On Discrete Time Reversibility modulo State Renaming and its Applications

Andrea Marin
Università Ca' Foscari Venezia, Italy
marin@dais.unive.it

Sabina Rossi
Università Ca' Foscari Venezia, Italy
srossi@dais.unive.it

ABSTRACT

Time reversibility plays an important role in the analysis of continuous and discrete time Markov chains (DTMCs). Specifically, the computation of the stationary distribution of a reversible Markov chain has been proved to be very efficient and does not require the solution of the system of global balance equations. A DTMC is reversible when the processes at forward and reversed time are probabilistically indistinguishable. In this paper we introduce the concept of ρ-*reversibility*, i.e., a notion of reversibility modulo a renaming of the states, and we contrast it with the previous definition of dynamic reversibility especially with respect to the assumptions on the state renaming function. We discuss the applications of discrete time reversibility in the embedded and uniformized chains of continuous time processes.

Categories and Subject Descriptors

C.4 [**Performance of Systems**]: *Modeling techniques*; D.4.8 [**Software**]: Performance—*Modeling and prediction, Stochastic analysis*

Keywords

Discrete Time Markov Chains, Reversibility

1. INTRODUCTION

Markov chains have been widely used to study the performances of computer hardware and telecommunication systems as well as software architectures. Markov models can be designed at a very low-level by mapping each state of the process to a state of the system and then by setting the transition rates/probabilities. However, this method is often time consuming and prone to errors due to the complexity of the considered systems. For these reasons, in the past decades, several formalisms have been developed to allow one to specify a stochastic model in a relative compact way by using important features such as the compositionality and/or the hierarchical approach [5, 1]. Nevertheless, compact descriptions do not necessary allow for efficient analysis because the stochastic process underlying even a simple high-level model may have a very high state space cardinality [14]. If we consider the steady-state analysis, the problem of deriving the stationary probability distribution requires to solve a set of $|S|$ linear equations (called *global balance equations*, GBE), where S is the state space of the chain.

The analysis of Markov chains can sometimes exploit some properties of the stochastic process that allow a numerical or analytical tractability of some performance indices' computation. For instance matrix geometrics/analytics methods take advantage of repeated patterns in the Markov chain process description [12], while lumping [8, 6] exploits the similarity in the stochastic behaviour of some states to reduce the state space. In this paper we focus on the techniques based on the idea of reversible processes (see, e.g., [7, 15, 4, 10, 11]). A Markov chain $X(t)$ is reversible when it is stochastically indistinguishable from the stochastic process at reversed time $X(\tau - t)$ for all τ. The computation of the stationary probabilities of reversible processes is extremely efficient since the system of GBE needs not to be solved and an explicit formula for each unnormalized state probability can be given in terms of the Markov chain's transition rates/probabilities. A remarkable example of reversible model is the set of $M/M/1$ queues with state dependent service rate. In this case the ratio between the stationary probabilities of state $n > 0$ and state 0 can be derived as the product of the transitions from 0 to n (λ^n in case of constant arrival rate λ) divided by the product of the transition rates from n to 0 ($\prod_{i=1}^{n} \mu(i)$, if $\mu(i)$ is the service rate in state i).

Contribution. In this paper we consider a notion of reversibility according to which a discrete time Markov chain $X(t)$ and its reversed $X(\tau - t)$ are probabilistically indistinguishable modulo an arbitrary renaming ρ of the states (ρ-*reversibility*). We show that this class of models enjoys properties that are very similar to those of the standard reversibility [7] and hence the notion of ρ-reversibility allows for an efficient computation of the stationary distribution. Similarly to [7, 15] and in contrast to [4], we show that we can decide this property solely on the analysis of the forward process without the need to derive $X(\tau - t)$. We discuss the properties of the renaming function ρ with attention to what has been previously done in the literature. Finally, we discuss the properties of reversibility modulo renaming both at continuous and discrete time and prove that a CTMC is ρ-reversible if and only if every corresponding uniformized chain is also ρ-reversible. This result does not

VALUETOOLS 2014, December 09-11, Bratislava, Slovakia

DOI 10.4108/icst.valuetools.2014.258200

hold for the embedded chain. We introduce the notion of *almost* ρ-reversible CTMC as a CTMC whose embedded chain is ρ-reversible and show that the conditions for almost ρ-reversibility are weaker than those required by the notion of ρ-reversibility at continuous time studied in [11].

Related work. In [7] the author deeply discusses the notion of time reversibility both for DTMCs and CTMCs, and several applications are illustrated. In particular the author is interested in a sound discussion of the so call *product-form* property of the Markov models. In [15, 3] the authors introduce the notion of *dynamic reversibility* to study physical systems such as the growth and decrescence of two dimensional crystals. We discuss our contribution with respect to such work in Section 3.1. In [10] we introduced the definition of autoreversibility that allows one to exploit the symmetrical structures of a class of CTMCs to derive the steady-state probabilities in an efficient way. Similarly to what is proposed here, a relation between a state and its inverse is taken in consideration. The idea of using a general permutation of states for comparing forward and reversed processes at continuous time is discussed in [11]. However, notice that the formulations of ρ-reversibility in continuous and discrete time differ for some peculiar aspects that we discuss in Section 4.

Structure of the paper. The paper is structured as follows. Section 2 introduces the notation and the definitions of reversible and reversed DTMC. Here, we also prove a new result to characterize reversed DTMCs. In Section 3 we introduce the notion of ρ-reversible DTMC and prove its properties. We also discuss the differences with the previous definition of dynamic reversibility. Section 4 discusses the properties of ρ-reversibility both at continuous and discrete time and states the relationships between a CTMC and the corresponding ρ-reversible uniformized and embedded discrete chains. Section 5 concludes the paper.

2. THEORETICAL BACKGROUND

In this section we review some aspects of the theory of discrete time Markov chains which will be required in the sequel (see [13] for more details).

Preliminaries. A *stochastic process* $X(t)$ in discrete time is a sequence of random variables taking values in a countable state space S for $t \in \mathbb{Z}$. If $(X(t_1), X(t_2), \ldots, X(t_n))$ has the same distribution as $(X(t_1+\tau), X(t_2+\tau), \ldots, X(t_n+\tau))$ for all $t_1, t_2, \ldots, t_n, \tau \in \mathbb{Z}$ then the stochastic process $X(t)$ is said to be *stationary*. A *Discrete Time Markov Chain* (DTMC) is a stochastic process $X(t)$ in discrete time, i.e., with $t \in \mathbb{Z}$, such that for all $t_1 < t_2 < \cdots t_n < t_{n+1}$ the joint distribution of $(X(t_1), X(t_2), \ldots, X(t_n), X(t_{n+1}))$ is such that

$$P(X(t_{n+1}) = i_{n+1} \mid X(t_1) = i_1, X(t_2) = i_2, \ldots, X(t_n) = i_n) \\ = P(X(t_{n+1}) = i_{n+1} \mid X(t_n) = i_n).$$

In other words, for a Markov chain its past evolution until the present state does non influence the conditional (on both past and present states) probability distribution of future behaviour. A Markov chain is *time homogeneous* if the conditional probability $P(X(t+1) = j \mid X(t) = i)$ does not depend upon t, and we use the notation

$$P(X(t+1) = j \mid X(t) = i) = p_{ij}$$

for $i, j \in S$. The value p_{ij} denotes the probability that the chain, whenever in state i, next makes a transition into state j, and is referred to as *one-step transition probability*. The square matrix $\mathbf{P} = (p_{ij})_{i,j \in S}$ is called *one-step transistion matrix*, and since when leaving state i the chain must move to one of the states $j \in S$, each row sums to one (e.g., forms a probability distribution): for each $i \in S$

$$\sum_{j \in S} p_{ij} = 1.$$

The *n-step transition probabilities* of the chain are given by

$$p^n_{ij} = P(X(t+n) = j \mid X(t) = i)$$

for $t, n \in \mathbb{Z}$ and $i, j \in S$, denoting the probability that a process in state i will be in state j after n additional transitions. A state j is said to be *accessible* from state i if $p^n_{ij} > 0$ for some $n \geq 0$. Two states i and j accessible to each other are said to *communicate* and to belong to the same *equivalence class*. A Markov chain is said to be *irreducible* if there is only one equivalence class, i.e., if all states communicate with each other. State i is said to have a *period* $d \geq 1$ if d is the greatest integer such that $p^n_{ii} = 0$ if n is not a multiple of d. A state with period 1 is said to be *aperiodic*. A Markov chain is said to be *aperiodic* if all states are aperiodic. The recurrence time T_i of a Markov chain is defined as $T_i = \min\{n \geq 1 \mid X_n = i \text{ given } X_0 = i\}$ (notice that T_i is a random variable.) A state i is said to be *recurrent* if $P(T_i < \infty) = 1$; otherwise, it is called *transient*. The mean recurrence time M_i of state i is defined as $M_i = E[T_i]$. A recurrent state i is called *positive recurrent* if $M_i < \infty$. A Markov chain is called *positive recurrent* if all of its states are positive recurrent. A time homogeneous DTMC is said *ergodic* if it is irreducible, aperiodic and positive recurrent. Henceforth, we assume the ergodicity of the DTMCs that we study. A chain satisfying all these assumptions admits a unique *stationary distribution*, that is the unique vector $\boldsymbol{\pi}$ of positive numbers π_i with $i \in S$ such that:

$$\boldsymbol{\pi} = \boldsymbol{\pi} P \quad \text{and} \quad \sum_{i \in S} \pi_i = 1. \tag{1}$$

Each π_i can be interpreted as the average proportion of time spent by the chain $X(t)$ in state i. A Markov chain with a stationary distribution is said to be in *steady state*.

Time reversibility. The analysis of an ergodic DTMC with stationary distribution can be greatly simplified if it satisfies the property that when the direction of time is reversed the behaviour of the chain remains the same.

Consider an ergodic DTMC in steady state, $X(t)$ with $t \in \mathbb{Z}$, defined on a state space S with transition probability matrix $\mathbf{P}$ and stationary distribution $\boldsymbol{\pi}$, and suppose that starting at some time we trace the sequence of states going backwards in time. We denote by $X^R(t)$ the reversed process of $X(t)$. It turns out that $X^R(t)$ is also a stationary CTMC.

Definition 1 (Reversibility [7]). *A DTMC $X(t)$ is* reversible *if for all $t_1, t_2, \ldots, t_n, \tau \in \mathbb{Z}$, $(X(t_1), \ldots, X(t_n))$ has the same distribution as $(X(\tau - t_1), \ldots, X(\tau - t_n))$.*

A reversible process is stationary. For a stationary Markov chain there exist simple necessary and sufficient conditions for reversibility expressed in terms of the equilibrium distribution $\boldsymbol{\pi}$ and the transition probabilities p_{ij}. These conditions are given in Proposition 1 and are called the *detailed balance equations*; they should be contrasted with the equilibrium equations, which are sometimes called the global balance equations.

Proposition 1 (Detailed balance equations [7]). *A stationary DTMC with state space S and transition probability matrix $\mathbf{P}$ is* reversible *if and only if there exists a collection of positive real numbers π_i, $i \in S$, summing to unity that satisfy the* detailed balance equations*:*

$$\pi_i p_{ij} = \pi_j p_{ji} \quad \textit{for all } i, j \in S. \tag{2}$$

In that case, π_i is the stationary probability of state i.

Equation (2) can be interpreted as stating that, for all states i and j, the flow from state i to state j is equal to that from state j to state i. If we can find nonnegative numbers, summing to 1, which satisfy Equation (2), then it follows that the Markov chain is reversible and the numbers represent its stationary probabilities. This is so since if

$$x_i p_{ij} = x_j p_{ji} \quad \text{and} \quad \sum_{i \in S} x_i = 1$$

then summing over i yields

$$\sum_{i \in S} x_i p_{ij} = x_j \sum_{i \in S} p_{ji} = x_j \quad \text{and} \quad \sum_{i \in S} x_i = 1$$

that are exactly the conditions in (1). Since the stationary probabilities π_i are the unique solution of the above system of equations, it follows that $x_i = \pi_i$ for all $i \in S$.

An important property of reversible DTMCs is the Kolmogorov's criterion which states that the reversibility of a Markov chain can be established directly from its transition probabilities.

Proposition 2 (Kolmogorov's criterion [13]). *A stationary DTMC with state space S and transition probability matrix $\mathbf{P}$ is* reversible *if and only if starting in state i, any path back to i has the same probability as the reversed path, for all $i \in S$, i.e., for every finite sequence of states $i_1, i_2, \ldots i_n \in S$,*

$$p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-1} i_n} p_{i_n i_1} = p_{i_1 i_n} p_{i_n i_{n-1}} \cdots p_{i_3 i_2} p_{i_2 i_1}. \tag{3}$$

Note that this criterion does not require the knowledge of the stationary distribution.

Reversed process. The transition probabilities of the *reversed process* $X^R(t)$ of a DTMC $X(t)$ can always be defined in terms of the stationary distribution of the process $X(t)$ as stated by the following proposition.

Proposition 3 (Reversed probabilities [13]). *Given a stationary DTMC $X(t)$ with state space S, transition probability matrix $\mathbf{P}$ and stationary distribution $\boldsymbol{\pi}$, the reversed process $X^R(t)$ is an ergodic DTMC with the same state space S, the same stationary distribution $\boldsymbol{\pi}$ and transition probability matrix $\mathbf{P}^R$ defined by:*

$$p^R_{ij} = \frac{\pi_j}{\pi_i} p_{ji}. \tag{4}$$

Observe that we can replace in Equation (4) any non-trivial solution of the GBE. Roughly speaking, we can say that the knowledge of the reversed process' transition probabilities allows for an efficient computation of the invariant measure of the process and vice versa the latter allows for an efficient definition of the reversed process.

The converse of Proposition 3 also holds.

Proposition 4 (Reversed process's equations [13]). *Consider a stationary DTMC $X(t)$ with state space S and transition probability matrix $\mathbf{P}$. If we can find a transition probability matrix $\mathbf{P}^R$ and a stationary distribution $\boldsymbol{\pi}$ such that*

$$\pi_i p_{ij} = \pi_j p^R_{ji} \tag{5}$$

then $\mathbf{P}^R$ is the transition probability matrix of the reversed process $X^R(t)$ and $\boldsymbol{\pi}$ is stationary distribution of both $X(t)$ and $X^R(t)$.

The importance of this proposition is that we can sometimes guess at the nature of the reversed chain and then use the set of Equations (5) to obtain both the stationary probabilities and the p^R_{ij}.

Following the lines of [4] developped for CTMC, we prove that the Kolmogorov's criteria can be generalized in order to encompass non-reversible DTMCs. The Discrete Time Generalized (DTG) Kolmogorov's criterion is stated below.

Proposition 5 (DTG-Kolmogorov's criterion). *A stationary DTMC with state space S and transition probability matrix $\mathbf{P}$ has reversed chain with transition matrix $\mathbf{P}^R$ if and only if for every finite sequence $i_1, i_2, \ldots i_n \in S$,*

$$p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-1} i_n} p_{i_n i_1} = p^R_{i_1 i_n} p^R_{i_n i_{n-1}} \cdots p^R_{i_3 i_2} p^R_{i_2 i_1}. \tag{6}$$

Proof. Consider finite sequence of states $i_1, i_2, \ldots i_n \in S$. By Proposition 4,

$$p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-1} i_n} p_{i_n i_1} = \frac{\pi_{i_1}}{\pi_{i_n}} p^R_{i_1 i_n} \frac{\pi_{i_n}}{\pi_{i_{n-1}}} p^R_{i_n i_{n-1}} \cdots \frac{\pi_{i_2}}{\pi_{i_1}} p^R_{i_2 i_1}$$

that, by simplifying, yields

$$p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-1} i_n} p_{i_n i_1} = p^R_{i_1 i_n} p^R_{i_n i_{n-1}} \cdots p^R_{i_3 i_2} p^R_{i_2 i_1}.$$

Conversely, observe that, since $X(t)$ is irreducible, for all $j, k \in S$ we can find a chain $j = j_0 \to j_1 \to \cdots \to j_{n-1} \to j_n = k$ (for $n \geq 1$) of one-step transitions.

Consider an arbitrary state $i_0 \in S$ as a reference state and $i \in S$. Let $i = i_n \to i_{n-1} \to \cdots \to i_1 \to i_0$ be a chain of one-step transitions in the forward process $X(t)$. We prove:

$$\pi_i = C_{i_0} \prod_{k=1}^{n} \frac{p^R_{i_{k-1} i_k}}{p_{i_k i_{k-1}}}, \tag{7}$$

where $C_{i_0} \in \mathbb{R}^+$. We prove that π_i is well-defined. Indeed, if $i = j_m \to j_{m-1} \to \cdots \to j_1 \to j_0 = i_0$ ($m \geq 1$) is another chain, we can always find a chain $i_0 = h_0 \to h_1 \to \cdots \to h_{l-1} \to h_l = i$. By hypothesis, we have:

$$\prod_{k=1}^{m} p_{j_k j_{k-1}} \prod_{k=1}^{l} p_{h_{k-1} h_k} = \prod_{k=1}^{l} p^R_{h_k h_{k-1}} \prod_{k=1}^{m} p^R_{j_{k-1} j_k}.$$

Moreover, considering the one-step chain $i = i_n \to i_{n-1} \to \cdots \to i_1 \to i_0 = h_0 \to h_1 \to \cdots \to h_{l-1} \to h_l = i$, we have:

$$\prod_{k=1}^{n} p_{i_k i_{k-1}} \prod_{k=1}^{l} p_{h_{k-1} h_k} = \prod_{k=1}^{l} p^R_{h_k h_{k-1}} \prod_{k=1}^{n} p^R_{i_{k-1} i_k}.$$

From the previous two equations we obtain:

$$\prod_{k=1}^{m} \frac{p^R_{j_{k-1} j_k}}{p_{j_k j_{k-1}}} = \prod_{k=1}^{n} \frac{p^R_{i_{k-1} i_k}}{p_{i_k i_{k-1}}}.$$

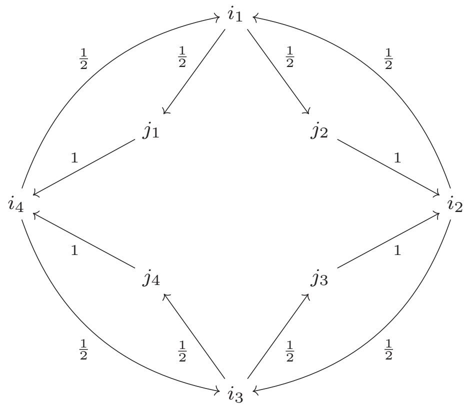

Figure 1: A ρ-reversible discrete time Markov chain

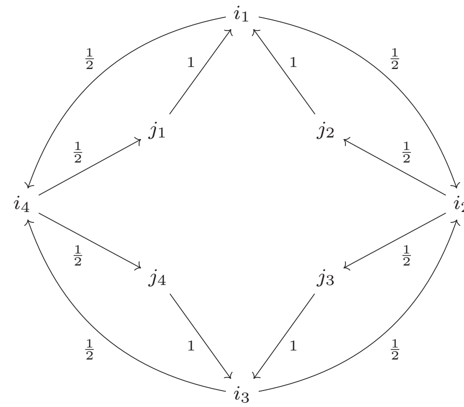

Figure 2: Reversed chain of the DTMC in Figure 1

Hence:

$$\pi_i = C_{i_0} \prod_{k=1}^{n} \frac{p^R_{i_{k-1} i_k}}{p_{i_k i_{k-1}}} ,$$

where $C_{i_0} \in \mathbb{R}^+$, is well-defined.

In order to prove that this is the stationary probability of state $i \in S$ we show that it satisfies the system of global balance equations for i. Indeed,

$$\pi_i = \sum_{j \in S} \pi_j p_{ji} ,$$

which can be written as:

$$1 = \sum_{j \in S} \frac{\pi_j}{\pi_i} p_{ji} .$$

By Proposition 4 this is equal to $\sum_{j \in S} p^R_{ij} = 1$ hence the equation above is an identity.

Now let $i, j \in S$ such that $p_{ji} > 0$. Then

$$\pi_j = C_{i_0} \frac{p^R_{ij}}{p_{ji}} \prod_{k=1}^{n} \frac{p^R_{i_{k-1} i_k}}{p_{i_k i_{k-1}}} = \pi_i \frac{p^R_{ij}}{p_{ji}} .$$

Hence, by Proposition 4, $\mathbf{P}^R$ is the transition probability matrix of $X^R(t)$. $\square$

3. REVERSIBILITY UNDER STATE PERMUTATION

Many stochastic chains are not reversible, however they may be reversible under some state permutation. In this section we generalize the notion of reversibility for DTMC and introduce a novel notion named *ρ-reversibility*.

Hereafter, a *renaming* ρ over the state space of a Markov chain is a bijection on S. For a Markov chain $X(t)$ with state space S we denote by $\rho(X)(t)$ the same process where the state names are changed according to ρ. More formally, let $\mathbf{P}$ and $\boldsymbol{\pi}$ be the transition probability matrix and the stationary distribution of $X(t)$, $\mathbf{P}'$ and $\boldsymbol{\pi}'$ be the transition probability matrix and the stationary distribution of $\rho(X)(t)$. It holds that, for all $i, j \in S$,

$$p_{ij} = p'_{\rho(i)\rho(j)} \quad \text{and} \quad \pi_i = \pi'_{\rho(i)}.$$

We introduce the notion of ρ-reversibility.

Definition 2 (ρ-reversibility). *Let $X(t)$ be an ergodic DTMC with state space S and ρ be a renaming on S. $X(t)$ is said to be* ρ-reversible *if for all $t_1, t_2, \ldots, t_n, \tau \in \mathbb{Z}$, $(X(t_1), \ldots, X(t_n))$ has the same stationary distribution as $(\rho(X)(\tau - t_1), \ldots, \rho(X)(\tau - t_n))$.*

Since, $X(t)$ and $X^R(t)$ have the same stationary distribution $\boldsymbol{\pi}$, it follows that if $X(t)$ is ρ-reversible then

$$\pi_i = \pi_{\rho(i)} \quad \text{for all } i \in S.$$

Example 1. *Consider the DTMC $X(t)$ depicted in Figure 1 and the corresponding reversed chain depicted in Figure 2. It is easy to see that $X(t)$ is ρ-reversible with the renaming ρ defined by:*

$$\rho(i_1) = i_2 \quad \rho(i_2) = i_1 \quad \rho(i_3) = i_4 \quad \rho(i_4) = i_3$$
$$\rho(j_1) = j_3 \quad \rho(j_2) = j_2 \quad \rho(j_3) = j_1 \quad \rho(j_4) = j_4.$$

Indeed, one can easly see that $X(t)$ and $\rho(X^R)(t)$ are probabilistically indistinguishable.

Example 2. *Consider the DTMC $X(t)$ depicted in Figure 3. One can easly prove that $X(t)$ is ρ-reversible under the renaming ρ defined by:*

$$\rho(1) = 3 \quad \rho(3) = 5 \quad \rho(5) = 7 \quad \rho(7) = 1$$
$$\rho(2) = 4 \quad \rho(4) = 6 \quad \rho(6) = 8 \quad \rho(8) = 2.$$

Analogously to Proposition 1 above, we prove necessary and sufficient conditions for ρ-reversibilty based on the existence of the solution of the linear system called *ρ-detailed balance equations*. The proof follows the lines in [7].

Proposition 6 (ρ-detailed balance equations). *Let $X(t)$ be an ergodic DTMC with state space S and transition probability matrix $\mathbf{P}$. Let ρ be a renaming on S. $X(t)$ is* ρ-reversible *if and only if there exists a collection of positive numbers π_i, $i \in S$, summing to unity that satisfy the following system of* ρ-detailed balance equations*:*

$$\pi_i p_{ij} = \pi_j p_{\rho(j)\rho(i)} \quad \text{for all } i, j \in S. \tag{8}$$

If such a solution $\boldsymbol{\pi}$ exists then it is the stationary distribution of both $X(t)$ and $\rho(X^R)(t)$ and $\pi_i = \pi_{\rho(i)}$ for all $i \in S$.

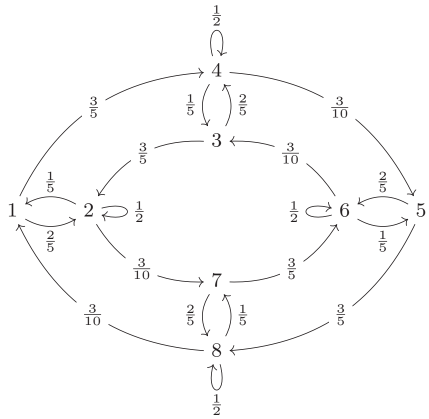

Figure 3: A ρ-reversible Markov chain

PROOF. Let $\mathbf{P}'$ be the transition probability matrix of $\rho(X^R)(t)$. We prove that if $X(t)$ is ρ-reversible then it satisfies the system of ρ-detailed balance equations (8) where $\boldsymbol{\pi}$ is the stationary distribution of $X(t)$. Indeed, observe that by Definition 2 of ρ-reversibility, $p'_{\rho(j)\rho(i)} = p^R_{ji}$. Moreover, since $X(t)$ and $\rho(X^R)(t)$ are probabilistically identical we have $p'_{\rho(j)\rho(i)} = p_{\rho(j)\rho(i)}$ and hence, $p^R_{ji} = p'_{\rho(j)\rho(i)} = p_{\rho(j)\rho(i)}$. Then, by Proposition 3 we obtain the desired result.

Now suppose that there exists a collection of positive numbers π_i, $i \in S$, summing to unity that satisfy the system of *ρ-detailed balance equations*. Then $\boldsymbol{\pi}$ is the stationary distribution of $X(t)$. This is proved by substituting the definition if π_i given by Equation (8) in the system of global balance equations of $X(t)$. Hence, we have:

$$\pi_i = \sum_{j \in S} \pi_j p_{ji} \,,$$

that divided by π_i gives:

$$1 = \sum_{j \in S} \frac{\pi_j}{\pi_i} p_{ji} = \sum_{j \in S} p_{\varrho(i)\varrho(j)} \,.$$

Since ϱ is a bijection, it this reduces to the identity $1 = 1$. Let us now prove that the *ρ-detailed balance equations* imply the fact that $X(t)$ is ρ-reversible. In fact, by Proposition 3

$$p^R_{\rho(i)\rho(j)} = \frac{\pi_j}{\pi_i} p_{\rho(j)\rho(i)} = p_{ij},$$

i.e., $X(t)$ and $\rho(X^R)(t)$ are probabilistically identical. $\square$

COROLLARY 1. *Let $X(t)$ be an ergodic DTMC with state space S, transition probability matrix $\mathbf{P}$ and stationary distribution $\boldsymbol{\pi}$. Let ρ be a renaming on the state space S. If the transition probabilities of $X(t)$ satisfy the following system of equations:*

$$\pi_i p_{ij} = \pi_j p_{\rho(j)\rho(i)} \quad \textit{for all } i, j \in S$$

then $X(t)$ is ρ-reversible.

By applying the Kolmogorov's criterion we obtain the following characterization of ρ-reversibility.

PROPOSITION 7. *Let $X(t)$ be an ergodic DTMC with state space S and transition probability matrix $\mathbf{P}$, and ρ be a renaming on S. $X(t)$ is* ρ-reversible *if and only if for every finite sequence $i_1, i_2, \ldots i_n \in S$,*

$$p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-1} i_n} p_{i_n i_1} = \\ p_{\rho(i_1)\rho(i_n)} p_{\rho(i_n)\rho(i_{n-1})} \cdots p_{\rho(i_3)\rho(i_2)} p_{\rho(i_2)\rho(i_1)}. \quad (9)$$

PROOF. Consider finite sequence of states $i_1, i_2, \ldots i_n \in S$. By Proposition 6,

$$p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-1} i_n} p_{i_n i_1} = \\ \frac{\pi_{i_1}}{\pi_{i_n}} p_{\rho(i_1)\rho(i_n)} \frac{\pi_{i_n}}{\pi_{i_{n-1}}} p_{\rho(i_n)\rho(i_{n-1})} \cdots \frac{\pi_{i_2}}{\pi_{i_1}} p_{\rho(i_2)\rho(i_1)}$$

that, by simplifying, yields

$$p_{i_1 i_2} p_{i_2 i_3} \cdots p_{i_{n-1} i_n} p_{i_n i_1} = \\ p_{\rho(i_1)\rho(i_n)} p_{\rho(i_n)\rho(i_{n-1})} \cdots p_{\rho(i_3)\rho(i_2)} p_{\rho(i_2)\rho(i_1)}.$$

Conversely, observe that, since $X(t)$ is irreducible, for all $j, k \in S$ we can find a chain $j = j_0 \to j_1 \to \cdots \to j_{n-1} \to j_n = k$ (for $n \geq 1$) of one-step transitions. From the hypothesis that $X(t)$ and $\rho(X^R)(t)$ are probabilistically identical, there is also a chain $\rho(k) = \rho(j_n) \to \rho(j_{n-1}) \to \cdots \to \rho(j_1) \to \rho(j_0) = \rho(j)$.

Consider an arbitrary state $i_0 \in S$ as a reference state and $i \in S$. Let $i = i_n \to i_{n-1} \to \cdots \to i_1 \to i_0$ and $\rho(i_0) \to \rho(i_1) \to \cdots \to \rho(i_{n-1}) \to \rho(i_n) = \rho(i)$ $(n \geq 1)$ be two chains of one-step transitions in $X(t)$. We prove that:

$$\pi_i = C_{i_0} \prod_{k=1}^{n} \frac{p_{\rho(i_{k-1})\rho(i_k)}}{p_{i_k i_{k-1}}} \,, \quad (10)$$

where $C_{i_0} \in \mathbb{R}^+$. We prove that π_i is well-defined. Indeed, if $i = j_m \to j_{m-1} \to \cdots \to j_1 \to j_0 = i_0$ $(m \geq 1)$ is another chain, we can always find a chain $i_0 = h_0 \to h_1 \to \cdots \to h_{l-1} \to h_l = i$. By hypothesis, we have:

$$\prod_{k=1}^{m} p_{j_k j_{k-1}} \prod_{k=1}^{l} p_{h_{k-1} h_k} = \prod_{k=1}^{l} p_{\rho(h_k)\rho(h_{k-1})} \prod_{k=1}^{m} p_{\rho(j_{k-1})\rho(j_k)} \,.$$

Moreover, considering the one-step chain $i = i_n \to i_{n-1} \to \cdots \to i_1 \to i_0 = h_0 \to h_1 \to \cdots \to h_{l-1} \to h_l = i$, we have:

$$\prod_{k=1}^{n} p_{i_k i_{k-1}} \prod_{k=1}^{l} p_{h_{k-1} h_k} = \prod_{k=1}^{l} p_{\rho(h_k)\rho(h_{k-1})} \prod_{k=1}^{n} p_{\rho(i_{k-1})\rho(i_k)} \,.$$

From the previous two equations we obtain:

$$\prod_{k=1}^{m} \frac{p_{\rho(j_{k-1})\rho(j_k)}}{p_{j_k j_{k-1}}} = \prod_{k=1}^{n} \frac{p_{\rho(i_{k-1})\rho(i_k)}}{p_{i_k i_{k-1}}} \,.$$

Hence:

$$\pi_i = C_{i_0} \prod_{k=1}^{n} \frac{p_{\rho(i_{k-1})\rho(i_k)}}{p_{i_k i_{k-1}}} \,,$$

where $C_{i_0} \in \mathbb{R}^+$, is well-defined.

In order to prove that this is the stationary probability of state $i \in S$ we show that it satisfies the system of global balance equations for i. Indeed,

$$\pi_i = \sum_{j \in S} \pi_j p_{ji} \,,$$

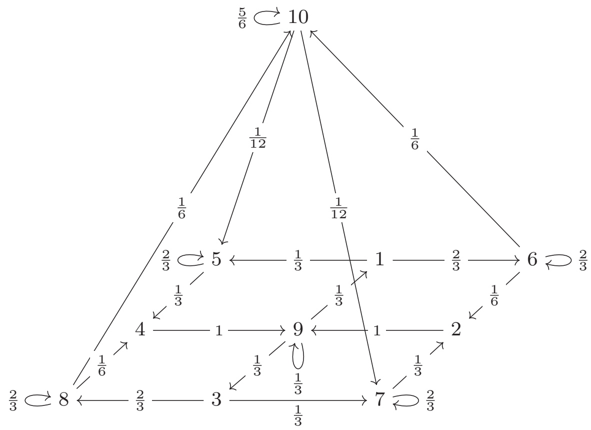

Figure 4: A ρ-reversible DTMC

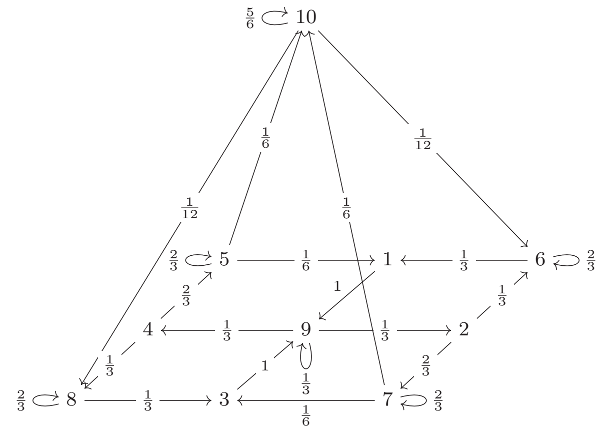

Figure 5: Reversed chain of the DTMC in Figure 4

which can be written as:

$$1 = \sum_{j \in S} \frac{\pi_j}{\pi_i} p_{ji} \,.$$

By Proposition 6 we have:

$$1 = \sum_{j \in S} \frac{p_{\rho(i)\rho(j)}}{p_{ji}} p_{ji} = \sum_{j \in S} p_{\rho(i)\rho(j)} = 1$$

that is an identity.

Now let $i, j \in S$ such that $p_{ji} > 0$. Then

$$\pi_j = C_{i_0} \frac{p_{\rho(i)\rho(j)}}{p_{ji}} \prod_{k=1}^{n} \frac{p_{\rho(i_{k-1})\rho(i_k)}}{p_{i_k i_{k-1}}} = \pi_i \frac{p_{\rho(i)\rho(j)}}{p_{ji}} \,.$$

Hence, by Proposition 6, $X(t)$ is ρ-reversible. $\square$

The stationary probability distribution of a ρ-reversible DTMC can be computed as follows.

Proposition 8. *Let $X(t)$ be an ergodic DTMC with state space S and transition probability matrix $\mathbf{P}$, ρ be a renaming on S, and $i_0, i_1, i_2, \dots i_n = i \in S$ be a finite sequence of states. If $X(t)$ is* ρ-reversible *then for all $i \in S$,*

$$\pi_i = C_{i_0} \prod_{k=1}^{n} \frac{p_{\rho(i_{k-1})\rho(i_k)}}{p_{i_k i_{k-1}}} \tag{11}$$

where $i_0 \in S$ is an arbitrary reference state and $C_{i_0} \in \mathbb{R}^+$.

Proof. See the proof of Proposition 7. $\square$

3.1 On the properties of function ρ

In this section we discuss some properties of the renaming function ρ for ρ-reversible DTMCs. In particular we show that a DTMC may be reversible modulo more than a single state renaming ρ. Recall that a permutation ρ on a set S admits a unique decomposition into *cycles* of different states:

$$(i, \rho(i), \rho(\rho(i)), \dots, \rho^n(i) \equiv i) \,.$$

Cycles of length one are called *fixed points* and those of length two *transpositions*. If ρ only consists of transpositions and fixed points then it is called *involution*, i.e., for all $i \in S$ we have $\rho(\rho(i)) = i$.

If a DTMC is ρ-reversible then it allows for a very efficient derivation of its stationary state probabilities. It is important to note that the existence of a bijection ρ with respect to which the process is ρ-reversible does not imply that ρ is unique. In case of multiple candidates for ρ the choice is arbitrary although it may be desirable to use the one whose cycles are longest in order to reduce the number of states to analyse (since all the states belonging to the same cycle share the same stationary probability) as shown by the following example.

Example 3. *Let us consider the DTMC depicted in Figure 1 and notice that the renaming under which $X(t)$ and $\rho(X^R)(t)$ are probabilistically identical is not unique. In this example $X(t)$ is also ρ'-reversible where ρ' is defined by:*

$$\begin{array}{llll} \rho'(i_1) = i_2 & \rho'(i_2) = i_3 & \rho'(i_3) = i_4 & \rho'(i_4) = i_1 \\ \rho'(j_1) = j_2 & \rho'(j_2) = j_3 & \rho'(j_3) = j_4 & \rho'(j_4) = j_1 . \end{array}$$

The renaming ρ given in Example 1 in an involution, while ρ' has longer cycles in its definition and thus shows immediately that the stationary distributions of states i_k (resp. j_k) are identical for $k = 1, \dots, 4$.

In [15, 3] the authors use the idea of *dynamic reversibility* to study physical systems by Markov chains. In these cases, the authors consider only involutions for the state renaming in the mapping between the forward and the reversed chain. So we think that understanding if using permutations instead of involutions effectively extends the class of tractable Markov chains is an intriguing problem. In many cases (see e.g., Examples 1 and 3), although a permutation of states may be found there exists also an involution and hence the benefit of using ρ with longer cycle are only computational. In Example 4 we present a ρ-reversible DTMC for which does not exist any involution ρ' such that $\rho'(X^R(t)) = X(t)$. As a consequence we show that using general permutations instead of simple involutions enlarges the class of models which are analysable with the theory of reversibility modulo state renaming.

Example 4. *Let us consider the DTMC depicted by Figure 4 whose reversed is shown in Figure 5. Observe that the forward and the reversed processes are identical with ρ*

defined by the cycles $(1,2,3,4);(5,6,7,8);(9);(10)$. *Let us prove that there cannot exist* ρ' *such that* ρ' *is formed by cycles of length* 1 *or* 2. *Observe state* 10, *and notice that if* ρ' *exists then* $\rho'(10)=10$ *since the residence time in the forward and reversed chain must be distributed identically and the self-loop with probability* $5/6$ *characterises only state* 10. *Observe the transition* $10 \to 5$ *in the forward DTMC. This implies two cases: either 1)* $\rho'(5)=6 \wedge \rho'(6)=5$ *or 2)* $\rho'(5)=8$ *and* $\rho'(8)=5$. *Assume case 1). Then we should find a node (which mimics node* 1*) in the forward process that has an incoming arc from* 6 *with probability* $1/6$ *and from* 5 *with probability* $1/3$. *It is easy to see that such a node does not exist. Assume case 2). Then, we should be able to find a node (which mimics node* 4*) that has an outgoing transition to* 8 *with probability* $2/3$ *and to* 5 *with probability* $1/3$. *It is easy to see that also this case is impossible.*

4. FROM CTMC TO DTMC

In the previous section we have considered stochastic processes that are discrete in both time and space, and that satisfy the Markov property. Here, we consider Continuous Time Markov Chains (CTMCs) and study the relation between the notion of ρ-reversibility defined in this setting [11] and the one presented above for DTMCs. We recall that the two main approaches to transform a CTMC into a DTMC are the uniformisation and the definition of the embedded chain (see, e.g., [14]). A CTMC is fully characterised by its infinitesimal generator matrix $\mathbf{Q}$ where q_{ij} is the transition rate from state i to j for $i \neq j$ while the diagonal elements $q_{ii}=-q_i$ are defined as the negative sum of the non-diagonal elements of each row. For ergodic chains, the steady-state distribution $\boldsymbol{\pi}^*$ is the unique vector of positive numbers π_i^* with $i \in S$, summing to unit and satisfying system of the global balance equations:

$$\boldsymbol{\pi}^*\mathbf{Q}=\mathbf{0}.$$

In [10, 11] we have introduced the counterparts of Propositions 6 and 7 for CTMCs which are stated below (noting that further conditions on the residence times are required with respect to the discrete case).

Proposition 9. *Let* $X(t)$ *be an ergodic CTMC and* ρ *be a renaming on its state space* S. *Then* $X(t)$ *is* ρ*-reversible if and only if there exists a collection of positive real number* π_i *summing to unity such that* $\pi_i q_{ij}=\pi_j q_{\rho(j)\rho(i)}$ *and* $q_i=q_{\rho(i)}$ *for every state* i. *If such a collection* π_i *exists, then it is the stationary distribution of* $X(t)$.

Proposition 10. *Let* $X(t)$ *be an ergodic CTMC and* ρ *be a bijection on its state space* S. *Then* $X(t)$ *is* ρ*-reversible if and only if for every finite sequence* $i_1,i_2,\ldots,i_n \in S$,

$$q_{i_1i_2}q_{i_2i_3}\cdots q_{i_{n-1}i_n}q_{i_ni_1} = q_{\rho(i_1)\rho(i_n)}q_{\rho(i_n)\rho(i_{n-1})}\cdots q_{\rho(i_2)\rho(i_1)}\,,$$

and $q_i=q_{\rho(i)}$ *for every state* $i \in S$.

Let $X(t)$ be a CTMC such that there exists a positive real number ϕ for which $q_i \leq \phi$ for all $i \in S$. We define the DTMC $X^U(t)$ with the same state space of $X(t)$ by uniformisation, i.e., we consider $\nu=\max\{q_i, i \in S\}$ and define the transition probabilities of $X^U(t)$ as follows:

$$p_{ij}=\begin{cases}\frac{q_{ij}}{\nu} & \text{if } i \neq j\\ 1-\frac{q_i}{\nu} & \text{if } i=j\,.\end{cases}$$

The steady-state distribution of $X^U(t)$ and $X(t)$ are the same and sometimes it may be more convenient to study the uniformised chain rather then that at continuous time. From the prospective of ρ-reversibility, the following result holds:

Proposition 11. *Let* $X(t)$ *be an ergodic CTMC and let* $X^U(t)$ *be a DTMC obtained by uniformisation. Then* $X(t)$ *is* ρ*-reversible if and only if* $X^U(t)$ *is* ρ*-reversible.*

Proof. $\Rightarrow$) If $X(t)$ is ρ-reversible then its stationary solution satisfies the detailed balance equation according to Proposition 9 for all state $i \neq j$. By definition of uniformisation, also the DTMC satisfies the corresponding equations for $i \neq j$. However, in the DTMC we have introduced new transitions, namely the self-loops, and we have to check that for each state we have: $\pi_i p_{ii}=\pi_i p_{\rho(i)\rho(i)}$, which is true if and only if $p_{\rho(i)\rho(i)}=p_{ii}$. This is true since, by Proposition 9, i and $\rho(i)$ have the same residence time distribution. $\Leftarrow$) If $X^U(t)$ is ρ-reversible then the detailed balance equations of Proposition 6 hold for all the state transitions $i \to j$. If $i \neq j$, clearly the detailed balance equations hold also for the CTMC. It remains to prove that the residence time distribution for state i and $\rho(i)$ are the same. Indeed, let us take the transition $i \to i$ in the DTMC and observe that the corresponding detailed balance equation implies $p_{ii}=p_{\rho(i)\rho(i)}$ that, by the definition of uniformisation, implies $q_i=q_{\rho(i)}$ in $X(t)$. □

Another possible way of analyzing a CTMC $X(t)$ is throughout the corresponding *embedded* Markov chain $X^E(t)$. If we consider the Markov process only at the moments upon which the state of the system changes, and we number these instances 0, 1, 2, etc., then we get a DTMC. This Markov chain has the transition probabilities p_{ij} for $i,j \in S$ as:

$$p_{ij}=\frac{q_{ij}}{\sum_{k\neq i}q_{ik}} \text{ for } j \neq i \tag{12}$$

and $p_{ii}=0$. If $\boldsymbol{\pi}$ is the steady-state distribution of the DTMC one may derive the distribution $\boldsymbol{\pi}^*$ of the corresponding CTMC as:

$$\pi_i^*=\frac{\pi_i q_i^{-1}}{\sum_{i\in S}\pi_i q_i^{-1}}\,. \tag{13}$$

In some cases, although the CTMC $X(t)$ is not ρ-reversible, its embedded chain $X^E(t)$ is. In this case we say that $X(t)$ is *almost* ρ*-reversible*. We can decide if $X(t)$ is almost ρ-reversible using the following proposition.

Proposition 12. *An ergodic CTMC* $X(t)$ *is almost* ρ*-reversible if and only if for every finite sequence of states it holds that:*

$$\frac{q_{i_1i_2}}{q_{i_1}}\frac{q_{i_2i_3}}{q_{i_2}}\cdots\frac{q_{i_{n-1}i_n}}{q_{i_{n-1}}}\frac{q_{i_ni_1}}{q_{i_n}} = \frac{q_{\rho(i_1)\rho(i_n)}}{q_{\rho(i_1)}}\frac{q_{\rho(i_n)\rho(i_{n-1})}}{q_{\rho(i_n)}}\cdots\frac{q_{\rho(i_2)\rho(i_1)}}{q_{\rho(i_2)}}\,.$$

Proof. The proof is trivial and follows from the definition of embedded DTMC and Proposition 7. □

Notice that if a CTMC is ρ-reversible then it is also almost ρ-reversible but the opposite is not true as shown by Example 5. The stationary distribution $\boldsymbol{\pi}^*$ of $X(t)$ satisfies the following set of detailed balance equations:

$$\pi_i^* q_{ij}=\pi_j^* q_{\rho(j)\rho(i)}\frac{q_j}{q_{\rho(j)}}\,.$$

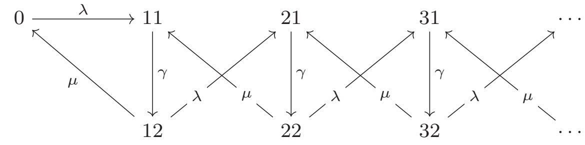

Figure 6: Infinite state space CTMC.

Example 5. *Let us consider the CTMC depicted by Figure 6 and prove that it is ρ-reversible under the assumption $\gamma = \lambda + \mu$ and according to the permutation ϱ defined as:*

$$\rho(s) = \begin{cases} 0 & \text{if } s = 0 \\ n2 & \text{if } s = n1, n \geq 1 \\ n1 & \text{if } s = n2, n \geq 1. \end{cases}$$

Note that, under the assumption $\gamma = \lambda + \mu$, for all the states s it holds $q_s = q_{\varrho(s)}$ and, by exploiting the regularity of the process, we have to check the following cycles:

- $0 \xrightarrow{\lambda} 11 \xrightarrow{\gamma} 12 \xrightarrow{\mu} 0$ *whose inverse is itself and hence the conditions of Proposition 10 are satisfied.*
- $n1 \xrightarrow{\gamma} n2 \xrightarrow{\lambda} (n{+}1)1 \xrightarrow{\gamma} (n{+}1)2 \xrightarrow{\mu} n1$ *whose inverse is* $n2 \xrightarrow{\lambda} (n+1)1 \xrightarrow{\gamma} (n+1)2 \xrightarrow{\mu} n1 \xrightarrow{\gamma} n2$ *whose product of the rates satisfies the conditions of Proposition 10.*

Therefore, the CTMC is ρ-reversible and we have $\pi_{n1} = \pi_{n2}$ for all $n \geq 1$. However, notice that the CTMC is almost ρ-reversible without requiring the condition $\gamma = \lambda + \mu$. We can therefore apply the detailed balance equations for almost ρ-reversible CTMCs to straightforwardly derive the invariant measure of the process (see e.g. for π_{12} and π_{11}):

$$\pi_{12} q_{12,0} = \pi_0 q_{0,11} \frac{q_0}{q_0} \quad \Rightarrow \quad \pi_{12} = \pi_0 \frac{\lambda}{\mu}$$
$$\pi_{11} q_{11,12} = \pi_{12} q_{11,12} \frac{q_{12}}{q_{11}} \quad \Rightarrow \quad \pi_{11} = \pi_0 \frac{\lambda(\lambda + \mu)}{\mu\gamma}.$$

Therefore, the closed form expression of the unnormalized steady-state distribution can be readily derived for $n > 0$:

$$\pi_{n1} = \pi_0 \left(\frac{\lambda}{\mu}\right)^n \frac{\lambda + \mu}{\gamma}, \quad \pi_{n2} = \pi_0 \left(\frac{\lambda}{\mu}\right)^n.$$

5. CONCLUSION

In this paper we have proposed a theory of reversibility modulo a renaming of states for discrete time Markov chains. We have shown that the class of ρ-reversible DTMCs enjoys a high numerical tractability since the unnormalized steady-state distribution can be derived as product of transition probabilities instead as solution of the system of linear equations GBEs. We have shown that in contrast to [15, 3, 10] the state renaming needs not to be an involution and moreover that assuming general bijections allows us to efficiently study models that cannot be analysed by involutions. Finally, we have used the results of ρ-reversibility for DTMC to give the definition of almost ρ-reversibility for CTMCs that allows us to enlarge the class of CTMCs that we are able to solve without computing the solution of the GBEs with respect to the previous class [11].

We leave as future work the investigation of the relations among ρ-reversibility and product-forms. In fact, if a CTMC is ρ-reversible then its steady-state distribution can be expressed as a ratio between two products of rates (see Proposition 8) and this clearly simplifies the task of obtaining a product-form solution for the model. Nevertheless, coherently with the product-form theory developed in the literature (see, e.g., [7, 4, 2, 9]), the formulation of conditions on the isolated components is desirable so that one has not to construct the whole joint process. In other words, we are interested in finding sufficient conditions under which the composition of high level stochastic models (e.g., Markovian process algebra components) originates a joint model which is ρ-reversible for some renaming ρ.

6. REFERENCES

[1] S. Balsamo and A. Marin. *Queueing Networks in Formal methods for performance evaluation*, chapter 2, pages 34–82. M. Bernardo and J. Hillston (Eds), LNCS, Springer, 2007.

[2] S. Balsamo and A. Marin. Performance engineering with product-form models: efficient solutions and applications. In *Proc. of ICPE*, pages 437–448, 2011.

[3] D. J. Gates and M. Westcott. Kinetics of polymer crystallization. discrete and continuum models. *Proc. R. Soc. Lond.*, 416:443–461, 1988.

[4] P. G. Harrison. Turning back time in Markovian process algebra. *Theoretical Computer Science*, 290(3):1947–1986, 2003.

[5] J. Hillston. *A Compositional Approach to Performance Modelling*. Cambridge Press, 1996.

[6] J. Hillston, A. Marin, S. Rossi, and C. Piazza. Contextual lumpability. In *7th International Conference on Performance Evaluation Methodologies and Tools, ValueTools '13*, pages 194–203, 2013.

[7] F. Kelly. *Reversibility and stochastic networks*. Wiley, New York, 1979.

[8] J. G. Kemeny and J. L. Snell. *Finite Markov Chains*. Springer, 1976.

[9] A. Marin, S. Balsamo, and P.G. Harrison. Analysis of stochastic Petri nets with signals. *Perf. Eval.*, 85(7):1520–1539, 2012.

[10] A. Marin and S. Rossi. Autoreversibility: exploiting symmetries in Markov chains. In *Proc. of IEEE MASCOTS'13*, pages 151–160, 2013.

[11] A. Marin and S. Rossi. On the relations between lumpability and reversibility. In *Proc. of IEEE MASCOTS'14*, pages 427–432, 2014.

[12] M. F. Neuts. *Structured stochastic matrices of M/G/1 type and their application*. Marcel Dekker, New York, 1989.

[13] S. M. Ross. *Stochastic Processes*. John Wiley & Sons, 2nd edition, 1996.

[14] W. J. Stewart. *Probability, Markov Chains, Queues, and Simulation*. Princeton University Press, UK, 2009.

[15] P. Whittle. *Systems in stochastic equilibrium*. John Wiley & Sons Ltd., 1986.