departure), forms a Markov chain with a transition operator $P = \|P_{ij}\|_{i,j\geq 0}$ where:

$$P_{ij} = \begin{cases} d_{i+1-j} = \frac{\lambda\gamma^{i+1-j}}{(\lambda+\gamma)^{i+2-j}}, & \text{if } 1 \leq j \leq i+1 \\ 1 - \sum_{k=0}^{i} d_k = \left(\frac{\gamma}{\gamma+\lambda}\right)^i, & \text{if } j = 0 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Suppose that the arrival flow of the $G/M/1$ system is close to the Poisson one. This proximity is then characterized by the metric:

$$w = w(H, E_\lambda) = \int_0^\infty |H - E_\lambda|(dt) \tag{4}$$

where $|a|$ is the variation of the measure a.

Designate by π_k^* and π_k the stationary distributions of the states of X_n^* and X_n. We have then:

$$\begin{cases} \pi_k^* = \lim_{n\to\infty} Pr(X_n^* = k), k = 0, 1, 2, ..., \\ \pi_k = \lim_{n\to\infty} Pr(X_n = k), k = 0, 1, 2, ..., \end{cases} \tag{5}$$

2.2 The strong stability criteria

Let $\mathcal{M} = \{\mu_j\}$ be the space of finite measures on $\mathbb{N}$, and $\mathcal{N} = \{f(j)\}$ the space of bounded measurable functions on $\mathbb{N}$. We associate with each transition kernel P the linear mapping:

$$(\mu P)_k = \sum_{j\geq 0} \mu_j P_{jk}. \tag{6}$$

$$(Pf)(k) = \sum_{i\geq 0} f(i) P_{ki}. \tag{7}$$

Introduce on $\mathcal{M}$ the class of norms of the form:

$$\|\mu\|_\upsilon = \sum_{j\geq 0} \upsilon(j)|\mu_j|. \tag{8}$$

Where υ is an arbitrary measurable function (not necessary finite) bounded below away from a positive constant. This norm induce in the space $\mathcal{N}$ the norm:

$$\|f\|_\upsilon = \sup_{k\geq 0} \frac{|f(k)|}{\upsilon(k)}. \tag{9}$$

Let us consider $\mathcal{B}$, the space of linear operators, with norm:

$$\|Q\|_\upsilon = \sup_{k\geq 0} \frac{1}{\upsilon(k)} \sum_{j\geq 0} \upsilon(j) Q_{kj} \tag{10}$$

DEFINITION 2.1. *The Markov chain X with a transition kernel P and an invariant measure π is said to be strongly υ-stable with respect to the norm $\|.\|_\upsilon$, if $\|P\|_\upsilon < \infty$ and each stochastic kernel $\mathbf{Q}$ on the space $(\mathbb{N}, \mathcal{B}(\mathbb{N}))$ in some neighborhood $\{\mathbf{Q} : \|\mathbf{Q} - \mathbf{P}\|_\upsilon < \epsilon\}$ has a unique invariant measure $\mu = \mu(Q)$ and $\|\pi - \mu\|_\upsilon \to 0$ as $\|\mathbf{Q} - \mathbf{P}\|_\upsilon \to 0$.*

THEOREM 2.1. *(see [7]) A Markov chain X, with transition kernel P, is strongly υ-stable, if and only if there exists a measure α and a non-negative measurable function h on $\mathbb{N}$ such that:*

1. *$\|P\|_\upsilon < \infty$;*
2. *$T = \mathbf{P} - h \circ \alpha > 0$;*
3. *$\exists\ m \geq 1$ and $\rho < 1$ such that $T^m \upsilon(x) \leq \rho\ \upsilon(x), \forall x \in \mathbb{N}$.*

THEOREM 2.2. *(see [7]) Let X be a strongly υ-stable Markov chain, with an invariant measure π and holding the theorem 1's conditions. If μ is the invariant measure of a kernel Q, then for the norm $\|Q - P\|_\upsilon$ sufficiently small, we have:*

$$\mu = \pi[I - \Delta R_0(I - \Pi)]^{-1} = \pi + \sum_{t\geq 1} \pi[\Delta R_0(I - \Pi)]^t$$

Where $\Delta = Q - P$, $R_0 = (I - T)^{-1}$ and $\Pi = \mathbb{1} \circ \pi$ is the stationary projector of the kernel P, $\mathbb{1}$ is the identity function, and I the identity kernel on $\mathcal{M}$.

CONSEQUENCE 1. *Under the Theorem 1's conditions,*

$$\mu = \pi + \pi\Delta R_0(I - \Pi) + \circ(\|\Delta\|_\upsilon^2)$$

for $\|\Delta\|_\upsilon$.

CONSEQUENCE 2. *Under the Theorem 1's conditions, for*

$$\|\Delta\|_\upsilon < \frac{(1 - \rho)}{c}$$

we have the estimation:

$$\|\mu - \pi\|_\upsilon \leq \|\Delta\|_\upsilon\ c\ \|\pi\|_\upsilon\ (1 - \rho - c\ \|\Delta\|_\upsilon)^{-1}$$

where

$$c = m\ \|\mathbf{P}\|_\upsilon^{m-1} (1 + \|\ \mathbb{1}\ \|_\upsilon\ \|\pi\|_\upsilon)$$

and

$$\|\pi\|_\upsilon \leq (\alpha\upsilon)(1 - \rho)^{-1}(\pi h)\ m\ \|\mathbf{P}\|_\upsilon^{m-1}.$$

2.3 Approximation of the $G/M/1$ system by the $M/M/1$ system

The proofs of the theorems given in this subsection can be found in [3].

2.3.1 Strong stability conditions

The first step consists on the determination of the strong υ-stability conditions of the $M/M/1$ system after a small perturbation of the arrival flow.

THEOREM 2.3. *Suppose that the charge $(\frac{\lambda}{\gamma})$ of the $M/M/1$ system is smaller than 1. Therefore, for all β such that $1 < \beta < \frac{\lambda}{\gamma}$, the imbedded Markov chain X_n is υ-strongly stable, after a small perturbation of the inter-arrival time, for $\upsilon(k) = \beta^k$.*

2.3.2 Estimation of the transition kernel deviation

To be able to estimate numerically the margin between the stationary distributions of the Markov chains X_n^* and X_n, we estimate the norm of the deviation of the transition kernel P^*.

THEOREM 2.4. *Let P (resp. P^*) be the transition operator of the imbedded chain in $M/M/1$ (resp. $G/M/1$) system. Then, for all β such that $1 < \beta < \frac{\gamma}{\lambda}$, we have:*

$$\|P^* - P\|_{\upsilon} \leq (1+\beta)w$$

where w is defined in (4).

2.3.3 Stability inequalities

This subsection consists on the determination of the deviation of the stationary distribution with respect to the norm $\|\|_{\upsilon}$.

THEOREM 2.5. *Suppose that in an $M/M/1$ system, the Markov chain X_n is strongly υ-stable, and*

$$w < \frac{(1-\rho)(\gamma-\lambda\beta)}{(1+\beta)(2\gamma-\lambda(1+\beta))} \tag{11}$$

Therefore:

$$\begin{aligned} Err &= \|\pi^* - \pi\|_{\upsilon} \qquad (12)\\ &\leq \frac{(1+\beta)(2\gamma-\lambda(1+\beta))(\gamma-\lambda)w}{\frac{(\beta-1)(\gamma-\lambda\beta)^3}{(\beta-1)\gamma+\lambda\beta} - (2\gamma-\lambda(1+\beta))(1+\beta)(\gamma-\lambda\beta)w} \end{aligned}$$

for all β such that $1 < \beta < \frac{\gamma}{\lambda}$ where π^* and π are defined in (5) and $\rho = \beta\frac{\lambda}{\gamma - \frac{\gamma}{\beta}+\lambda}$.

3. KERNEL DENSITY ESTIMATION METHOD

Let $X_1, ..., X_n$ be a sample coming from a random variable X of density function f and distribution F. The Rosenblatt kernel estimator (see [9]) of the density $f(x)$ for each point $x \in \mathbb{R}$ is given by:

$$f_n(x) = \frac{1}{nh_n}\sum_{j=1}^{n} K(\frac{x-X_j}{h_n}) \tag{13}$$

Where K is a symmetric density function called kernel and h_n is the smoothing parameter (or bandwidth).

In the practice, when we use the kernel density method to estimate a probability density function of *iid* observations, it's necessary to choose the kernel function K and the bandwidth h_n. The optimal choice of (K, h_n) is generally made following the criteria of minimization of the mean square error (MSE) given by:

$$MSE(f_n(x)) = \mathbb{E}(f_n(x) - f(x))^2, \tag{14}$$

or of the mean integrated square error (MISE) given by:

$$MISE(f_n(x)) = \mathbb{E}\int_{-\infty}^{+\infty}(f_n(x) - f(x))^2 dx. \tag{15}$$

Several studies have been elaborated to discuss the good choice of the two parameters of this method (K, h_n). Many among them, for example [6], show that the choice of the kernel function K is not very important and that it's completely satisfactory to choose the kernel function for the suitability of the computer calculation such the gaussian kernel.

3.1 Bandwidth choice

In the practice, the critical step in the kernel density estimation is the choice of the smoothing parameter (bandwidth) h_n which controls the smoothness of the kernel estimator (1). This problem has been widely studied and many methods have been proposed. Most of them suppose that f is a smooth function over the real line $\mathbb{R}$. The methods proposed in the literature can be divided into two classes [8]:

3.1.1 First generation methods (or classical methods)

Most of them have been developed before 1990. The most popular are: ”rules of thumb”, ”least squares cross-validation” and ”biased cross-validation”.

3.1.2 Second generation methods (or plug-in methods)

The most of them have been elaborated after 1990. The bias of the kernel estimator (1) is written in function of the unknown density f and is usually approximated by the developments in Taylor series. A pilot estimate of f is then injected in order to derive an estimator of the bias and after that an estimator of MISE given in (15). The optimal bandwidth minimizes this last estimated measure. The most known are: ”Solve-the-Equation Plug-In Approach” and ”Smoothed Bootstrap”.

3.2 Boundary effects

Several results are known in the literature when the density function is defined on the real line $\mathbb{R}$ [5, 6, 8, 9]. In the case of a density function defined on a bounded support, the boundary effects are present. To resolve this problem, many recent methods have been elaborated [10, 4].

3.2.1 Schuster estimator "mirror image"

Schuster [10] suggests to create the mirror image of the data in the other side of the boundary and then apply the estimator (1) for the set of the initial data and their reflection. $f(x)$ is then estimated, for $x \geq 0$, as follows:

$$\hat{f}_n(x) = \frac{1}{nh_n}\sum_{j=1}^{n}[K(\frac{x-X_j}{h_n}) + K(\frac{x+X_j}{h_n})] \tag{16}$$

3.2.2 Asymmetric Gamma kernel estimator

A simple idea to avoid the problem of boundary effects, is the use of a flexible kernel, which never assign a weight out of the support of the density function and which correct automatically and implicitly the boundary effects. We can cite the asymmetric kernels [4] given by the following form:

$$\hat{f}_b(x) = \frac{1}{n}\sum_{i=1}^{n} K(x,b)(X_i), \tag{17}$$

where b is the bandwidth and the asymmetric kernel K can be taken as a Gamma density K_G with the parameters $(x/b + 1, b)$ given by:

$$K_G(\frac{x}{b}+1, b)(t) = \frac{t^{x/b}e^{-t/b}}{b^{x/b+1}\Gamma(x/b+1)}, \tag{18}$$

In this paper, we choose the Epanechnikov kernel [6], given by:

$$K(y) = \begin{cases} 0.75(1-x^2), & \text{if } |y| < 1; \\ 0, & \text{otherwise,} \end{cases} \quad (19)$$

and the bandwidths h_n and b are chosen to minimize the criteria of the "lest squares cross-validation" [5] given by:

$$LSCV(h_n) = \int f_n(x)^2 dx - \frac{2}{n}\sum_{i=1}^{n} f_{h_n,-i}(X_i), \quad (20)$$

where $f_{h_n,-i}(x_i)$ is given as follows:

$$f_{h_n,-i}(x_i) = \frac{1}{(n-1)h_n} \sum_{\substack{j=1 \\ j \neq i}}^{n} K\left(\frac{x_i - X_j}{h_n}\right). \quad (21)$$

4. KERNEL DENSITY METHOD FOR THE APPROXIMATION OF THE $G/M/1$ SYSTEM BY THE $M/M/1$ SYSTEM USING THE STRONG STABILITY METHOD

We want to apply the kernel density method to estimate numerically the proximity of the $G/M/1$ and $M/M/1$ systems, by evaluating the variation distance w defined in (4) and the error Err defined in (12) between the stationary distributions of the tow according systems when applying the strong stability method.

To realize this work, we follow the general following steps:

1) Generation of a sample of size n of general probability distribution G with theoretical density $g(x)$.

2) Use of the kernel density method to estimate the theoretical density function $g(x)$ by a function noted in general $g_n^*(x)$.

3) Verification, in this case, of the strong stability conditions given in the subsection (2.3).

4) Analysis of the results and determination of the variation distance w and the error Err (defined respectively in the formulas (4) and (12)) in the sense of stability.

Consider the two systems to approximate, $G/M/1$ $(FIFO, \infty)$ and $M/M/1$ $(FIFO, \infty)$. The variation distance w which characterizes the proximity of these systems is given by:

$$w = w(G, E_\lambda) = \int |G - E_\lambda|(dt) = \int |g_n^* - e_\lambda|(t)dt \quad (22)$$

The arrival rate is given by:

$$\lambda = 1/\mathbb{E}(T) = 1/\int t dG(t) = 1/\int t g(t) dt = 1/\int t g_n^*(t) dt. \quad (23)$$

The stationary distribution π_i of the $M/M/1$ system is given by:

$$\pi_i = (1-\rho)\rho^i \qquad i = 0, 1, ... \quad (24)$$

With $\rho = \frac{\lambda}{\gamma}$ is the charge of the $M/M/1$ system, λ is the mean rate of the inter-arrival duration and γ is the service mean time; and the stationary distribution α_i of the $G/M/1$ system is given by:

$$\alpha_i = (1-x)x^i \quad \forall i \geq 0 \quad (25)$$

Table 1: w and Err measures for different samples

	Exp(1) $\gamma = 10$	Weibull(2,0.5,0) $\gamma = 10$	Gamma(1,3) $\gamma = 2$
Inter-arrival mean time λ	0.9190	1.8244	0.2750
Charge ρ of the system	$0.0919 < 1$	$0.1824 < 1$	$0.1375 < 1$
Domain of stability	$1 < \beta < 10.8811$	$1 < \beta < 5.4814$	$1 < \beta < 7.2740$
Variation distance w	0.2444	0.3502	0.1615
Error Err on stationary distributions			

With x the solution (found numerically by the fixed-point method) of the system:

$$x = \int e^{-\gamma t(1-x)} g(t) dt \quad (26)$$

where g is the density function of the general law G.

4.1 Simulation study 1

For the general law G, we generate samples of size $n = 50$ of different laws. We take the number of simulations $R = 100$. For each case of law, we replace the function $g_n^*(t)$ defined in step 2) above by the density function $g_n(t)$ found by applying the Rosenblatt estimator given in the formula (13) to estimate the theoretical density $g(t)$ of each sample. The programming with Matlab 7.0 gives us the results in the table 1.

4.1.1 Discussion

According to the Table 1, the application of the kernel density estimation method with the use of the Rosenblatt estimator (13) for the approximation of the $G/M/1$ system by the $M/M/1$ system when using the strong stability method don't allow us to determine the error on the stationary distributions between the two systems. This is due to the importance of the value of the variation distance w (for example $w = 0.2444$ for Exp(1) and $w = 0.3502$ for Weibull(2,0.5,0)). affirm and reinforce the order of the importance of the smallness of the perturbation done in the study of the strong stability of the systems.

4.2 Simulation study 2

In a first case, we consider a $G/M/1$ system such that the density function of the general law G is given by:

$$g(x) = \begin{cases} \frac{1}{2}e^{-x} + e^{-2x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases} \quad (27)$$

In a second case, we use the density function $g_n(x)$ found by applying the Rosenblatt estimator given in the formula (13) to estimate the theoretical density function $g(x)$.
In a third case, we use the density function $\tilde{g}_n(x)$ found by applying the Schuster estimator given in the formula (16) to estimate the theoretical density function $g(x)$.
In a fourth case, we use the density function $\hat{g}_n(x)$ found by applying the asymmetric kernel estimator given in the formula (17) with the use of the Gamma kernel given in

Table 2: Parameters of the ideal system $M/M/1$

Inter-arrival mean time	3/4
Mean rate of the inter-arrival time λ	4/3
Charge ρ of the system	2/15

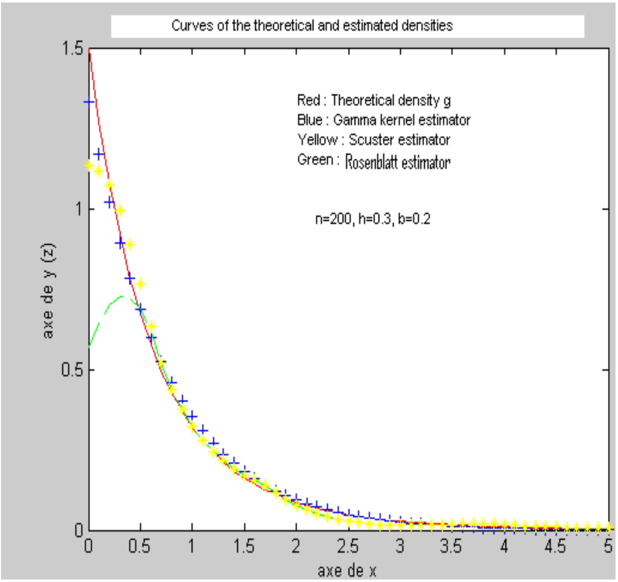

Figure 1: Curves of the theoretical and estimated densities

the formula (18) to estimate the theoretical density function $g(x)$.
For the last three cases, we take:

- The sample size $n = 200$.
- The number of simulations $R = 100$.

In all the cases, we introduce the service mean time: $\gamma = 10$.

We first determine the mean rate inter-arrival time:

$$\lambda = \frac{1}{\int xg(x)dx} \tag{28}$$

We obtain the parameters of the ideal system $M/M/1$ in the table 2.

The curves of the theoretical and estimated densities are given in the figure 1. The programming with Matlab 7.0 gives us the results in Table 3.

4.2.1 Discussion

Figure (1) shows that the use of Gamma kernel or Schuster estimators improves the quality of the estimation.

Table 3: w and Err measures for different estimators

	$g(x)$	$g_n(x)$	$\tilde{g}_n(x)$	$\hat{g}_n(x)$
Variation distance w	0.0711	0.2104	0.0895	0.0792
Error Err on the stationary distributions	0.21		0.35	0.26

We note in the Table 3 that the approximation error on the stationary distributions of the $G/M/1$ and $M/M/1$ systems is given when applying the kernel density method by considering the correction of the boundary effects such in the case of using the Schuster estimator ($Err = 0.35$) or in the case of using the asymmetric Gamma kernel estimator ($Err = 0.26$). But, when applying the kernel density method without taking in consideration the correction of the boundary effects such in the case of using the Rosenblatt estimator, the approximation error (Err) on the stationary distributions of the quoted systems could not be given.

5. CONCLUSION

In this paper, we show the applicability of the strong stability method to evaluate an approximation error on the stationary distributions of the G/M/1 and M/M/1 systems when the general law of arrivals G is unknown so its density function must be estimated by the means of the kernel density estimation method. The strong stability method states that the perturbation done must be small, in the sense that the general law G must be close but not equal to the Poisson one. Consequently, the density function of the law G must be close to the density function of the exponential law which is defined on a bounded support $[0, \infty[$. Thus, the boundary effects must be taken in consideration when using the kernel density method.

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