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On the Degrees of Freedom for Multi-Hop Wireless Networks under Layered TDD Constraint

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Abstract

Degrees of freedom (DoF) of a network build a new scaling law characterizing the scalability of capacity at high signal-to-noise region. In this paper, we extend our recent work from cascaded network to the general K-hop layered network. The main framework is based on the assumption of layered TDD, where all nodes at each layer work with the same on/off status. By this approach we decompose the DoF analysis into two steps: 1) apply the result of cascaded networks; 2) analyze / design the transmission of each hop. The upper and lower bounds on DoF are deduced. By viewing the network as cascaded X channels, we find an inner bound of the DoF region, applicable to many message topologies. The detail of message splitting is demonstrated. Finally ultimate analysis shows if the number of antennas/nodes at each relay layer goes to infinity, the lower bound reaches the upper bound. As a by-product, when K > 2 the network can alleviate the effect of TDD with the increase of relay antennas/nodes.

Keywords: Multi-hop networks, degree of freedom (DoF), layered TDD, decode-and-forward, Fibonacci.

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1. Introduction

For wireless networks, it is important to find/build a scaling law, which conventionally means the asymptotic capacity (or rate) as the number of nodes increases under fixed signal-to-noise ratio (SNR) [1–6]. Recently, another kind of scaling law has been built to characterize the scalability of a network's capacity by the concept of degrees of freedom (DoF), along with the development of multiple-antenna theory and techniques [7, 8]. This new scaling law is about the asymptotic behavior of DoF when SNR increases to infinity, where multiple-antenna node scenario is usually considered, including single-antenna node as a special case. We only consider the DoF scaling law in this paper.

DoF concept might stem from the pre-log factor or multiplexing gain in single-user multiple-input multiple-output (MIMO) systems [7–9], representing the maximum number of independent message streams which can be simultaneously transmitted from the source to the destination with arbitrary low error-probability. If perfect channel state information (CSI) is available at the receiver/transmitter and there are M antennas at the transmitter and N antennas at the receiver, its DoF is $d = \min(M, N)$, showing linear scaling of its capacity expression: $C = d\log(\rho) + o(\log(\rho))$, with the definition of SNR as $\rho = P/\sigma_{\mathrm{n}}^2$, where P is the transmit power and σ_{n}^2 is the noise power. For multi-user unicast MIMO systems, DoF region is used to characterize the mutual limitation on all desired messages, while sum DoF is an approximation of the sum rate. Different node/link/message topology and antenna configuration will result in different DoF regions. The maximum sum DoF (often called DoF for brevity) indeed represents the scaling factor, while DoF region provides more details for understanding a network.

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Besides the single-user channel, many results on DoF of single-hop wireless networks have been obtained. For the multiuser multiple access channel (MAC) [10] and broadcast channel (BC) [11, 12] the DoF is equal to that of single-user MIMO with the same number of transmit and receive antennas if channel state information (CSI) is available at receiver for MAC and transmitter for BC. There might be DoF loss if the BC transmitter has no CSI. For these basic scenarios, centralized processing can be done at least at one end of the link(s) and the spatial/antenna resource can be fully exploited by using simple method such as zero-forcing approach. However, for any network with multiple transmitters and multiple receivers, only distributed processing is possible, which will generally cause some DoF loss. Assuming global CSI at all transmitters and receivers, interference alignment (IA) technique provides a new method to manage the effect of interference to the minimum extent. The idea behind IA is to allocate the signal spaces for all messages such that the interferences at each receiver can be aligned/overlapped into a common space as well as keeping its desired messages decodable. There have been many works studying the DoF aspect of interference channel (IC) [13–15] and X channel (XC) (also called X network) [16–19]. A general $M \times N$ XC (M transmitters and N receivers) [18, 19] with single antenna at each node has a sum DoF of $\frac{MN}{M+N-1}$. Later, the DoF of an $M \times N$ MIMO X network with A antennas at each node is shown [20] to be $AMN/(M + N - 1)$. The IA technique had been surveyed and summarized by Syed. A. Jafar in monograph [21].

However, the DoF behavior of general multi-hop networks is often very hard to investigate. Since line networks are often more accessible for analysis and are fundamental building blocks for complicated networks, the assumption of network architectures structured as line directed chains has been chosen for various problems [22]. Most works choose layered networks for consideration, which is the direct extension from cascaded network by splitting each node into multiple subnodes. Here cascaded networks is composed by multiple nodes connected one by one as a chain. For multi-hop wireless networks, there is a basic upper bound given by the famous max-flow-min-cut theorem of Ford-Fulkerson [23], which comes from the wired network. Some recent works including [24–28] analyze the achievable capacity or DoF of specific wireless networks, most of which consider the interference relay networks and two-hop layered networks. These works make significant progress for understanding the fundamental limits of wireless multi-hop networks. Some important concepts and techniques such as interference neutralization (IN) and aligned IN have been proposed. Nevertheless, there are very few general results. Moreover, full-duplex mode is often assumed in most works, which makes the achievable schemes for multi-hop networks quite impractical. We know that full-duplex mode brings serious self-interference so that practical systems often instead choose half-duplex mode to reduce the cost, including frequency-division-duplex (FDD) and time-division-duplex (TDD). The network behavior will be significantly affected by half-duplex constraint.

The upper bounds on the capacity in general multi-terminal networks with finite number of states under TDD constraint are derived and applied to the cascaded relay network [29]. Then in [30], this upper bound is shown to be achievable for fixed channel. Later, [31] proves the achievable DF rate of a two-hop layered network with single source and single destination under TDD constraint is half of that of the corresponding channel without relay layer, when the number of relay nodes goes to infinity. To improve the spectral efficiency, [32] proposes two-way relaying and two-path relaying protocols which in fact change the original message topology of the networks. Unlike cascaded network where transmit-receive is only allowed between neighboring layers, some linear models, such as [22, 33, 34], assume a node can transmit to or receive from multiple nodes in the line networks, i.e., allowing cross-layer communication. On the other hand, for QoS and optimization, queuing and game theories provide powerful tools for performance analysis. There are many related papers such as [35, 36] in this research topic, which can also be exploited for further study.

In this paper, we consider the DoF of arbitrary layered multi-hop wireless networks under layered TDD constraint. Based on our recent investigation on single-node cascaded network, Fibonacci sequence has been connected to the number of feasible network states (FNS), and a lower bound has been found by maximizing the decode-and-forward (DF) rate with proper scheduling of all FNS. Based on these results, we study the DoF issue for multi-hop network with multiple-node layers. The basic idea is to view the network as cascaded XC. Upper bound is obtained by using antenna splitting argument. Then we obtain an achievable lower bound by combining the general result of XC and maximum DF rate. This approach is applicable to lots of message topologies including the interference relay network. The idea of message splitting is also demonstrated. We further analyze the ultimate sum DoF when the number of antennas at each relay layer goes to infinity. Compared with the two-hop case, we find that networks with at least three hops can obtain a larger ultimate sum DoF, which is in fact as the single-user MIMO scenario, doubled at most when the source and destination layers have equal number of antennas.

This paper is organized as follows. In Section 2, we introduce the system model. Section 3 provides basic results for cascaded directed networks, while general case with multiple-node layers is studied in Section 4. Ultimate DoF analysis is discussed in Section 5. Finally, Section 6 concludes this paper.

2. System model

The multi-hop layered network model is shown in Fig. 1. There are $K + 1$ layers cascaded one by one: source (S) layer, relay (R) layers ($R_1, \dots, R_{K-1}$), and destination (D) layer. Each layer has one or more wireless nodes. Nodes only at source layer want to send some unicast messages to some nodes only at destination layer. For simplicity, we denote the nodes at layer k as set $V_k, \forall k \in \{0, 1, \dots, K\}$. Thus there are $|V_k|$ nodes for layer k. The hop between layers V_{k-1} and V_k is denoted by $\mathcal{H}_k, \forall k = 1, 2, \dots, K$, whose input and output signals are $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$, respectively. We use V_k^m to denote the mth node at layer k, which has A_k^m antennas. Assume all relay nodes use directional antenna or technique such as beam-forming to guarantee the radio wave is kept forward in single direction: $S \to D$ and the backward radio

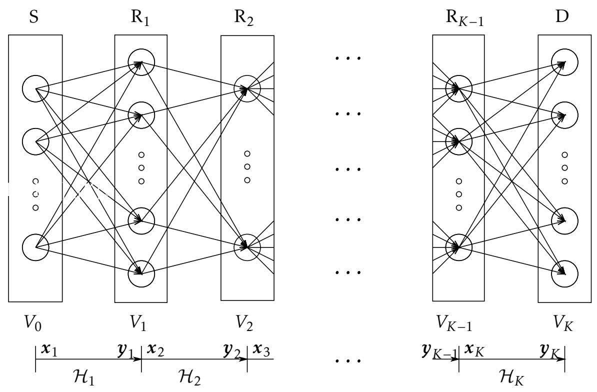

Figure 1. System model of multi-hop layered wireless networks. S: source; R: relay; D: destination. V_k denotes the set of nodes at the kth layer. Messages are forwarded hop by hop. $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ are the input and output of the kth hop $\mathcal{H}_k$.

wave in the direction of $\mathcal{D} \to \mathcal{S}$ is weak enough to be negligible. Furthermore, there is no direct link between non-neighboring layers, i.e., direct communication is only allowed between neighboring layers V_{k-1} and V_k, $\forall k = 1, \dots, K$.

We use binary variable δ_k to identify the state of hop k:

$$\delta_k = \begin{cases} 1 & \mathcal{H}_k \text{ active} \\ 0 & \mathcal{H}_k \text{ inactive} \end{cases}$$

Here state of hop $\mathcal{H}_k$ is inactive or active means all links between V_{k-1} and V_k are shut down or working, respectively. In other words, we assume the link topology between any two neighboring layers is of full connection and has the same on/off status. Then layered TDD (L-TDD) is assumed, i. e., each relay layer cannot send and receive at the same time, which implies that $\mathcal{H}_{k-1}$ and $\mathcal{H}_k$ cannot be simultaneously active. This L-TDD constraint can be expressed as

$$\delta_{k-1}\delta_k = 0, \quad \forall k = 2, 3, \dots, K. \tag{1}$$

Due to the above constraint some transmission patterns are forbidden for the network, which can be indicated by the conception of network (hop) state.

Definition 1. *Network state is defined by*

$$\boldsymbol{q} \triangleq \begin{bmatrix} \delta_1 & \delta_2 & \dots & \delta_K \end{bmatrix} \tag{2}$$

which is a $1 \times K$ binary row vector.

Any non-zero network state is called **feasible** if and only if it satisfies the above L-TDD constraint (1). We call it as FNS for brevity in the following context.

Assume channel coefficients between neighboring nodes $k-1$ and k are non-zero, finite. Since we focus on DoF where the SNR goes to infinity, the transmit power P_k of every node V_k is constrained to P and the additive Gaussian white noise for each receiver is set to be σ_n^2. With full power P, the capacity of $\mathcal{H}_k$ is C_k. DF relay strategy is considered in this paper. Thus, C_k gives the maximum DF rate over the corresponding hop. For brevity, we denote the network by $\mathcal{N}(C_1, C_2, \ldots, C_K)$.

We require that each source node has at least one message to be sent to at least one of the destination nodes, and each destination node has at least one desired message. Otherwise we can delete that node if it has no message to transmit or receive. Denote the message between source V_0^i and destination V_K^j by $W^{[ji]}$, if it exists. Assume all the active messages are mutually independent. For message $W^{[ji]}$, its DoF d_{ji} is defined by

$$d_{ji} \triangleq \left\{ d_{ji} \in \mathbb{R}_+ | \forall w_{ji} \in \mathbb{R}_+, \sum_{j=1}^{|V_K|} \sum_{i=1}^{|V_0|} w_{ji} d_{ji} \leq \right.$$
$$\left. \limsup_{\rho \to \infty} \sup_{[(R_{ji}(\rho))] \in \mathcal{C}(\rho)} \sum_{j=1}^{|V_K|} \sum_{i=1}^{|V_0|} \frac{w_{ji} R_{ji}(\rho)}{\log(\rho)} \right\} \tag{3}$$

where w_{ji} is the weight factor, $R_{ji}(\rho) = \frac{\log(|W^{[ji]}(\rho)|)}{\kappa_0}$ is the rate of the codeword encoding the message $W^{[ji]}$, κ_0 is the length of the codeword, $\mathcal{C}(\rho)$ denotes the capacity region of the network, and ρ is the SNR. We further use $\mathcal{D}$ to denote the DoF region:

$$\mathcal{D} = \left\{ d_{ji} | \forall j \in [1, |V_K|], \forall i \in [1, |V_0|] \right\}. \tag{4}$$

The (sum) DoF of the network is given by

$$d_\Sigma = \sum_{j,i} d_{ji}. \tag{5}$$

We want to improve the information transmission rate from S to D. i.e., achieving a higher DF rate / DoF of the whole network. For this goal, we need to schedule the activeness of all FNS. So all channel coefficients are required to keep fixed during the scheduling process.

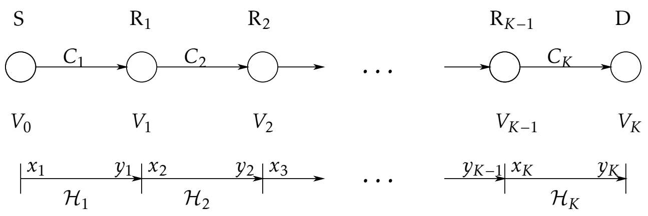

Figure 2. System model of K-hop cascaded directed relay networks with single node at each layer, and component link capacity set $\{C_1, C_2, \cdots, C_K\}$.

After that, the channel coefficients can be different. This implies that our results are also applicable to quasi-static or block fading channels.

It should be pointed out that due to the lack of direct links between non-neighboring layers, messages from S to D might not be transmitted in their original forms. To accomplish the communication task, we may need to reform the messages including splitting and reorganization to make it suitable for the next hop transmission.

3. Basic results for networks with single node at each layer

When each layer has only one node, layered network is simplified to cascaded network. Then we denote the node at layer k by $V_k, \forall k = 0, 1, \ldots, K$, and there are A_k antennas at V_k. The network model is shown in Fig. 2.

In this section, we first show some basic concepts and then use them to analyze the DoF.

3.1. Preliminary

The results provided in this subsection have been accepted for publication [37], so we only give a brief introduction without proof.

On the number of all feasible network states.

Definition 2 (S-sequence). *Let F_n be the Fibonacci sequence, i. e., $F_n = F_{n-1} + F_{n-2}$ with seed values $F_0 = 0$ and $F_1 = 1$. Define a new sequence $S_n = S_{n-1} + F_n$ with seed value $S_0 = 0$. For simplicity, we call it as S-sequence.*

The above S-sequence is shown by the Table 1.

The value of S_n can be computed in closed-form, which can be done by exploiting the relation between

Table 1. Fibonacci sequence and S-sequence

n	0	1	2	3	4	5	6	7	8	...
F_n	0	1	1	2	3	5	8	13	21	...
S_n	0	1	2	4	7	12	20	33	54	...

S-sequence and Fibonacci number. By Binet's formula $F_n = \frac{\psi^n - \varphi^n}{\psi - \varphi} = \frac{\psi^n - \varphi^n}{\sqrt{5}}$, the closed-form expression of the above S-sequence can be proven to be:

$$S_n = \frac{\psi^{n+2} - \varphi^{n+2}}{\psi - \varphi} - 1 = \frac{\psi^{n+2} - \varphi^{n+2}}{\sqrt{5}} - 1, \tag{6}$$

where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio and $\psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi = -\frac{1}{\varphi}$.

We have connected the number of all feasible network states with the above S-sequence, as stated by the following lemma.

Lemma 1. *The number of all feasible network states for K-hop cascaded networks with TDD constraint is S_K.*

Furthermore, the physical meaning of Fibonacci number for the cascaded directed network is that when the number of hops increases from $K-1$ to K, the number of new FNS (i.e., with hop $\mathcal{H}_K$ active) is F_K.

After all feasible network states are obtained, we can arrange them into a $S_K \times K$ matrix $\boldsymbol{Q}_K = \begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}} & \boldsymbol{q}_2^{\mathrm{T}} & \dots & \boldsymbol{q}_{S_K}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, called FNS matrix, each row of which represents a FNS. Here $\boldsymbol{q}_m^{\mathrm{T}}$ is the transpose of the mth feasible network state.

We can obtain the FNS matrix $\boldsymbol{Q}_K$ in a recursive way as

$$\boldsymbol{Q}_K = \begin{bmatrix} \boldsymbol{Q}_{K-1} & \boldsymbol{0}_{S_{K-1}} \\ \boldsymbol{0}_{K-1}^{\mathrm{T}} & 1 \\ \begin{bmatrix} \boldsymbol{Q}_{K-2} & \boldsymbol{0}_{S_{K-2}} \end{bmatrix} & \boldsymbol{1}_{S_{K-2}} \end{bmatrix}, \tag{7}$$

with seeds $\boldsymbol{Q}_1 = 1$ and $\boldsymbol{Q}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, where $\boldsymbol{0}_k$ and $\boldsymbol{1}_k$ are zero and all-ones column vectors with k elements, respectively. For example, we have

$$\boldsymbol{Q}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{Q}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \tag{8}$$

Maximum achievable DF rate. Our target is to maximize the network rate by scheduling all FNS with DF relay strategy. To accomplish this goal, it is reasonable to assume that each transmit node utilizes its maximal power P and each active hop $\mathcal{H}_k$ achieves its full rate C_k. All hops maintain a constant capacity C_k, $\forall k = 1, 2, \dots, K$ during the whole scheduling period time. Thus all messages can be correctly delivered without re-transmission.

Let the mth ($\forall m \in \{1, 2, \dots, S_K\}$) FNS $\boldsymbol{q}_m$, i.e., the m row in $\boldsymbol{Q}_K$, keeps active in t_m seconds. Here a feasible network state is called **active** if its time allocation is non-zero, otherwise inactive. We then arrange these time allocation variables into an $S_K \times 1$ vector as $\boldsymbol{t} = \begin{bmatrix} t_1 & t_2 & \dots & t_{S_K} \end{bmatrix}^{\mathrm{T}}$.

To obtain a stable solution, each hop should carry the same amount of messages (denoted by r) for information balance during a complete scheduling cycle. Here a complete scheduling cycle is a period which contains all active FNS, each of which is used only once. r will be the network rate if the cycle time is normalized to 1.

Based on the above discussion, we can formulate the scheduling problem into a linear program (LP)

$$\begin{aligned} \max \quad & r \\ \text{s.t.} \quad \boldsymbol{D}\boldsymbol{Q}_K^{\mathrm{T}}\boldsymbol{t} &= r\boldsymbol{1} \qquad (9) \\ \sum_{m=1}^{S_K} t_m &= 1 \qquad (10) \\ \boldsymbol{t} &\geq \boldsymbol{0} \qquad (11) \end{aligned}$$

where $\boldsymbol{D} = \mathrm{diag}([C_1, C_2, \dots, C_K])$ is a diagonal matrix, (9) is for rate balance, (10) is used to normalize the time allocation, and (11) stands for non-negativity. Here we omit the dimension of $\boldsymbol{0}$ and $\boldsymbol{1}$ for brevity.

The above LP problem can be efficiently solved by many existing numerical techniques, while closed-form solution can be obtained if $K \leq 3$. Generally, we can get at least one numerical optimal solution $\boldsymbol{t}^*$, which provides a stable scheduling scheme for all feasible network states to maximize the transmission rate of the whole network, denoted by r^*.

Next, we have found the maximum DF rate for cascaded directed network $\mathcal{N}(C_1, C_2, \ldots, C_K)$ under TDD constraint, which is stated by the following lemma.

Lemma 2. *The maximum achievable rate of K-hop cascaded directed relay network $\mathcal{N}(C_1, C_2, \ldots, C_K)$ with DF and TDD is given by*

$$r^*_{cas} = \min\left\{\frac{C_1C_2}{C_1+C_2}, \frac{C_2C_3}{C_2+C_3}, \ldots, \frac{C_{K-1}C_K}{C_{K-1}+C_K}\right\}. \quad (12)$$

And the optimal scheduling scheme is given by solving the above LP problem.

In the quasi-static or block fading scenario, the overall achievable DF rate depends on the fading characterization of the network.

Upper bound. The upper bound on the network capacity is given by [29], which has the same expression as (12). This means that DF with FNS scheduling is optimal, when all links keep constant capacities.

3.2. DoF result

Based on the above results, we can analyze the DoF of cascaded directed network. Since there is only one message from S to D, we simply use d to denote the network DoF.

From the single-user MIMO theory [7–9], the DoF is determined by the rank of its channel. Rank deficiency will cause DoF loss. In order to maximize the DoF utilization, we assume all component channels are of full rank. Thus the DoF will be further determined by the minimum number of antennas at transmitter and receiver. Then we have, $\forall k = 1, 2, \ldots, K$

$$d_k = \min\{A_{k-1}, A_k\} \quad (13)$$
$$C_k = d_k \log\rho + o(\log\rho). \quad (14)$$

In the full-duplex (FD) scenario, the DoF is determined by the maximum-flow-min-cut theorem, which turns out to be

$$d^{FD}_{cas} = \min\{A_0, A_1, \ldots, A_K\} \quad (15)$$

With TDD constraint, we have known that the DF with FNS scheduling is optimal to achieve the network capacity, as long as the component links keep constant capacities. Therefore, the network DoF can be easily obtained, as stated by the following lemma.

Lemma 3. *The DoF of cascade directed K-hop network with constant $C_k, \forall k = 1, 2, \ldots, K$ under TDD constraint is given by*

$$d^{TDD}_{cas} = \min\left\{\frac{d_1d_2}{d_1+d_2}, \frac{d_2d_3}{d_2+d_3}, \ldots, \frac{d_{K-1}d_K}{d_{K-1}+d_K}\right\}. \quad (16)$$

For quasi-static or block fading channels, the above d is still applicable, if only all component channels are of full rank.

4. Extension to multiple-node layer case

For multiple-node layer case, there might be more than one messages. So we use sum DoF d_Σ (5) to investigate the transmission ability of the whole network, and use DoF region to characterize the relationship among all messages. Multiple-node layer makes the network much complicated to analyze, since there might be quite a lot of feasible network states resulted by different on/off status of nodes in each layer.

Recalling the L-TDD constraint, which sets all nodes at each layer working at the same on/off status. This assumption reduces the number of all FNS of layered networks to that of cascaded networks. Due to L-TDD constraint, we can: 1) simplify the analysis of layered networks by regarding each layer as a super node and applying the results for cascaded networks; 2) improve the network design by specifying each component channel with its best scheme. Such a decomposition approach provides a new framework which is applicable to all kind of layered networks.

4.1. Main results

Upper bound on DoF. In fact, the first step of above approach implies that we can obtain an upper bound, which is organized as the following theorem.

Theorem 1. *Sum DoF of the directed multi-hop networks with L-TDD constraint is upper bounded by (16) and (13), where* $A_k = \sum_{m=1}^{|V_k|} A_k^m, \forall k = 0, 1, \ldots, K$.

Proof. We begin with a cascaded network with only one node at each layer. From Lemma 3, we know that the DoF of cascaded network with TDD constraint (1) is given by (16), where V_k is a node with A_k antennas. Now we split each node in this network into several subnodes. In particular, the node $V_k, \forall k = 0, 1, \ldots, K$, is split into $|V_k|$ subnodes, among which the mth subnode is allocated with A_k^m antennas, $\forall m = 1, 2, \ldots, |V_k|$, satisfying $\sum_{m=1}^{|V_k|} A_k^m = A_k$. In this way, we obtain the argued layered network. Because antenna splitting does not increase the capacity and DoF, the DoF result of cascaded network 3 provides an upper bound on the sum DoF of the new network with multiple-node layer. □

For example, the upper bound on the DoF of $2 \times 2 \times 2$ layered network with single-antenna nodes is only 1 due to TDD constraint, in contrast with 2 in full duplexing mode.

An achievable lower bound on sum DoF. The second step of the above framework needs to specify each component channel of the layered multi-hop network. For $\mathcal{H}_k$, if $|V_{k-1}| = 1$ or $|V_k| = 1$, i.e., at least one end has only one node, we can use the corresponding SU/BC/MAC results; if there are multiple nodes at both ends, we can use the existing IC/XC results. The former case obviously has a higher DoF. If all hops $\mathcal{H}_k, \forall k = 1, 2, \ldots, K$ have at least one end with single node, the above upper bound is achievable by straightforward extension of Lemma 3.

In the next context, we assume that there are multiple nodes at each layer, resulting that each component channel has multiple transmitters and multiple receivers. We have known that XC with single-antenna nodes has relatively high DoF and is applicable to arbitrary number of nodes at both ends. Therefore we choose XC as the component channel for each hop to provide the lower bound on the sum DoF of the whole network, which turns out to be a cascaded XC structure.

Theorem 2. *The DoF region* $\mathcal{D} = \left\{d_{ji} | \forall i \in [1, |V_0|], \forall j \in [1, |V_K|]\right\}$ *is achievable if*

$$d_\Sigma^{L-TDD} \le \alpha \tag{17a}$$

$$\sum_{j=1}^{|V_K|} d_{ji} \le \frac{\alpha A_0^i}{\sum_{m=1}^{|V_0|} A_0^m} \quad \forall i \in [1, |V_0|] \tag{17b}$$

$$\sum_{i=1}^{|V_0|} d_{ji} \le \frac{\alpha A_K^j}{\sum_{m=1}^{|V_K|} A_K^m} \quad \forall j \in [1, |V_K|] \tag{17c}$$

where

$$\alpha \triangleq \min\left\{\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \frac{\alpha_2\alpha_3}{\alpha_2+\alpha_3}, \ldots, \frac{\alpha_{K-1}\alpha_K}{\alpha_{K-1}+\alpha_K}\right\} \tag{18}$$

$$\alpha_k \triangleq \frac{\sum_{m=1}^{|V_{k-1}|} A_{k-1}^m \sum_{m=1}^{|V_k|} A_k^m}{\sum_{m=1}^{|V_{k-1}|} A_{k-1}^m + \sum_{m=1}^{|V_k|} A_k^m - 1}, \forall k \in [1, K]. \tag{19}$$

Proof. According to [18], the single-antenna single-hop XC with M transmitters and N receivers has a sum DoF of

$$d_\Sigma^{XC} = \frac{MN}{M+N-1},$$

which can be achieved perfectly or asymptotically by a symmetric and uniform DoF allocation

$$d_{ji} = \frac{1}{M+N-1}, \forall j = 1, 2, \ldots, N; \forall i = 1, 2, \ldots, M.$$

We use this result to show the achievability of the given DoF region.

Now by regarding the kth hop as a $\sum_{m=1}^{|V_{k-1}|} A_{k-1}^m \times \sum_{m=1}^{|V_k|} A_k^m$ virtual XC with single-antenna nodes, we know that a sum DoF of $d_k = \frac{\sum_{m=1}^{|V_{k-1}|} A_{k-1}^m \sum_{m=1}^{|V_k|} A_k^m}{\sum_{m=1}^{|V_{k-1}|} A_{k-1}^m + \sum_{m=1}^{|V_k|} A_k^m - 1}$ is achievable for this channel from [18]. Then we transform the argued network into a cascaded network of XC with single-node layers, whose kth hop has capacity of $C_k = d_k \log\rho + o(\log\rho)$. Applying Lemma 3, we obtain the achievable sum DoF with L-TDD constraint as α, (18).

Moreover, the achievability proof of the XC with single antenna implicitly requires equal DoF allocation for each message, which indicates that each virtual

Table 2. Examples of the source and destination DoF conditions for single-antenna node and $|V_0| = |V_K| = 3$.

Index	DoF for all active messages
1	$d_{11} = d_{22} = d_{33}$
2	$d_{11} = 2d_{22} = 2d_{23} = 2d_{32} = 2d_{33}$
3	$d_{12} = d_{13} = d_{21} = d_{23} = d_{31} = d_{32}$
4	$d_{11} = d_{22} = d_{33} = d_{21} = d_{12} = d_{31} = d_{13} = d_{32} = d_{23}$

single-antenna source/destination node has equal outgoing/ingoing sum DoF. In other words, the DoF resource should be uniformly allocated to each source/destination node according to their number of antennas. Since the achievable sum DoF of the whole network is α, we have

$$\sum_{j=1}^{|V_K|} d_{ji} \leq \frac{\alpha A_0^i}{\sum\limits_{m=1}^{|V_0|} A_0^m}, \quad \forall i \in [1, |V_0|]$$

for the source layer and

$$\sum_{i=1}^{|V_0|} d_{ji} \leq \frac{\alpha A_K^j}{\sum_{m=1}^{|V_K|} A_K^m} \quad \forall j \in [1, |V_K|]$$

for the destination layer. □

The above lower bound α on DoF provides a baseline to measure the scalability of a given network configuration. For example, the $2 \times 2 \times 2$ interference network with single-antenna nodes can obtain a DoF of 2/3 under L-TDD constraint. In fact, (17) gives an inner bound of the DoF region.

DoF improvement is available if some component channel satisfies special requirement. For example, if all nodes on both sides of $\mathcal{H}_k$ has the same number of antennas, say A (>1). Then we can use the corresponding DoF result

$$\bar{\alpha}_k = \frac{A|V_{k-1}||V_k|}{|V_{k-1}| + |V_k| - 1} \tag{20}$$

to replace α_k in (19) and still compute the network DoF as (18). Specially, if all nodes in the network are equipped with A antennas and there are equal N nodes at each layer, the lower bound on DoF becomes $\frac{AN^2}{4N-2}$, which is higher than the baseline: $\frac{AN^2}{4N-2/A}$.

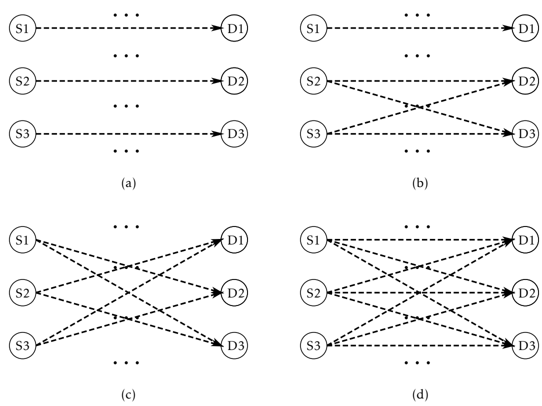

Figure 3. Message topology demonstration for the listed examples. 3 sources; 3 destinations. (a) "≡"; (b) "$1 + X_2$"; (c) "$X_3 -$ ≡"; (d) "X_3".

Further improvement is possible if we use advanced interference management methods such as IA and IN, which is beyond the scope of this paper.

4.2. Examples

Applicable message topology. The source and destination DoF conditions (17b)(17c) for the achievability include quite a lot of message topologies. It can be easily verified that the parallel/interference relay network with equal number of antennas at each transmit/receive node is a special case of our model. Table 2 shows some examples which satisfy these conditions for a network with single-antenna nodes and $|V_0| = |V_K| = 3$.

The message topologies listed in Table 2 are demonstrated in Fig. 3. Here "≡" denotes the 3-user interference relay network, "1" denotes a point-to-point message topology, "X_2" and "X_3" denote 2×2 and 3×3 XC message topology, respectively.

It is interesting to point out that the source and destination DoF conditions (17b) (17c) for the achievability introduce some flexibility: by shutting down one or more antennas at some nodes, the achievable scheme can be applied to more message topologies.

About message splitting. Example 1 in Table 2 is also applicable to multiple-antenna node case. For example, a network with $K = 2$, two source nodes with $A_0^1 = 2$ and $A_0^2 = 1$, two single-antenna nodes at each relay layer, and three single-antenna nodes at the destination layer,

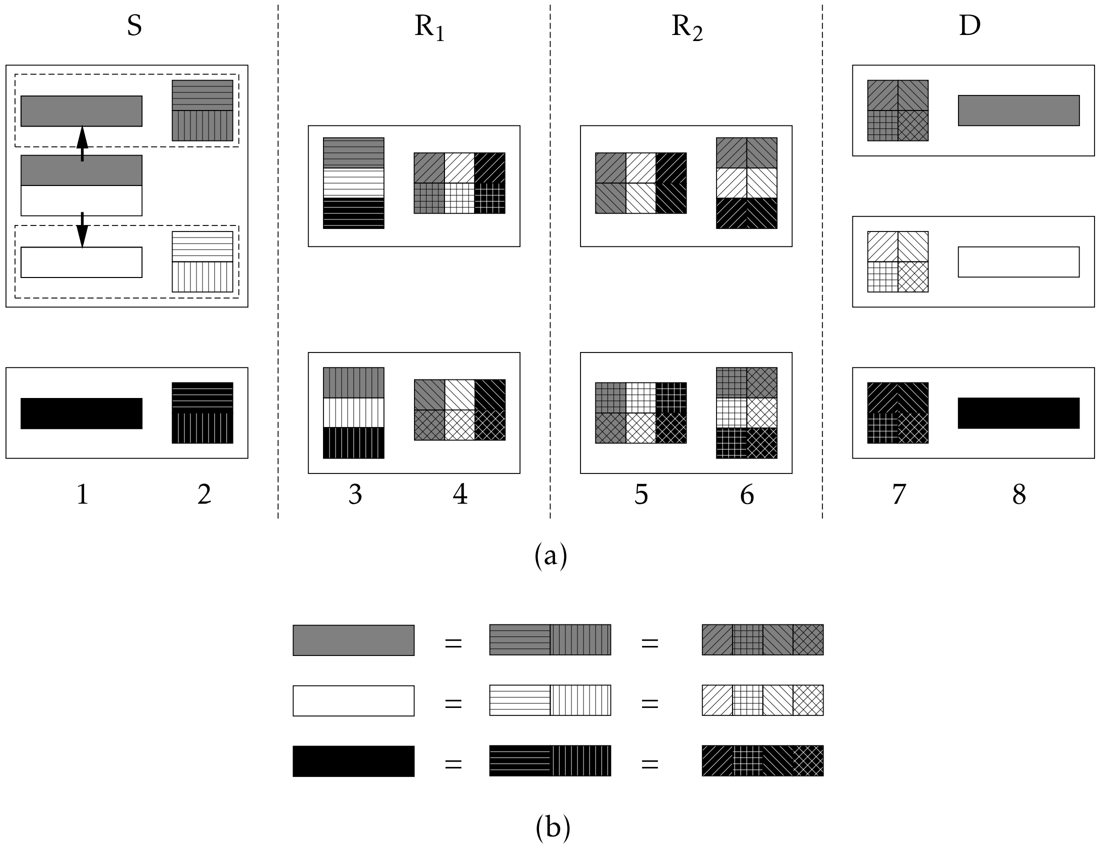

Figure 4. Demonstration of the message splitting processing. (a) Message processing steps: 1. original messages at layer S; 2. split and transmitted messages at layer S; 3. received and decoded messages at layer R_1; 4. split, reorganized, and forwarded messages at layer R_1; 5. received and decoded messages at layer R_2; 6. reorganized and forwarded messages at layer R_2; 7. received and decoded messages at layer D; 8. reorganized orginal messages at layer D. (b) Message relationship.

demonstrated in Fig. 4. We will show the process of message splitting, which can be quite complicated if the numbers of hops, nodes, and antennas become large. At each layer, solid frame box denotes a physical node, while dashed box denotes a virtual distributed node. For each message box, its height represents the DoF for the corresponding message, while its width represents the block length. For the split message boxes except the last hop, same marker style (horizontal, vertical, diagonal,anti-diagonal) indicates that these messages are desired at the same receiver. The filling colors of the message box (gray, white, black) indicate the different destination nodes. We notify that the last relay layer does not split but only reorganize the decoded messages. The ordering of the split sub-messages should be carefully handled at each destination node in order to recover the original messages.

5. Ultimate sum DoF

In this section, we will discuss the ultimate sum DoF when each relay layer has infinite nodes (e.g., for single-antenna node case) or antennas (e.g., for single multiple-antenna node case). In both cases, the total number of antennas at each relay layer goes to infinity. We organize the result as the following corollary.

Corollary 1 (Ultimate sum DoF). *If* $\sum_{m=1}^{|V_K|} A_K^m \to \infty, \forall k = 1, 2, \ldots, K-1,$ *the network has the following ultimate sum DoF:*

$$d_{\Sigma}^{*} = \begin{cases} \frac{\sum_{m=1}^{|V_0|} A_0^m \sum_{m=1}^{|V_2|} A_2^m}{\sum_{m=1}^{|V_0|} A_0^m + \sum_{m=1}^{|V_2|} A_2^m} & K{=}2 \\ \min\left\{\sum_{m=1}^{|V_0|} A_0^m, \sum_{m=1}^{|V_K|} A_K^m\right\} & K{>}2 \end{cases} \quad (21)$$

Proof. We will prove it by checking the upper and lower bounds on the sum DoF for the two conditions about the number of hops respectively.

Upper bound. By Theorem 1 the upper bound on sum DoF of the network is

$$d^{upper} = \min\left\{\frac{d_1 d_2}{d_1+d_2}, \frac{d_2 d_3}{d_2+d_3}, \ldots, \frac{d_{K-1} d_K}{d_{K-1}+d_K}\right\}$$

where

$$d_k \triangleq \min\{A_{k-1}, A_k\}, \forall k = 1, 2, \ldots, K$$

and

$$A_k = \sum_{m=1}^{|V_k|} A_k^m, \forall k = 0, 1, \ldots, K.$$

(1) When $K = 2$, we have $A_1 \to \infty$, A_0 and A_2 are constant. So $d_1 = A_0$ and $d_2 = A_2$ hold. Then the above upper bound simplifies to

$$d_2^{upper} = \frac{A_0 A_2}{A_0 + A_2} = \frac{\sum_{m=1}^{|V_0|} A_0^m \sum_{m=1}^{|V_2|} A_2^m}{\sum_{m=1}^{|V_0|} A_0^m + \sum_{m=1}^{|V_2|} A_2^m}. \quad (22)$$

(2) When $K > 2$, we have $A_1 \to \infty, \ldots, A_{K-1} \to \infty$, while A_0 and A_K are constant. The upper bound turns to

$$\begin{aligned} d_K^{upper} &= \min\left\{\frac{A_0\infty}{A_0+\infty}, \frac{\infty\infty}{\infty+\infty}, \ldots, \frac{\infty A_K}{\infty + A_K}\right\} \\ &= \min(A_0, A_K) \\ &= \min\left(\sum_{m=1}^{|V_0|} A_0^m, \sum_{m=1}^{|V_K|} A_K^m\right) \quad (23) \end{aligned}$$

Lower bound. We use Theorem 2 to compute the lower bound on sum DoF.

(1) When K=2, we have:

$$\alpha_1 = \frac{\sum_{m=1}^{|V_0|} A_0^m \sum_{m=1}^{|V_1|} A_1^m}{\sum_{m=1}^{|V_0|} A_0^m + \sum_{m=1}^{|V_1|} A_1^m - 1} \to \sum_{m=1}^{|V_0|} A_0^m$$

and

$$\alpha_2 = \frac{\sum_{m=1}^{|V_1|} A_1^m \sum_{m=1}^{|V_2|} A_2^m}{\sum_{m=1}^{|V_1|} A_1^m + \sum_{m=1}^{|V_2|} A_2^m - 1} \to \sum_{m=1}^{|V_2|} A_2^m$$

as $\sum_{m=1}^{|V_1|} A_1^m \to \infty$. Thus the lower bound will be

$$d_2^{lower} = \alpha = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} \to \frac{\sum_{m=1}^{|V_0|} A_0^m \sum_{m=1}^{|V_2|} A_2^m}{\sum_{m=1}^{|V_0|} A_0^m + \sum_{m=1}^{|V_2|} A_2^m},$$

which meets the upper bound (22).

(2) When $K > 2$, by Theorem 2, the following sum DoF is achievable

$$\begin{aligned} d_K^{lower} &= \min\left\{\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \frac{\alpha_2\alpha_3}{\alpha_2+\alpha_3}, \ldots, \frac{\alpha_{K-1}\alpha_K}{\alpha_{K-1}+\alpha_K}\right\} \\ &\to \min\left\{\sum_{m=1}^{|V_0|} A_0^m, \sum_{m=1}^{|V_K|} A_K^m\right\} \quad (24) \end{aligned}$$

where we use the fact that $\alpha_1 \to \sum_{m=1}^{|V_0|} A_0^m$, $\alpha_K \to \sum_{m=1}^{|V_K|} A_K^m$, and $\alpha_k = \frac{\sum_{m=1}^{|V_{k-1}|} A_{k-1}^m \sum_{m=1}^{|V_k|} A_k^m}{\sum_{m=1}^{|V_{k-1}|} A_{k-1}^m + \sum_{m=1}^{|V_k|} A_k^m - 1} \to \infty$, $\forall k = 2, 3, \ldots, K-1$. We can see that this lower bound (24) reaches the upper bound (23).

In summary, the lower bound meets the corresponding upper bound for the both cases in ultimate scenario, which completes the proof. □

Remark. Based on Corollary 1, we find when $K > 2$, if the total number of antennas at each relay layer goes to infinity, the whole network will act as a point-to-point MIMO channel with $\sum_{m=1}^{|V_0|} A_0^m$ transmit antennas and $\sum_{m=1}^{|V_K|} A_K^m$ receive antennas, or equivalently as if the layered TDD constraint does not exist. Compared with the two-hop case $K = 2$, a three-hop network can obtain a larger sum DoF, doubled at most when the source layer and destination layer have equal number of antennas. This property indicates some insights. Given enough number of relay nodes (equivalently number of antennas), we can arrange their positions or transmission procedure to build a three-hop network instead of two-hop to maximize the sum DoF. On the other side, the increase of number of hops does not help improve the sum DoF when $K > 3$ in the ultimate case. In summary, when $K > 2$ the network can alleviate the effect of TDD with the increase of relay antennas/nodes in contrast with the case of $K = 2$.

This phenomenon can be clearly shown in Fig. 5, where the two-source two-destination network is composed with single-antenna nodes. For convenience we assume the number of nodes at each relay layer is

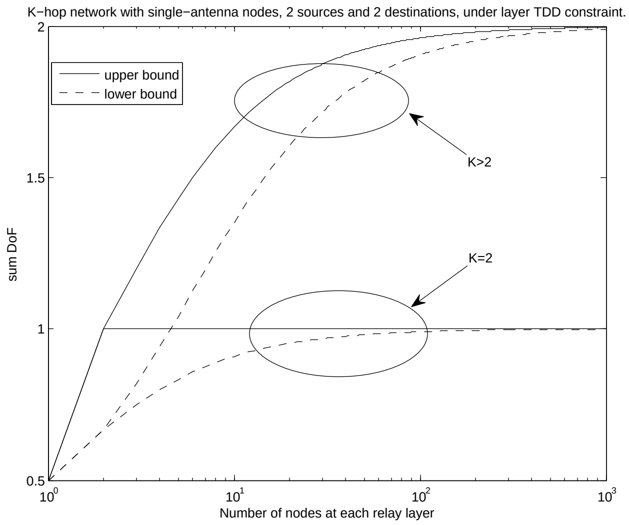

Figure 5. DoF property of directed multi-hop networks under layered TDD constraint

equal. The solid line shows the upper bound on the sum DoF, while the slashed line is for the lower bound obtained by the cascaded XC structure.

6. Conclusion

We analyzed the DoF scaling law of multi-hop wireless network with layered TDD constraint. By utilizing the basic result for cascaded network with single node at each layer, we extended it to the general case where each layer can have multiple nodes. Besides the upper bound, an achievable lower bound on the sum DoF is obtained by choosing XC as component channels, i.e., analyzing the network in the cascaded XC framework. In the ultimate case, we found that networks with no less than three hops have a larger sum DoF than the corresponding two-hop networks and can approach the DoF of single-user MIMO channel when the number of relay nodes/antennas goes to infinity, as if the layered TDD constraint does not exist. So our results might indicate that the future research could be around 3-hop networks with advanced interference management techniques such as IA and IN.

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