Fast Evaluation of the Moments of the Interval Availability of Large Markov Models

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ABSTRACT

This paper proposes a new numerically stable method for evaluating the first K moments ($K \geq 1$) of the interval availability for repairable systems. Its time complexity is $\mathcal{O}(|\mathcal{S}|KdN_{max})$, where $|\mathcal{S}|$ is the cardinality of the Markov model state space; d is the average degree of connectivity of the Markov chain states; and N_{max} represents the number os transition of the Markov chain in $[0, t]$). This time complexity is $O(L|\mathcal{S}|)$ times faster than the best existing method, without extra memory requirements, where L is the number of different time intervals for which the moments are evaluated.

Keywords

Markov models, availability, cumulative operational time, repairable systems, transient evaluation.

1. INTRODUCTION

Systems that can recover from failure during their utilization period are known as repairable systems. Thus, during the utilization period $[0, t]$ a repairable system alternates between situations in which it is in operational state (i.e. a situation considered to be satisfactory according to user-defined criteria) and others in which it is non-operational state (that is, in a failed state).

One of performance metrics which most impact the quality of the service offered by repairable systems is $O(t)$, the stochastic process corresponding to the accumulated operational time of the system during the utilization period $[0, t]$. An equivalent metric to $O(t)$ is the interval availability A(t), defined by $O(t)/t$, that is, the amount of time the system is in operational condition per time unit.

One method to characterize a stochastic process is by evaluating its moments, see for example [1], [6]. Therefore, several previous works aiming to evaluate the interval availability of repairable systems have focused on the evaluation of the first K moments of O(t) (or $A(t)$) ([2], [3], [4], [5], [6]). The asymptotic costs (complexity) of the methods proposed to date [[2], [3], [4], [5], [6]] are summarized in Table 1.

In Table 1, S is the state space of the Markov chain that models the system under analysis; T is the set of transitions between states; $|S|$ and $|T|$ are the cardinality of S and T, respectively; d is the average degree of connectivity of the Markov chain states ($1 \leq d \leq |S|$); N_{max} is the maximum number of transitions made by the Markov chain in $[0, t]$ and K is the number of the highest-order moment evaluated.

The method proposed in [2] aimed to find the distribution of $O(t)$, including the calculation of its first moment. The factor $C'(1 \leq C' \leq N_{\max})$ in the temporal complexity (Table 1) is evaluated taking into account the probability of the paths traveled by the stochastic process. No method is proposed to evaluate higher order moments of $O(t)$. In [3],[5], the k^{th} moment of $R(t)$, the total cumulative reward of a Markov chain, is calculated using a numerical convolution technique. $O(t)$ is evaluated as the special case of $R(t)$ in which the rewards associated with the chain states must be either 0 or 1. In [5] a symbolic expression for the first K^{th} moments of $RTA(t)$ was also given. In [4], the distribution and first moment of $O(t)$ were evaluated using the uniformization method (first employed in [2]). The algorithm proposed in [4] improves on the efficiency of the other methods [2], [3], [5] for the cases where the Markov chain enters into a stationary state within the $[0, t]$ time interval. The parameter H in the time cost of this method is the iteration in which the stationary state is reached. Finally, in [6], the k^{th}, $1 \leq k \leq K$, moment of $R(t)$ was calculated by using the Laplace transform after randomizating the Markov chain. This is the most efficient method to date.

In this paper a new numerically stable method for evaluating the first K moments of $O(t)$ is proposed, where K is an arbitrary natural number. The proposed method is $O(L|S|)$ times faster than the best existing method, without extra memory requirements, where L is the number of time intervals for which the moments are evaluated. This gain in processing speed is specially significant in large Markov models, where $|S|$ can exceed 10^6 states. In this case, the method proposed in this paper is about six orders of magnitude faster than the one of [6].

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Table 1: Complexity of methods for evaluating the first k moments of $O(t)$, $1 \leq k \leq K$

Method	Measure evaluated	Time cost	Spatial cost
De Souza e Silva and Gail [2]	$E(O^1(t))$	$O(\|S\|C'(N_{max} - C'))$	$O(\|S\|N_{max})$
Iyer, Donatiello and Heidelberger [3]	$E(O^k(t))$	$O(\|S\|^4 \cdot K^2)$	$O(\|T\| + \|S\|K)$
Sericola [4]	$E(O^1(t))$	$O(\|T\| \cdot min(H, N_{max}))$	$O(\|T\|)$
Telek and Rácz [6]	$E(O^k(t))$	$O(\|T\| \cdot \|S\|KN_{max})$	$O(\|T\| + \|S\|K))$
Method proposed in this paper	$E(O^k(t))$	$O(\|T\|KN_{max})$	$O(\|T\| + \|S\|K))$

The remainder of this paper is organized as follows: Section 2 describes the system's model and derives the method for evaluating the first K moments of operational time $E[O^k(t)]$ and $E[A^k(t)]$, $1 \leq k \leq K$; Section 3 illustrates the method feasibility throughout a numerical example; and finally, Section 4 concludes the paper.

2. EVALUATION OF THE K^{TH} MOMENT OF THE OPERATIONAL TIME

Consider a repairable system whose behavior may be modeled by a continuous time Markov chain $\{X(t), t \geq 0\}$ with a finite state space $\mathcal{S}$ and a transition rate matrix $\mathbf{Q}$. $\mathcal{S}$ is divided into two disjoint sets denominated $\mathcal{S}_O$ and $\mathcal{S}_F$. $\mathcal{S}_O = \{s_i : s_i \text{ is an operational state}\}$, contains all the states considered by the user to be operational, while $\mathcal{S}_F = \{s_i : s_i \text{ is a failed state}\}$, contains those considered to be non-operational (failed states).

Let $|\mathcal{S}|$ and $|\mathcal{S}_O|$ be the cardinality of the sets $\mathcal{S}$ and $\mathcal{S}_O$, respectively ($|\mathcal{S}| > |\mathcal{S}_O|$) and let $I(c)$ be an indicator function defined as follows:

$$I(c) = \begin{cases} 1, & \text{if condition } c \text{ is true} \\ 0, & \text{otherwise} \end{cases}$$

Let:

- $a(\tau)$ be the point availability of the system evaluated at time instant τ. That is:

$$a(\tau) = I(X(\tau) \in \mathcal{S}_O) \tag{1}$$

- $O(t)$ be the cumulative operational time of the system during the observation period [0,t], defined by:

$$O(t) = \int_0^t a(\tau) d\tau \tag{2}$$

Note that (2) implies that $0 \leq O(t) \leq t$.

- $A(t)$ be the interval availability of the system evaluated in the interval $[0, t]$. $A(t)$ is defined as

$$A(t) = \frac{O(t)}{t}, \qquad t > 0 \tag{3}$$

- $E[O^k(t)], E[A^k(t)]$; $k \geq 1$, be the k^{th} moment of $O(t)$ and $A(t)$, respectively.

To evaluate $E[O^k(t)]$, we begin in similar fashion to the procedure used in [2],[4], and [6], among others. First, the uniformization technique is applied to matrix $\mathbf{Q}$, thus obtaining the transition matrix $\mathbf{P}$ and a Poisson process $N(t)$ whose parameter is Λt, where Λ is the uniformization rate of $\mathbf{Q}$. We then condition on n, the number of transitions made by the Poisson process $N(t)$ in [0,t] and apply the total probabilities theorem. The same theorem is applied again, this time conditioning on the sample path $\vec{e}(n) = \langle e_{(1)}, e_{(2)}, \ldots, e_{(n)}, e_{(n+1)} \rangle$ traveled by the uniformized Markov chain, where $e_{(l)}$ is the l^{th} state the chain visits. This procedure allow us to conclude that:

$$E[O^k(t)] = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \sum_{\forall \vec{e}(n)} P(\vec{e}(n)) E\left[O^k(t) | \vec{e}(n)\right] \tag{4}$$

where $P(\vec{e}(n))$ is the probability that the Markov chain transit the sample path $\vec{e}(n)$.

Now consider Figure 1. The top graph exemplifies a possible sample path $\vec{e}(n)$ followed by the uniformized Markov chain, given that $N(t) = n$. On the other hand, the bottom graph shows the instantaneous availability associated with that path. To simplify the example it was assumed that $\mathcal{S}_O = \{s_0, s_1, \ldots, s_{|\mathcal{S}_O|-1}\}$ and $\mathcal{S}_F = \{s_{|\mathcal{S}_O|}, s_{|\mathcal{S}_O|+1}, \ldots, s_{|\mathcal{S}|-1}\}$.

Note also that the area under the curve of the bottom graph corresponds to $[O(t)/\vec{e}(n)]$. This illustrates the well-established result [2]:

$$[O(t)|\vec{e}(n)] = \sum_{l=1}^{n+1} V_l(n,t) \cdot I\Big(e_{(l)} \quad \in \quad \mathcal{S}_O\Big) \tag{5}$$

where $V_l(n,t) = \tau_l - \tau_{l-1}$ and τ_l $(1 \leq l \leq n)$ is the instant when the uniformized Markov chain makes the l^{th} transition. To complete the definition of $V_l(n,t)$ for the cases where $l = 1$ and $l = n+1$, we add the definitions: $\tau_0 = 0$ and $\tau_{n+1} = t$.

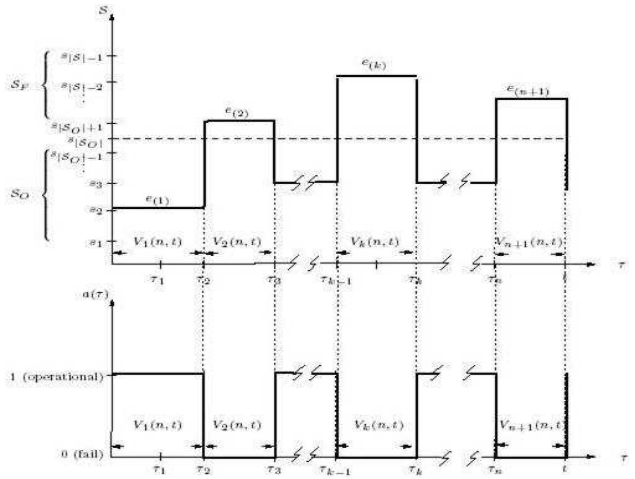

Figure 1: Upper graph: example of a path $\vec{e}(n)$. Lower graph: $a(\tau)$ associated to $\vec{e}(n)$

From now on, the paper describes a new derivation of $E[O^k(t)]$.
Substituting (5) into (4), we obtain

$$E[O^k(t)] = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \sum_{\forall \vec{e}(n)} P(\vec{e}(n)) \cdot E\left[\left(\sum_{l=1}^{n+1} V_l(n,t) \cdot I(e_{(l)} \in \mathcal{S}_O)\right)^k\right] \tag{6}$$

To evaluate $P(\vec{e}(n))$ recurrently, we separate the set of possible sample path according to its last visited state. To this end we use the fact that $\sum_{\forall \vec{e}(n)} = \sum_{\forall s_i \in \mathcal{S}} \sum_{\substack{\forall\ \vec{e}(n)\ \mid \\ e_{(n+1)} = s_i}}$, where "|" stands for "such that", and "$\vec{e}(n) \mid e_{(n+1)} = s_i$" is a sample path traveled by the uniformized Markov chain that ends in the state s_i (that is: $e_{(n+1)} = s_i$). This equality allows us to rewrite (6) as:

$$E[O^k(t)] = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \sum_{\forall s_i \in \mathcal{S}} \sum_{\substack{\forall\ \vec{e}(n)\ \mid \\ e_{(n+1)} = s_i}} P(\vec{e}(n)) \cdot E\left[\left(\sum_{l=1}^{n+1} V_l(n,t) \cdot I(e_{(l)} \in \mathcal{S}_O)\right)^k\right] \tag{7}$$

Let $|\vec{e}(n)|$ be the number of operational states visited by the path $\vec{e}(n)$ during the interval $[0,t]$, that is:

$$|\vec{e}(n)| = \sum_{l=1}^{n+1} I(e_{(l)} \in \mathcal{S}_O) \tag{8}$$

Note that for any sample path $\vec{e}(n)$, $0 \leq |\vec{e}(n)| \leq n+1$.

It is well known that $\sum_{l=1}^{n+1} V_l(n,t) I(e_{(l)} \in \mathcal{S}_O)$ [8] has the same distribution of the $|\vec{e}(n)| - th$ order statistic of n independent random variables, uniformly distributed over the interval $[0,t]$. Let $f_{U_{(h)}(n,t)}(\tau)$ be the probability density function of the h^{th} order statistic of n (≥ 1) independent uniform random variables on the interval $[0,t]$. Exploiting the fact that $f_{U_{(h)}(n,t)}(\tau) = h\binom{n}{h}\frac{\tau^{h-1}(t-\tau)^{n-h}}{t^n}$ [7] an applying the definition of the k^{th} moment, we can then write:

$$E\left[\left(\sum_{l=1}^{n+1} V_l(n,t) \cdot I(e_{(l)} \in \mathcal{S}_O)\right)^k\right] = |\vec{e}(n)|\binom{n}{|\vec{e}(n)|}\frac{1}{t^n} \cdot \int_0^t \tau^{k+|\vec{e}(n)|-1}(t-\tau)^{n-|\vec{e}(n)|} d\tau$$

Upon solving the integral we have:

$$E\left[\left(\sum_{l=1}^{n+1} V_l(n,t) \cdot I(e_{(l)} \in \mathcal{S}_O)\right)^k\right] = t^k \frac{\binom{|\vec{e}(n)|-1+k}{|\vec{e}(n)|-1}}{\binom{n+k}{n}};\ n \geq 1 \tag{9}$$

Substituting (9) into (8), we then obtain:

$$E\Big[O^k(t)\Big] = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \sum_{\forall s_i \in \mathcal{S}} \sum_{\substack{\forall\ \vec{e}(n)\ \mid \\ e_{(n+1)} = s_i}} P(\vec{e}(n)) t^k \cdot \frac{\binom{|\vec{e}(n)|-1+k}{|\vec{e}(n)|-1}}{\binom{n+k}{n}} \tag{10}$$

Equation (10) constitutes a new method of evaluating $E[O^k(t)]$. This solution displays a desirable feature: each of its terms has a simple probabilistic interpretation, meaning that the entire right-hand expression has a simple interpretation as well. However, it also exhibits a high computational cost and may have numerical stability problems due to the binomial coefficients.

To overcome these drawbacks, a new method which allows the evaluation of the right-hand side of equation (10) in an efficient and stable manner will be derived in the following.

First, given that $E[O(t)] = tE[A(t)]$ it is also true that

$$E[O^k(t)] = t^k E[A^k(t)] \tag{11}$$

Equations (10) and (11) imply that:

$$E[A^k(t)] = \sum_{n=0}^{\infty} e^{-\Lambda t}\frac{(\Lambda t)^n}{n!}\sum_{\forall s_i\in\mathcal{S}} \cdot \sum_{\substack{\forall\ \vec{e}(n)\ |\\ e_{(n+1)}=s_i}} P(\vec{e}(n))\frac{\binom{|\vec{e}(n)|-1+k}{|\vec{e}(n)|-1}}{\binom{n+k}{n}} \quad (12)$$

Since $0 \le E[A(t)] \le 1$, the k^{th} moment of interval availability satisfies $0 \le E[A^k(t)] \le 1$. Analogously, because $0 \le E[O(t)] \le t$, the k^{th} moment of cumulative operational time satisfies $0 \le E[O^k(t)] \le t^k$. This implies that from a numerical point of view it is more convenient to evaluate equation (12) than (10). In what follows, therefore, we shall concentrate on the evaluation of $E[A^k(t)]$.

2.1 Evaluation of $E[A^k(t)]$

We begin defining:

$$A_i^k(n) = \sum_{\substack{\forall\ \vec{e}(n)\ |\\ e_{(n+1)}\ =\ s_i}} P(\vec{e}(n))\frac{\binom{|\vec{e}(n)|-1+k}{|\vec{e}(n)|-1}}{\binom{n+k}{n}} \quad (13)$$

$$0 \le i \le |\mathcal{S}|; \quad k \ge 1; n \ge 0$$

The advantage to defining the $A_i^k(n)$ terms is that they can be evaluated efficiently using a set of recurrent equations. To accomplish this we first apply the Chapman-Kolmogorov equation, which gives us

$$A_i^k(n) = \sum_{\substack{\forall s_j|\\ P_{j,i}\neq 0}} P_{j,i} \sum_{\substack{\forall\vec{e}(n-1)|\\ e_{(n)}=s_j}} P(\vec{e}(n-1))\frac{\binom{|\vec{e}(n)|-1+k}{|\vec{e}(n)|-1}}{\binom{n+k}{n}} \quad (14)$$

$$0 \le i \le |\mathcal{S}|; \quad k \ge 1; n \ge 0;$$

where $P_{j,i}$ is the probability that the uniformized Markov chain changes its state in one step from, s_j to s_i, and $\vec{e}(n-1)$ is obtained from the vector $\vec{e}(n)$ by extracting the $e_{(n+1)}$ component, in this case s_i.

To recurrently evaluate the right-hand side of equation (13) we must express the second summation of (14) in a convenient form. Taking into account that the path $\vec{e}(n)$ has n transition associated to it while path $\vec{e}(n-1)$ has only $(n-1)$, and that if $s_i \in \mathcal{S}_O$ then $|\vec{e}(n)| = |\vec{e}(n-1)|+1$ whereas if $s_i \in \mathcal{S}_F$, then $|\vec{e}(n)| = |\vec{e}(n-1)|$, we rewrite equation (14) as follows:

$$\begin{aligned} A_i^k(n) &= I(s_i\in\mathcal{S}_F)\sum_{\substack{\forall s_j|\\ P_{j,i}\neq 0}} P_{j,i}\sum_{\substack{\forall\vec{e}(n-1)|\\ e_{(n)}=s_j}} P(\vec{e}(n-1))\frac{n}{n+k} \\ &\cdot\frac{\binom{|\vec{e}(n-1)|-1+k}{|\vec{e}(n-1)|-1}}{\binom{n-1+k}{n-1}} \\ &+I(s_i\in\mathcal{S}_O)\sum_{\substack{\forall s_j|\\ P_{j,i}\neq 0}} P_{j,i}\sum_{\substack{\forall\vec{e}(n-1)|\\ e_{(n)}=s_j}} P(\vec{e}(n-1)) \\ &\cdot\left[\frac{n}{n+k}\frac{\binom{|\vec{e}(n)|-2+k}{|\vec{e}(n)|-2}}{\binom{n-1+k}{n-1}}\right. \\ &\left.+\frac{k}{n+k}\frac{\binom{|\vec{e}(n)|-1+(k-1)}{|\vec{e}(n)|-1}}{\binom{n+(k-1)}{n}}\right] \end{aligned} \quad (15)$$

Equation (15) is in turn rewritten so that the class of terms defined in equation (13) is easily recognizable:

$$\begin{aligned} A_i^k(n) &= I(s_i\in\mathcal{S}_F)\sum_{\substack{\forall s_j|\\ P_{j,i}\neq 0}} P_{j,i}\frac{n}{n+k} \\ &\cdot\sum_{\substack{\forall\vec{e}(n-1)|\\ e_{(n)}=s_j}} P(\vec{e}(n-1))\frac{\binom{|\vec{e}(n-1)|-1+k}{|\vec{e}(n-1)|-1}}{\binom{n-1+k}{n-1}} \\ &+I(s_i\in\mathcal{S}_O)\sum_{\substack{\forall s_j|\\ P_{j,i}\neq 0}} P_{j,i}\frac{n}{n+k} \\ &\cdot\sum_{\substack{\forall\vec{e}(n-1)|\\ e_{(n)}=s_j}} P(\vec{e}(n-1)\frac{\binom{|\vec{e}(n-1)|-2+k}{|\vec{e}(n-1)|-2}}{\binom{n-1+k}{n-1}} \\ &+I(s_i\in\mathcal{S}_O)\frac{k}{n+k} \\ &\cdot\left\{\sum_{\substack{\forall s_j|\\ P_{j,i}\neq 0}} P_{j,i}\sum_{\substack{\forall\vec{e}(n-1)|\\ e_{(n)}=s_j}} P(\vec{e}(n-1))\right\} \\ &\cdot\frac{\binom{|\vec{e}(n)|-1+(k-1)}{|\vec{e}(n)|-1}}{\binom{n+(k-1)}{n}} \end{aligned} \quad (16)$$

Notice that the second summation in both the first and second terms of equation (16) equals $A_j^k(n-1)$ (see equation (13)). Also, the factor between braces in the last term of (16)

is equal to $\sum_{\forall \vec{e}(n)} P(\vec{e}(n))$, which implies that the expression following the open brace of the same term equals $A_i^{k-1}(n)$. These equivalencies allow us to reexpress equation (16) in the following manner:

$$\begin{aligned} A_i^k(n) &= \sum_{\forall s_j \in \mathcal{S}} P_{j,i} \Big\{ I(s_i \in \mathcal{S}_F) \frac{n}{n+k} A_j^k(n-1) \\ &\quad + I(s_i \in \mathcal{S}_O) \frac{n}{n+k} A_j^k(n-1) \Big\} \\ &\quad + I(s_i \in \mathcal{S}_O) \frac{k}{n+k} A_i^{k-1}(n) \qquad (17) \\ &\quad 0 \le i \le |\mathcal{S}| - 1;\ k \ge 1;\ 1 \le n \end{aligned}$$

Since $I(s_i \in \mathcal{S}_F) + I(s_i \in \mathcal{S}_O) = 1$, equation (17) can also be rewritten as

$$\begin{aligned} A_i^k(n) &= I(s_i \in \mathcal{S}_O) \frac{k}{(n+k)} A_i^{k-1}(n) \\ &\quad + \frac{n}{(n+k)} \sum_{\forall s_j \in \mathcal{S}} P_{j,i} A_j^k(n-1) \qquad (18) \\ &\quad 0 \le i \le |\mathcal{S}| - 1;\ k \ge 1;\ n \ge 1 \end{aligned}$$

Equation (15) allow a recurrent evaluation of the terms $A_i^k(n)$ as a function of the parameters h and k.

The inicial conditions of the recurrence of equation (18) are determined by the definition of the recurrence given in equation (13). Then: $A_i^0(n) = \sum_{\substack{\vec{e}(n)| \\ e_{(n+1)}=s_i}} P(\vec{e}(n))$. Which in turn equals $\pi_i(n)$, the probability that the Markov chain enters state s_i immediately after performing the n^{th} transition. These equivalencies may be summed up in the equation:

$$A_i^0(n) = \pi_i(n); \quad 0 \le i \le |\mathcal{S}| - 1; \quad n \ge 0 \qquad (19)$$

where $\pi_i(n)$ is evaluated recurrently using the Chapman-Kolmogorov equation, so that

$$\pi_i(n) = \sum_{\forall s_j | P_{j,i} \neq 0} P_{j,i} \pi_j(n-1); \quad 0 \le i, j \le |\mathcal{S}| - 1;\ n \ge 1 \qquad (20)$$

Analogously, it is easy to conclude that:

$$A_i^k(0) = \pi_i(0) I(s_i \in \mathcal{S}_O); \quad 0 \le i \le |\mathcal{S}| - 1; \quad k \ge 1 \qquad (21)$$

Equations (19) and (21) are the initial conditions of the recurrent equation (18).

Returning to our main problem, which is the evaluation of $E[A^k(t)]$, we define $A^k(n)$ as:

$$A^k(n) = \sum_{\forall s_i \in \mathcal{S}} A_i^k(n) \qquad (22)$$

Then substituting (22) into (12), we obtain:

$$E[A^k(t)] = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} A^k(n) \qquad (23)$$

As usual when applying the uniformization technique, in order to actually evaluate (22) the first summation is limited to a maximum of N_{max} terms, where N_{max} is a previously evaluated number that limits the error to a predetermined maximum $\varepsilon(N_{max})$ [4]. With this modification we arrive at our final solution, which is:

$$E[A^k(t)] = \sum_{n=0}^{N_{max}} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} A^k(n) + \varepsilon(N_{max}) \qquad (24)$$

Table 2 brings together the equations comprising the method just outlined for evaluating the first K moments of the internal availability and the cumulative operational time.

Table 2: Summary of the proposed method.

Equation (21) $A_i^k(0) = \pi_i(0) I(s_i \in S_0)$; $0 \le i \le \lvert\mathcal{S}\rvert - 1; k \ge 1$
Equation (20) $\pi_i(n) = \sum_{\forall s_j \lvert P_{j,i} \neq 0} P_{j,i} \pi_j(n-1)$; $0 \le i \le \lvert\mathcal{S}\rvert - 1; \quad n \ge 1$
Equation (19) $A_i^0(n) = \pi_i(n)$; $0 \le i \le \lvert\mathcal{S}\rvert - 1; \quad n \ge 0$
Equation (18) $A_i^k(n) = I(s_i \in \mathcal{S}_O) \frac{k}{(n+k)} A_i^{k-1}(n)$ $+ \frac{n}{n+k} \sum_{\substack{\forall s_j \lvert \\ P_{j,i} \neq 0}} P_{j,i} A_j^k(n-1)$; $0 \le i \le \lvert\mathcal{S}\rvert - 1; k \ge 1; n \ge 1$
Equation (22) $A^k(n) = \sum_{\forall s_i \in \mathcal{S}} A_i^k(n)$
Equation (24) For each different value of t compute: $E[A^k(t)] = \sum_{n=0}^{N_{max}} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} A^k(n) + \varepsilon(N_{max})$
Equation (11) $E[O^k(t)] = t^k [A^k(t)]$

The method summarized in Table 2 possesses certain highly desirable characteristics, which are:

Probabilistic Interpretation

As shown in equations (9), (10), (11) and (13), the $A_i^k(n)$ ($k \ge 1$) terms correspond to the k^{th} moment of transient availability for the special case in which the uniformized

Markov chain performs n transitions and the last state visited is s_i. Additionaly, as stated in equation (20), $A_i^0(n)$ is equal to $\pi_i(n)$. This means that each term that form part of the final method to evaluate $E[A^k(t)]$ has its own probabilistic interpretation.

Numerical Stability
As noted in the discussion following equation (12), the k^{th} moment of interval availability is a value bounded between 0 and 1. Also, since the $A_i^k(n)$ terms are positive values (see equation (13)) and are components of $E[A^k(t)]$, each of them is necessarily bounded by 0 and 1 as well. Furthermore, we observe in equation (18) that they are all obtained by summing two terms that in turn are the product of two values bounded by 0 and 1. These characteristics demonstrate that $E[A^k(t)]$ is obtained solely by summing or multiplying values bounded between 0 and 1, thereby ruling out numerical stability problems in the use of the proposed method.

Spatial Cost
For each step in the algorithm (a given value of n), the main memory cost is the need to store the $A_i^k(n)$ terms, with $0 \leq i \leq |\mathcal{S}|-1$ and $0 \leq k \leq K$, which require $\mathcal{O}(|\mathcal{S}|K)$ memory locations. The stochastic matrix $\mathbf{P}$ that models the system must also be stored. Since matrix $\mathbf{P}$ is normally a sparse matrix, the use of sparse matrix techniques ensures that only $O(|\mathcal{S}|d)$ memory locations are needed, where d is the mean number of states adjacent to a state in $\mathbf{P}$. The algorithm thus requires $O(|\mathcal{S}|(d+K))$ memory locations in all. Using the fact that $|T| = |S|d$, this complexity reduces to $O(|T| + |S|K)$.

Time Cost
According to (18) the evaluation of each of the $A_i^k(n)$ terms ($0 \leq i \leq |\mathcal{S}|-1$; $0 \leq n \leq N_{max}$ and $0 \leq k \leq K$) requires $|\mathcal{S}|$ multiplication or addition operations. Again, however, by resorting to sparse matrix techniques the evaluation of each of these terms will require only d operations. The foregoing leads us to conclude that the method uses $O(|\mathcal{S}|dKN_{max}) = O(|T|KN_{max})$ multiplication or addition operations. In addition, it should be noticed that the evaluation of the $A_i^k(n)$ terms does not depend on the length of the utilization time. Thus, the time complexity of the proposed method depends only on the value of the longest utilization time (represented by the value of N_{max}) for which $[A^k(t)]$ is evaluated and not L, the number of different utilization intervals for which the interval availability is evaluated. The existing methods instead, need to be executed once for each of the L different values. As a result, the temporal cost of all the methods listed in Table 1 must be multiplied by the factor L.

This fact implies that the proposed method is $O(L|\mathcal{S}|)$ times faster than the best current method ([6]), without any extra memory requirement.

3. NUMERICAL EXAMPLE

As a way of example, the proposed method was applied to a multiprocessor system made of C identical processors, M identical memory units and B identical buses. The multiprocessor system is in operational state as long as at least 1 processor, 1 memory unit and 1 bus are working. The failure and reparation rate of each system component (under non-operational states it is assumed that components do not fail) are as follows:

	Failure rate	Reparation rate
Processor	$\mu_C = 1/2$ year^{-1}	$\lambda_C = 1/20\ min^{-1}$
Memory unit	$\mu_M = 1/3$ year^{-1}	$\lambda_M = 1/10\ min^{-1}$
Bus	$\mu_B = 1/3$ year^{-1}	$\lambda_B = 1/60\ min^{-1}$

Model. The duration of the reparation and failure times are assumed to be governed by an exponential distribution. Thus, the multiprocessor system can be modelled by a Markov chain whose states are given by the triplet (c,m,b) where c, m and b represent the number of operational CPUs (processors), the number of operational memory units and the number of operational buses, respectively. The operational states with the lowest and highest number of operative components are the states (1,1,1) and (C,M,B), respectively. For this example C=36, M=144 and B=72. This model of the system has 391.392 states and 2.630.880 transitions between states.

Model Solution. A system like the one described in this section is normally used for several years. Therefore, an utilization period of approximately 2 years ($t = 10^6$ minutes, to be precise) was used in this numerical example. The uniformization rate was $\Lambda \approx 0.17$ (then $\Lambda t = 1.7 \cdot 10^5$). Figure 2 and Table 3 show the interval availability for different utilization periods, shorter or equal to 2 years. It should be noticed that, due to the efficiency of the proposed method, these results were obtained in a few hours in a personal computer, in spite of the large Markov chain utilized (in the order of 10^5 states and 10^6 transitions) and the long utilization period (in the order of 10^6 units of time). >From Figure 2 it can be seen that the interval availability of the system decreases as the utilization period increases. When the utilization period is close to 2 years, the interval availability tends to stabilize because the length of the failure-reparation cycle is in the order of the utilization period. For utilization periods shorter than 2 years instead, the interval availability is significantly higher because during such utilization periods failures barely occur.

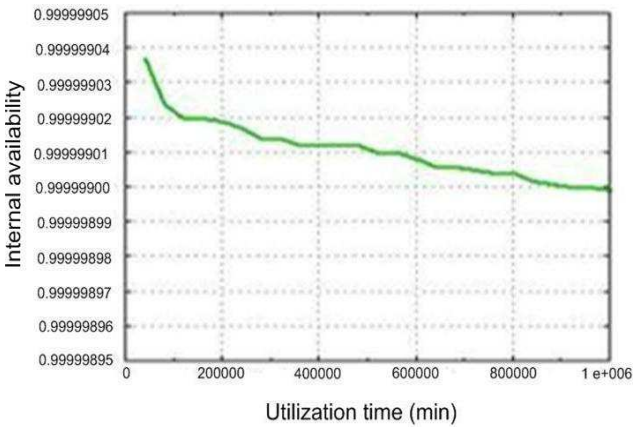

Figure 2: Interval availability for different utilization time intervals

Utilization time [min]	Mean Operational time [min]	Mean Interval availability
40000	39999,9615	0,999999037
200000	199999,8039	0,999999019
400000	399999,6048	0,999999012
600000	599999,4048	0,999999008
800000	799999,2033	0,999999004
1000000	999998,9992	0,999998999

Table 3: Operational time and interval availability for different values of the utilization time

4. CONCLUDING REMARKS

A new method was presented for evaluating the first K moments of the interval availability for repairable systems. The proposed solution is $O(L|\mathcal{S}|)$ times faster than the best existing algorithm without any extra memory cost and it is numerically stable. The time-domain approach of this new solution is based on the uniformization technique and allows all of its terms to be given a probabilistic interpretation.

This new method, with its low computational cost, allows that problems with large Markov model space can be solved in a matter of few minutes on a standard current-model PC.

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