Spatial Fairness in Multi-Channel CSMA Line Networks

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ABSTRACT
In this paper we consider a multi-channel random-access carrier-sense multiple access (CSMA) line network with n saturated links, where each link can be active on at most one of the C available channels at any time. Using the product form solution of such a network, we develop fast algorithms to compute the per-link throughputs and use these to study the spatial fairness in such a network. We consider both standard CSMA networks and CSMA networks with so-called channel repacking.

Recently it was shown that fairness in a single channel CSMA line network can be achieved by means of a simple formula for the activation rates, which depends solely on the number of interfering neighbors. In this paper we show that this formula still achieves fairness in the multi-channel setting under heavy and low traffic, but no such simple formula seems to exist in general. On the other hand, numerical experiments show that the fairness index when using the simple single channel formula in the multi-channel setting is very close to one, meaning this simple formula also eliminates most of the spatial unfairness in a multi-channel network.

Keywords
CSMA, multi-channel, fairness, mac

1. INTRODUCTION
Random-access carrier-sense multiple access (CSMA) networks have received considerable attention over the past few decades and various stochastic models have been developed and studied in great detail, e.g., [5, 9, 8, 16, 17], we refer to [6, 15] for a detailed literature overview. Most of these studies have focused on the channel throughput, stability, packet delay and/or fairness (between different links) in case of a single channel CSMA network. Studies on multi-channel CSMA networks are far less abundant and include [3, 12], where [3] focuses on throughput optimality and stability and [12] on computing the throughput in large circular and line networks where all the links make use of equal activation rates.

Spatial unfairness in single channel CSMA networks is fairly well understood [8, 18] as links at the border of the network have a restricted neighborhood and thus a higher probability to access the channel. In large line networks these border effects do not propagate inside the network as opposed to more general network topologies. Recently it was shown that spatial unfairness in line networks of limited size can also be eliminated by adapting the activation rates (i.e., mean backoff times) using a simple formula [18]. More specifically, all links achieve the same long run average throughput if the activation rate of link i is of the form $\alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$, for any α, where $\gamma(i)$ is the number of interfering neighbors of link i. Further, these activation rates are the only ones that achieve fairness due to [17].

In this paper we study spatial fairness of multi-channel CSMA line networks (of moderate size). We consider a similar network model as in [12], which differs from [2] in the sense that we limit ourselves to line networks, assume that all links have access to all the channels, that interference is the same on each channel, that links can only be active on a single channel at a time and that channels are selected uniformly at random when the backoff timer expires. While we can relax some of these assumptions, this might considerably increase the time complexity of the algorithms developed to compute the per-link throughputs.

The following contributions are made in this paper. First, we develop fast algorithms to compute the per-link throughputs, where the time complexity grows linear in the number of links, by exploiting the product form solution of the network. Second, we prove that the simple formula $\alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$ to achieve fairness in a single channel network still guarantees fairness in the multi-channel setting under heavy and low traffic, that is, if α is either very small or large. Third, by considering a special case, we show that a simple formula to achieve spatial fairness in the multi-channel setting which depends only on the number of interfering neighbors does not exist. Finally, by making use of the fast algorithms developed to compute the per-link throughput, we show that while the simple formula $\alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$ does not eliminate all spatial unfairness in the multi-channel setting, it does eliminate most of the unfairness as the Jain's fairness index [11] is typically very close to one.

VALUETOOLS 2014, December 09-11, Bratislava, Slovakia

DOI 10.4108/icst.valuetools.2014.258164

Apart from the standard multi-channel CSMA model we also consider a multi-channel CSMA network with channel repacking. Channel repacking basically means that a channel is assigned to a link whenever its backoff timer expires and there is either a channel available or one can be made available by reassigning some of the channels already in use. While this is hard to achieve in practice, especially on a general network topology, we mainly study this variant as we felt that a simple formula to achieve fairness is more likely to exist in this case. The results however indicate that this is not the case and all the findings listed above for the standard CSMA network also apply to the network with repacking.

2. MODEL DESCRIPTION

Consider a CSMA line network consisting of C channels, n links and an interference range of β, meaning a link cannot be simultaneously active with one of its β left or right neighbors on the same channel. Assume a link can only be active on one channel at a time and packet lengths follow an exponential distribution (with mean 1). It is worth noting that the results presented in this paper remain valid for more general packet length distributions, i.e., phase-type distributions, due to the underlying loss network (see for instance [5, 16, 3] for more details). Backoff timers are assumed to follow an exponential distribution, the average of which is specified further on. Links are assumed to be saturated at all times, that is, each link has at least one packet ready for transmission at any time. We consider two systems: with and without channel repacking.

In case of *channel repacking* we assume that each link maintains a single backoff timer and is assigned a channel when the timer expires provided that a channel is available or *one can be made available* by reassigning some of the already assigned channels.

Without channel repacking we still maintain a single backoff timer per link, but when it expires a channel is selected uniformly at random among the C channels and is assigned in case it is not being used by any of the interfering links. As we assume exponential backoff times, this is equivalent to maintaining C timers, one for each channel.

To emphasize the difference between both systems, assume $C = 2$, $\beta = 1$, link 1 is using channel 1 and link 3 is using channel 2. In this case link 2 can become active with channel repacking (as user 1 simply needs to switch channels), while it cannot without channel repacking.

Due to the exponential nature of the packet lengths and backoff times it is easy to see that the evolution of both systems can be captured by a continuous-time Markov chain. More specifically, for the system *with repacking* all feasible states are given by $\bar{\Omega}_n$ the set of all binary strings of length n such that there are at most C ones in any sequence of $\beta+1$ consecutive bits. Note that due to repacking it suffices to keep track of the links that are active, meaning there is no need to keep track of the channel ids. Let $\bar{w}_i = 1$ if link i is active on some channel and set $\bar{w}_i = 0$ otherwise.

If we denote ν_i as the activation rate of link i, we obtain a loss network and the steady state probability $\bar{\pi}(\bar{w})$ of being in state $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_n) \in \bar{\Omega}_n$ can be expressed as

$$\bar{\pi}(\bar{w}) = \bar{Z}_\nu^{-1} \prod_{i=1}^{n} \nu_i^{\bar{w}_i}, \tag{1}$$

where $\bar{Z}_\nu = \sum_{\bar{w}\in\bar{\Omega}_n} \prod_{i=1}^n \nu_i^{\bar{w}_i}$ is the normalizing constant and $\nu = (\nu_1, \ldots, \nu_n)$. When $\beta < C$, all the links can be active simultaneously in case of repacking and $\bar{\Omega}_n$ is simply the set of all binary strings of length n. Hence, without loss of generality we may assume that $\beta \geq C$ in case of repacking.

Without repacking we clearly do need to keep track of the ids of the channels in use as they may affect whether or a link can become active (as in the example before). Thus the set of all feasible states is given by Ω_n, the set of all strings of length n over the alphabet $\{0, 1, \ldots, C\}$ such that any sequence of $\beta+1$ consecutive symbols does not contain more than one $c > 0$. Let $C\nu_i$ be the parameter of the exponential distribution of the backoff timer of link i. It is easy to see and well known [3, 12] that this system corresponds to a loss network and the steady state probability $\pi(w)$ of being in state $w = (w_1, \ldots, w_n) \in \Omega_n$ can be expressed as

$$\pi(w) = Z_\nu^{-1} \prod_{i=1}^{n} \nu_i^{1[w_i>0]}, \tag{2}$$

where $Z_\nu = \sum_{w\in\Omega_n} \prod_{i=1}^n \nu_i^{1[w_i>0]}$ is the normalizing constant, $\nu = (\nu_1, \ldots, \nu_n)$ and $1[A] = 1$ if A is true and $1[A] = 0$ otherwise.

Throughout the paper we add a bar to a variable or symbol whenever it is related to the system with repacking, unless it concerns a common parameter such as C, β, etc.

3. MATRIX EXPRESSIONS FOR THE NORMALIZING CONSTANT

In this section we derive a matrix expression for the constants Z_ν and $\bar{Z}_\nu$. Using these expressions we can compute the normalizing constant of the system with repacking in $O(n\binom{\beta+1}{C})$ time and of the system without repacking in $O(n \min(2^\beta, (\beta+1)^C))$ time.

3.1 With Channel Repacking

THEOREM 1. *The normalizing constant $\bar{Z}_\nu$ can be written as*

$$\bar{Z}_\nu = \left(\prod_{i=1}^{n}(1+\nu_i)\right) \bar{P}_n(C, \beta+1, \nu),$$

where $\bar{P}_n(C, \beta+1, \nu)$ is the probability that we have at most C successes in any $\beta+1$ consecutive Bernoulli trials when performing a total of n independent Bernoulli trials where the i-th trial has success probability $p_i = \nu_i/(1+\nu_i)$.

PROOF. The result is immediate by noting $\bar{Z}_\nu$ can be written as

$$\bar{Z}_\nu = \frac{\sum_{\bar{w}\in\bar{\Omega}_n} \prod_{i=1}^n \left(\frac{\nu_i}{1+\nu_i}\right)^{1[\bar{w}_i=1]} \left(1 - \frac{\nu_i}{1+\nu_i}\right)^{1[\bar{w}_i=0]}}{\prod_{i=1}^n \frac{1}{1+\nu_i}}.$$

□

Probabilities of the type $\bar{P}_n(C, \beta+1, \nu)$ have been studied previously in the area of reliability theory [7], in fact the result in Theorem 2, where $\beta = C$, is equivalent to the method presented in [10] for the so-called consecutive-k-out-of-n:F system. We will generalize this method to any $\beta \geq C$ which implies that the our proposed method is also useful to analyze the reliability of a consecutive-k-out-of-m-from-n:F system.

THEOREM 2. *When $\beta = C$, we can express $\bar{P}_n(\beta, \beta+1, \nu)$ as*

$$\bar{P}_n(\beta, \beta+1, \nu) = e_1^* \left(\prod_{i=1}^{n} \bar{M}_{\beta,\beta+1}(\nu_i) \right) e,$$

where e_1^* is first row of the size $\beta+1$ identity matrix, e is a column vector of ones and

$$\bar{M}_{\beta,\beta+1}(\nu_i) = \frac{1}{1+\nu_i} \left[\begin{array}{c|cccc} 1 & \nu_i & & & \\ \vdots & & \ddots & & \\ 1 & & & & \nu_i \\ \hline 1 & 0 & & \dots & 0 \end{array} \right],$$

for $i = 1, \dots, n$.

PROOF. When $\beta = C$ we can have at most β successes in a row. To obtain an expression for $\bar{P}_n(\beta, \beta+1, \nu)$ we construct a time-inhomogeneous Markov chain with $\beta+1$ transient, labeled 0 to β, and one absorbing state. We start in state 0 and the i-th transition corresponds to performing the i-th Bernoulli trail. The $\beta+1$ transient states keep track of the number consecutive successes, meaning a success increases the state by 1, while a failure resets the state to 0. If a success occurs in state β, meaning we have more than β successes in a row, we move to the absorbing state. The probability $\bar{P}_n(\beta, \beta+1, \nu)$ can be expressed as the probability of being in a transient state at time n. $\square$

This theorem allows us to compute $\bar{Z}_\nu$ in $O(n\beta)$ time when $\beta = C$.

In order to generalize the previous idea, we introduce the matrices $\bar{M}_{C,\beta+1}(\nu_i)$ of size $\sum_{k=0}^{C} \binom{\beta-C+k}{k}$. The rows and columns of $\bar{M}_{C,\beta+1}(\nu_i)$ are labeled by the strings $\bar{w} \in \bar{\Omega}_{C,\beta}$ with

$$\bar{\Omega}_{C,\beta} = \cup_{k=0}^{C} \{\bar{w} \in \{0,1\}^{\beta-C+k} | \sum_i \bar{w}_i = k\}.$$

Note the length of $\bar{w} \in \bar{\Omega}_{C,\beta}$ is limited by β. Let $l(\bar{w})$ be the length of $\bar{w}$ and $z(\bar{w})$ the position of the first zero (which exists for $\beta > C$), e.g., $l((1,1,0,1,0,1)) = 6$ and $z((1,1,0,1,0,1)) = 3$, then

$$(1+\nu_i)\left(\bar{M}_{C,\beta+1}(\nu_i)\right)_{\bar{w},\bar{w}'} = \begin{cases} 1 & \bar{w}' = (\bar{w}_{z(\bar{w})+1}, \dots, \bar{w}_{l(\bar{w})}, 0), \\ \nu_i & l(\bar{w}) < \beta, \bar{w}' = (\bar{w}_1, \dots, \bar{w}_{l(\bar{w})}, 1), \\ 0 & \textit{otherwise}. \end{cases}$$

THEOREM 3. *When $\beta > C \geq 1$, we can express $\bar{P}_n(C, \beta+1, \nu)$ as*

$$\bar{P}_n(C, \beta+1, \nu) = e_1^* \left(\prod_{i=1}^{n} \bar{M}_{C,\beta+1}(\nu_i) \right) e,$$

where e_1^* is first row of the identity matrix, e is a column vector of ones. Further, the matrices $\bar{M}_{C,\beta+1}(\nu_i)$ are of size $\binom{\beta+1}{C}$.

PROOF. We rely on a time-inhomogeneous Markov chain as before and label the transient states by the strings in $\bar{\Omega}_{C,\beta}$. The binary string $\bar{w} \in \bar{\Omega}_{C,\beta}$ reflects the outcome of all the previous trials that occurred after the $(\beta+1-C)$-last failure. A new success is only allowed if the $(\beta+1-C)$-last failure occurred strictly less than β trials ago and simply adds a 1 to the state. If a failure occurs we can forget about the outcome of all the trials up until and including the first 0 in $\bar{w}$, while adding a 0.

It is easy to see that $|\bar{\Omega}_{C,\beta}| = \binom{\beta+1}{C}$ as

$$\sum_{k=0}^{C} \binom{\beta-C+k}{k} = \sum_{k=0}^{C} \binom{\beta-k}{C-k} = \binom{\beta+1}{C},$$

where the latter equality follows from repeatedly applying $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. $\square$

Note, each row of $\bar{M}_{C,\beta+1}(\nu_i)$ contains at most 2 nonzero entries, meaning multiplying $\bar{M}_{C,\beta+1}(\nu_i)$ with a column vector requires at most $2|\bar{\Omega}_{C,\beta}|$ floating point operations, which means the time complexity to compute $\bar{Z}_\nu$ is bounded by $O(n\binom{\beta+1}{C})$.

Remark. When $\nu_i = \sigma$, for $i = 1, \dots, n$, it is also possible to express $\bar{Z}_n$ as

$$\bar{Z}_n = \sum_{j=0}^{n} \bar{B}_n(\beta+1, C, j)\sigma^j, \tag{3}$$

where $\bar{B}_n(m, C, j)$ denotes the number of binary strings of length n with exactly j ones such that no m consecutive bits contain more than C ones. When $\beta = C$, we are thus interested in the number of binary strings of length n with at most C consecutive ones. As shown in [1, Theorem 3.3], $B_n(C+1, C, j)$ can be expressed as

$$\bar{B}_n(C+1, C, j) = \binom{n-j+1}{j}_C,$$

where $\binom{n}{i}_s$ is the generalized binomial coefficient defined by the recursion:

$$\binom{n+1}{i}_s = \sum_{k=0}^{s-1} \binom{n}{i-k}_s,$$

and $\binom{n}{0}_s = 1$. Note, when $s = 1$ these are the usual binomial coefficients, while for $s > 1$ they can be expressed in terms of the usual ones [4, p.19] by

$$\binom{n}{i}_s = \sum_{k=0}^{\lfloor i/s \rfloor} (-1)^k \binom{n}{k} \binom{n+i-sk-1}{n-1}.$$

For $\beta > C$ counting these strings is equivalent to solving the so-called generalized birthday problem. Rather involved closed form expressions for $\bar{B}_n(m, C, j)$ were derived in [14] when $j/2 < C$ and in [13, Theorem 1] for the general case. The latter however are expressed as a large sum of determinants and therefore does not result in an efficient manner to compute $\bar{B}_n(m, C, j)$.

3.2 Without Channel Repacking

Consider a $(C+1)$-sided coin with outcomes $0, 1, \ldots, C$ and assume that the probability of having outcome c, for $c \in \{1, \ldots, C\}$, equals p, while the outcome 0 has the remaining probability $1 - Cp$, for some $p \in (0, 1/C)$. Let $\mathcal{S}_\beta$ be the set of all binary strings of length β that contain at most C ones. To define the set of matrices $M_{C,\beta+1}(\nu_i)$ of size $|\mathcal{S}_\beta| = \sum_{k=0}^{\min(C,\beta)} \binom{\beta}{k} \leq 2^\beta$, we label the rows and columns of $M_{C,\beta+1}(\nu_i)$ by the strings in $\mathcal{S}_\beta$. For $z \in \mathcal{S}_\beta$, let $n(z)$ be the value of the binary number represented by z, e.g., $n((0, 1, 0, 1)) = 5$, and define

$$(1+C\nu_i)\left(M_{C,\beta+1}(\nu_i)\right)_{z,z'} = \begin{cases} 1 & n(z) < 2^{\beta-1}, n(z') = 2n(z), \\ 1 & n(z) \geq 2^{\beta-1}, n(z') = 2n(z) - 2^\beta, \\ \nu_i(C-k) & \sum_{i=1}^{\beta} z_i = k, n(z) < 2^{\beta-1}, \\ & n(z') = 2n(z) + 1, \\ \nu_i(C-k) & \sum_{i=1}^{\beta} z_i = k, n(z) \geq 2^{\beta-1}, \\ & n(z') = 2n(z) - 2^\beta + 1, \\ 0 & otherwise. \end{cases} \tag{4}$$

The normalization constant Z_ν can be expressed as follows using the matrices $M_{C,\beta+1}(\nu_i)$:

THEOREM 4. *The normalizing constant Z_ν can be written as*

$$Z_\nu = \left(\prod_{i=1}^{n}(1 + C\nu_i)\right) P_n(C, \beta + 1, \nu),$$

and

$$P_n(C, \beta + 1, \nu) = e_1^* \left(\prod_{i=1}^{n} M_{C,\beta+1}(\nu_i)\right) e,$$

where e_1^* is first row of the size $|\mathcal{S}_\beta|$ identity matrix, e is a column vector of ones.

PROOF. The proof is similar to the proof of Theorem 1 by noting that $P_n(C, \beta + 1, \nu)$ is the probability that when flipping n coins with $C+1$ sides, where $p = \nu_i/(1+C\nu_i)$ for coin i, no sequence of $\beta + 1$ consecutive flips results in two or more identical outcomes equal to some $c > 0$.

To express $P_n(C, \beta+1, \nu)$ we construct a time-inhomogeneous Markov chain (as in the proof of Theorem 3) with $|\mathcal{S}_\beta|$ transient, labeled $z \in \mathcal{S}_\beta$, and one absorbing state. We start in state $(0, \ldots, 0)$ and the i-th transition corresponds to performing the i-th $(C+1)$-sided coin flip. The transient states keep track of the position of the outcomes $c > 0$ in the last β coin flips. If we are in transient state z and the outcome of coin flip i is 0, we simply shift z to the left, drop the leading bit and add a zero to the right. If the outcome is $c > 0$ and $\sum_{i=1}^{\beta} z_i = k$ there is a probability $(C-k)/C$ that the outcome differs from the k outcomes with $c > 0$ in the last β coin flips. If the outcome differs, we shift z to the left, drop the leading bit and add a one to the right, otherwise we jump to the absorbing state. The probability $P_n(C, \beta+1, \nu)$ can be expressed as the probability of being in a transient state at time n. □

Note, each row of $M_{C,\beta+1}(\nu_i)$ contains at most 2 nonzero entries, meaning multiplying $M_{C,\beta+1}(\nu_i)$ with a column vector requires at most $2|\mathcal{S}_\beta|$ floating point operations, which means the time complexity to compute Z_ν is bounded by $O(n \min(2^\beta, (\beta + 1)^C))$ as $\sum_{k=0}^{\beta} \binom{\beta}{k} = 2^\beta$ and $\sum_{k=0}^{C} \binom{\beta}{k} \leq (\beta + 1)^C$.

4. COMPUTING LINK THROUGHPUTS

In case of channel repacking, denote the long-run average throughput of link j as $\bar{\theta}_j(\nu)$. It corresponds to the long-run fraction of time that link j is active on some channel. To express $\bar{\theta}_j(\nu)$ define the matrices $\bar{N}_{C,\beta+1}(\nu_i)$ as

$$(1 + \nu_i)\left(\bar{N}_{C,\beta+1}(\nu_i)\right)_{w,w'} = \begin{cases} \nu_i & l(w) < \beta, w' = (w_1, \ldots, w_{l(w)}, 1), \\ 0 & otherwise, \end{cases}$$

i.e., they are obtained by setting all the entries of $\bar{M}_{C,\beta+1}(\nu_i)$ that correspond to a failure to zero.

THEOREM 5. *The throughput $\bar{\theta}_j(\nu)$ of node j can be computed as*

$$\bar{\theta}_j(\nu) = \frac{\bar{P}_n^{(j)}(C, \beta + 1, \nu)}{\bar{P}_n(C, \beta + 1, \nu)},$$

where

$$\bar{P}_n^{(j)}(C, \beta + 1, \nu) = e_1^* \left(\prod_{i=1}^{j-1} \bar{M}_{C,\beta+1}(\nu_i)\right) \bar{N}_{C,\beta+1}(\nu_j) \left(\prod_{i=j+1}^{C} \bar{M}_{C,\beta+1}(\nu_i)\right) e.$$

PROOF. Using the expression for the steady state we get

$$\bar{\theta}_j(\nu) = \bar{Z}_\nu^{-1} \sum_{\bar{w} \in \Omega} \prod_{i=1}^{n} \nu_i^{\bar{w}_i} 1[\bar{w}_j = 1].$$

The result now follows from Theorem 1 and by noting that $\bar{P}_n^{(j)}(C, \beta + 1, \nu)$ represents the probability that we have at most C successes in any $\beta + 1$ consecutive Bernoulli trials when performing a total of n independent Bernoulli trials where the i-th trial has success probability $p_i = \nu_i/(1 + \nu_i)$ and the j-th trial is successful. □

By first computing the vectors $e_1^* \prod_{i=1}^{j-1} \bar{M}_{C,\beta+1}(\nu_i)$ as well as the vectors $\prod_{i=j+1}^{C} \bar{M}_{C,\beta+1}(\nu_i)e$, for $j = 1, \ldots, n$, we can compute the vector of throughputs $\bar{\theta}(\nu) = (\bar{\theta}_1(\nu), \ldots, \bar{\theta}_n(\nu))$ in $O(n\binom{\beta+1}{C})$ time.

For the system without channel repacking we can proceed in exactly the same way to compute the vector $\theta(\nu) =$

$(\theta_1(\nu), \ldots, \theta_n(\nu))$ of channel throughputs, by defining the matrices $N_{C,\beta+1}(\nu_i)$ as

$$(1+C\nu_i)\left(N_{C,\beta+1}(\nu_i)\right)_{z,z'} = \begin{cases} \nu_i(C-k) & \sum_{i=1}^{\beta} z_i = k, n(z) < 2^{\beta-1}, \\ & n(z') = 2n(z)+1, \\ \nu_i(C-k) & \sum_{i=1}^{\beta} z_i = k, n(z) \geq 2^{\beta-1}, \\ & n(z') = 2n(z) - 2^{\beta} + 1, \\ 0 & \textit{otherwise}. \end{cases} \quad (5)$$

i.e., they are obtained by setting all the entries of $M_{C,\beta+1}(\nu_i)$ that correspond to outcome 0 to zero.

Remark. In [12] the authors also propose the use of a matrix product to compute the throughput $\theta_j(\nu)$ of link j, but they focus on large networks with equal activation rates. Further, the matrices used are considerably larger than the ones used in our approach. For instance, for $C = 3$ and $\beta = 2$ matrices of size 13 are used, while in our case size $\sum_{k=0}^{2}\binom{2}{k} = 4$ suffices.

5. FAIRNESS

Let $\gamma(i)$ be the number of links that interfere with link i. The main result in [18] showed that in case of a single channel, i.e., $C = 1$, fairness can be achieved in a line network consisting of n links if $\nu_i = \alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$, for $i = 1, \ldots, n$, for any choice of α. The following section indicates that this choice of ν_i still guarantees fairness in case of multiple channels, i.e., $C \geq 1$, under heavy and low traffic with and without repacking.

5.1 Heavy and low traffic

We start by considering the case where the number of channels C is at most $\beta + 1$.

THEOREM 6. *Let $n > \beta \geq 1$, $C \leq \beta + 1$ and set $\nu_i = \alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$, then*

$$\lim_{\alpha\to\infty} \theta_j(\nu) = \lim_{\alpha\to\infty} \bar{\theta}_j(\nu) = \frac{C}{\beta+1},$$

for $j = 1, \ldots, n$

PROOF. We restrict ourselves to the system *without* channel repacking. The argument for the system with repacking proceeds similarly. When α becomes large $\nu_i \approx \alpha^{\gamma(i)-\gamma(1)+1}$ and the product form in (2) implies that the Markov chain spends most of its time in the states w that maximize

$$val(w) \stackrel{def}{=} \sum_{i=1}^{n} (\gamma(i) - \gamma(1) + 1) 1[w_i > 0].$$

We will argue that there are $C!\binom{\beta+1}{C}$ states w for which $val(w)$ is maximized and that each $j \in \{1, \ldots, n\}$ is active in exactly $C!\binom{\beta}{C-1}$ of these states. This results in a throughput of $\binom{\beta}{C-1}/\binom{\beta+1}{C} = C/(\beta+1)$ for each link.

Define the following subset of Ω_n of size $C!\binom{\beta+1}{C}$:

$$\mathcal{M}_n = \{w \in \Omega_n | \sum_{i=1}^{\beta+1} 1[w_i > 0] = C, w_j = w_{j-(\beta+1)}, j > \beta+1\}.$$

Note for $w \in \mathcal{M}_n$ any set of $\beta + 1$ consecutive elements contains C *distinct* positive elements. Further, $w_j > 0$ in exactly $C!\binom{\beta}{C-1}$ states $w \in \mathcal{M}_n$, as there are $\binom{\beta}{C-1}$ ways to select the remaining $C - 1$ positive elements in the first $\beta + 1$ positions. To complete the proof we now show that $val(w) = (n-\beta)C$ for $w \in \mathcal{M}_n$ and $val(w) < (n-\beta)C$ for $w \notin \mathcal{M}_n$.

For $n = \beta + 1$ it is clear that $val(w) = C$ for $w \in \mathcal{M}_n$ as $\gamma(i) = \beta$ for all $i \in \{1, \ldots, n\}$ and $val(w)$ is therefore equal to the number of ones in w. If we add a link to a line network of n links, we see that $\gamma(n-\beta+1), \ldots, \gamma(n)$ increase by one, while $\gamma(i)$ remains identical for $i \leq n - \beta$ and $\gamma(n+1) = \gamma(1)$. Further, any state w can have at most C positive elements in the last $\beta + 1$ positions, thus

$$val(w_1, \ldots, w_{n+1}) \leq val(w_1, \ldots, w_n) + C.$$

Hence, $val(w) \leq (n-\beta)C$ for all $w \in \Omega_n$. When $w \in \mathcal{M}_n$ the last $\beta + 1$ positions of $w = (w_1, \ldots, w_n)$ contain exactly C positive elements and $w_{n+1} = w_{n-\beta}$, thus each time we add an element to $w \in \Omega_n$ such that $w_{n+1} = w_{n-\beta}$, its value increases by C. This implies that $val(w) = (n-\beta)C$ for $w \in \mathcal{M}_n$.

Assume $w \notin \mathcal{M}_n$ and w contains less than C positive elements in the first $\beta + 1$ positions. In this case $val(w) < (n-\beta)C$ as $val(w_1, \ldots, w_{\beta+1}) < C$ and adding a single element can only increase the value by C. If w does contain exactly C positive elements in the first $\beta + 1$ positions, let j be smallest index such that $w_j \neq w_{j-(\beta+1)}$. In this case we must have $w_j = 0$ and $w_{j-(\beta+1)} > 0$, as we otherwise get $C + 1$ positive elements in $(w_{j-\beta}, \ldots, w_j)$ and thus a repetition of the same positive value within a set of $\beta + 1$ consecutive values. Thus, when adding w_j, the value of $(w_1, \ldots, w_{j-1})$ increases by $C - 1$ instead of C, which implies that $val(w)$ must be less than $(n-\beta)C$. $\square$

When $C \geq \beta + 1$ the throughput $\theta_j(\nu)$ approaches one even for $\nu_i = \alpha$ as α tends to infinity, as Ω_n contains states where all the links are active on some channel and these will dominate as α becomes large.

It is easy to see that setting $\nu_i = \alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$ also guarantees fairness in low traffic conditions (i.e., as α goes to zero) as there is no need to have multiple channels in this case.

5.2 Intermediate traffic

For intermediate rates, setting $\nu_i = \alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$ does not guarantee fairness except when $C = 1$ as will become clear from the following two propositions.

PROPOSITION 1. *Let $\beta = n - 2$ and let $\phi = \nu_2 = \ldots = \nu_{n-1}$, then fairness is achieved in a system* **with repacking** *if and only if*

$$\nu_1 = \nu_n =$$
$$\frac{1}{2}\left(\sqrt{(1-\phi\bar{S}_2(\phi)/\bar{S}_1(\phi))^2+4\phi} - [1-\phi\bar{S}_2(\phi)/\bar{S}_1(\phi)]\right), \tag{6}$$

with $\bar{S}_k(y) = \sum_{i=0}^{C-k}\binom{n-3}{i}y^i$ *for* $k \geq C$ *and* $\bar{S}_k(y) = 0$ *for* $k > C$.

Proof. Clearly $\nu_1 = \nu_n$ due to the symmetry of the system, while $\phi = \nu_2 = \ldots = \nu_{n-1}$ implies that $\bar{Z}_\nu^{-1}\bar{\theta}_1(\nu) = \bar{Z}_\nu^{-1}\bar{\theta}_i(\nu)$, for $i = 2, \ldots, n-1$, can be written as

$$\nu_1(1+\nu_1)\sum_{i=0}^{C-1}\binom{n-2}{i}\phi^i =$$
$$\phi\left((1+\nu_1)^2\sum_{i=0}^{C-2}\binom{n-3}{i}\phi^i + \binom{n-3}{C-1}\phi^{C-1}\right),$$

as link 1 can be simultaneously active with link n and at most $C-1$ intermediate links and if an intermediate link i is active with at most $C-2$ other intermediate links both link 1 and n can be active, while they must both be silent if there are $C-1$ other active intermediate links. In other words, ν_1 is the positive solution of a quadratic equation and (7) follows by noting that $\sum_{i=0}^{C-1}\binom{n-2}{i}\phi^i - \phi\sum_{i=0}^{C-2}\binom{n-3}{i}\phi^i = \sum_{i=0}^{C-1}\binom{n-3}{i}\phi^i$. $\square$

Proposition 2. *Let* $\beta = n-2$ *and let* $\phi = \nu_2 = \ldots = \nu_{n-1}$*, then fairness is achieved in a system* **without repacking** *if and only if*

$$\nu_1 = \nu_n =$$
$$\frac{\sqrt{(1+\phi S_2(\phi)/S_1(\phi))^2+4\phi} - [1-\phi S_2(\phi)/S_1(\phi)]}{2(1+S_2(\phi)/S_1(\phi))}, \tag{7}$$

with $S_k(y) = \sum_{i=0}^{C-k}\frac{C!}{(C-k-i)!}\binom{n-3}{i}y^i$ *for* $k \geq C$ *and* $S_k(y) = 0$ *for* $k > C$.

Proof. Clearly $\nu_1 = \nu_n$ due to the symmetry of the system, while $\phi = \nu_2 = \ldots = \nu_{n-1}$ implies that $Z_\nu^{-1}\theta_1(\nu) = Z_\nu^{-1}\theta_i(\nu)$, for $i = 2, \ldots, n-1$, can be written as

$$\nu_1(1+\nu_1)C\sum_{i=0}^{C-1} i!\binom{C-1}{i}\binom{n-2}{i}\phi^i +$$
$$\nu_1^2 C(C-1)\sum_{i=0}^{C-2} i!\binom{C-2}{i}\binom{n-2}{i}\phi^i =$$
$$\phi C\sum_{i=0}^{C-1} i!\binom{C-1}{i}\binom{n-3}{i}\phi^i +$$
$$\phi(\nu_1^2+2\nu_1)C(C-1)\sum_{i=0}^{C-2} i!\binom{C-2}{i}\binom{n-3}{i}\phi^i +$$
$$\phi\nu_1^2 C(C-1)(C-2)\sum_{i=0}^{C-3} i!\binom{C-3}{i}\binom{n-3}{i}\phi^i,$$

as link 1 can be simultaneously active with link n and at most $C-1$ or $C-2$ intermediate links depending on whether

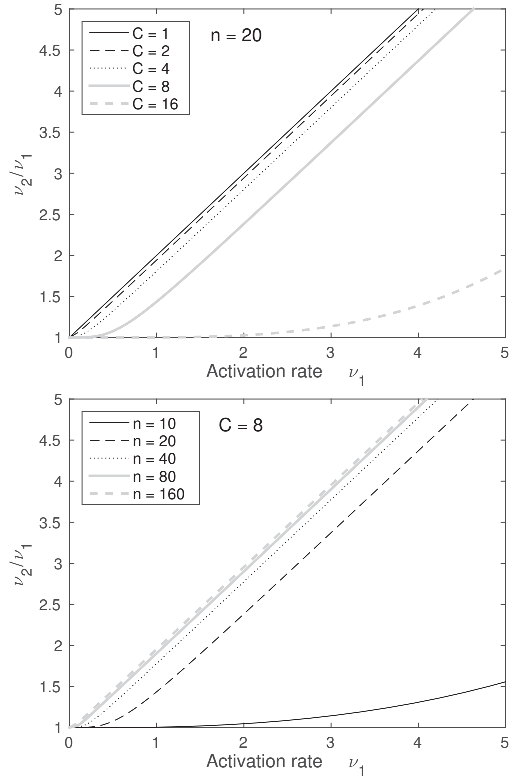

Figure 1: With repacking: ratio of ν_2 and ν_1 to achieve fairness as a function of ν_1 when $\beta = n-2$. For $C > 1$ channel this ratio is no longer a linear function of ν_1 and depends on n.

link 1 and n use the same or a different channel. If a intermediate link i is active with at most $C-2$ other intermediate links both link 1 and n can be active (either on the same or a different channel), while they must both be silent if there are $C-1$ other active intermediate links. In other words, ν_1 is the positive solution of a quadratic equation and (7) follows by noting that

$$\sum_{i=0}^{C-k}\frac{C!}{(C-k-i)!}\binom{n-2}{i}\phi^i -$$
$$\phi\sum_{i=0}^{C-k-1}\frac{C!}{(C-k-1-i)!}\binom{n-3}{i}\phi^i =$$
$$\sum_{i=0}^{C-k}\frac{C!}{(C-k-i)!}\binom{n-3}{i}\phi^i,$$

due to Pascal's triangle identity. $\square$

When $C = 1$, the both results reduce to $\nu_1 = (\sqrt{1+4\phi}-1)/2$, meaning $\phi = \nu_1(1+\nu_1)$ and ν_2/ν_1 grows linearly as a function of ν_1. Figures 1 and 2 indicate that when $C > 1$ the ratio ν_2/ν_1 needed to achieve fairness is no longer a linear function of ν_1 and this ratio depends on the network size n. The results do seem to indicate that if $n \gg C$ the fair ratio is close to $(1+\nu_1)$, which is the fair ratio for $C = 1$.

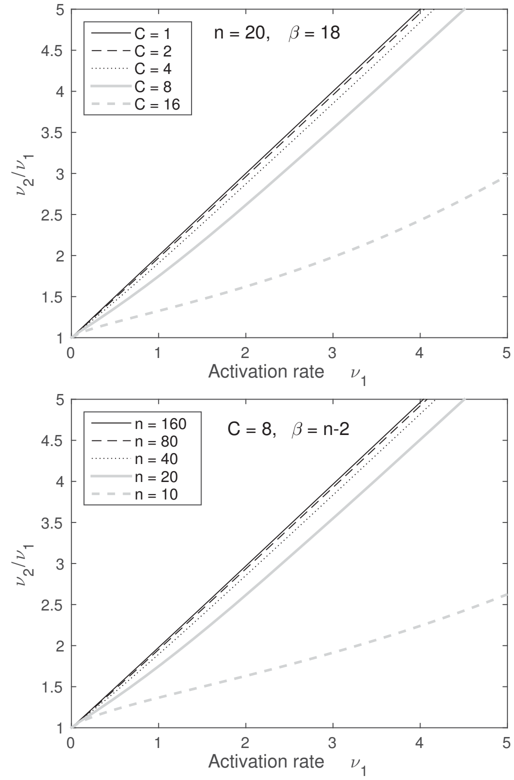

Figure 2: Without repacking: ratio of ν_2 and ν_1 to achieve fairness as a function of ν_1 when $\beta = n - 2$. For $C > 1$ channel this ratio is no longer a linear function of ν_1 and depends on n.

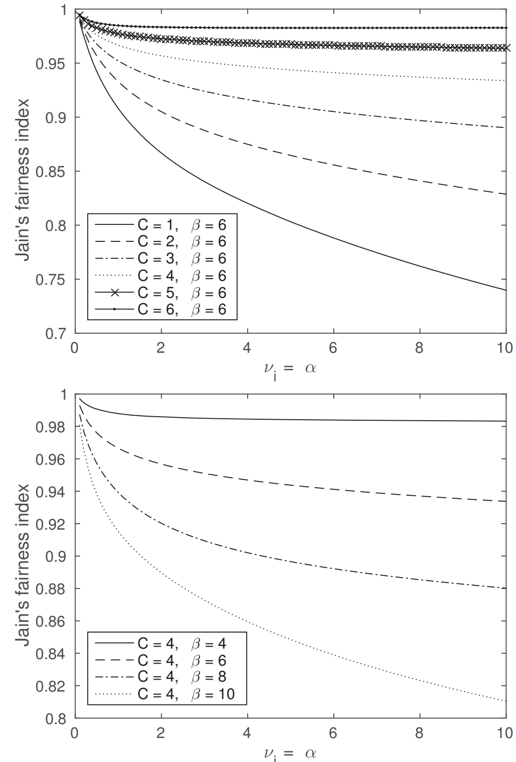

Figure 3: Without repacking: fairness index as a function of the activation rate $\nu_i = \alpha$ with $n = 40$ and either C or β fixed.

6. NUMERICAL RESULTS

In this section we investigate the impact of having multiple channels on the fairness in the network. We limit ourselves to the system without channel repacking as this is the most relevant from a practical point of view and numerical experiments not shown here confirm that the main conclusions for the system with repacking are in fact similar. To express the fairness of the system we make use of Jain's well-known fairness index [11], which is computed as

$$\mathcal{J}(\theta(\nu)) = \frac{\left(\sum_{j=1}^{n} \theta_j(\nu)\right)^2}{n \sum_{j=1}^{n} \theta_j(\nu)^2}.$$

We start by considering the case where all the links make use of the same activation rate, that is, $\nu_i = \alpha$ for $i = 1, \ldots, n$.

Figure 3 depicts the fairness index in a line network consisting of $n = 40$ links as a function of the activation rate α for different combinations of C and β. This figure demonstrates that fairness improves as the number of channels C increases with β fixed, while increasing β with C fixed increases unfairness. This is quite expected as decreasing C or increasing β implies that a link is more severely influenced by the activity of its neighboring links. We also note that the unfairness is quite severe as the index is well below one (unless C is close to β) and worsens as links become more aggressive, i.e., α increases.

We now repeat the same experiment, but instead of using equal rates we set the activation rate $\nu_i = \alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$, which guarantees fairness in low and heavy traffic as proven in Theorem 6 and fairness in all cases when $C = 1$ due to [18].

Figure 4 depicts the fairness index in a line network consisting of $n = 40$ links as a function of the parameter α. The first thing to note is that the index is now very close to one (above 0.995), meaning while these activation rates only guarantee fairness in the single channel setup, the unfairness is very limited in the multi-channel setup. We further note that as opposed to the equal rate case, fairness slightly decreases with the number of available channels C in most cases. Further, when C is fixed, having more or less interference, that is, increasing β, may result in either an increase or a decrease in fairness depending on the value of α.

Figure 5 further demonstrates that setting ν_i equal to $\alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$ results in a drastic improvement of the network fairness compared to using fixed activation rates. The fairness index in this particular case increases from 0.8583 to 0.9998. Note that the choice $\nu_i = 0.5(1.5)^6 \approx 5.7$ corresponds to the rate of the links in the middle of the network when $\nu_i = \alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$ and $\alpha = 0.5$.

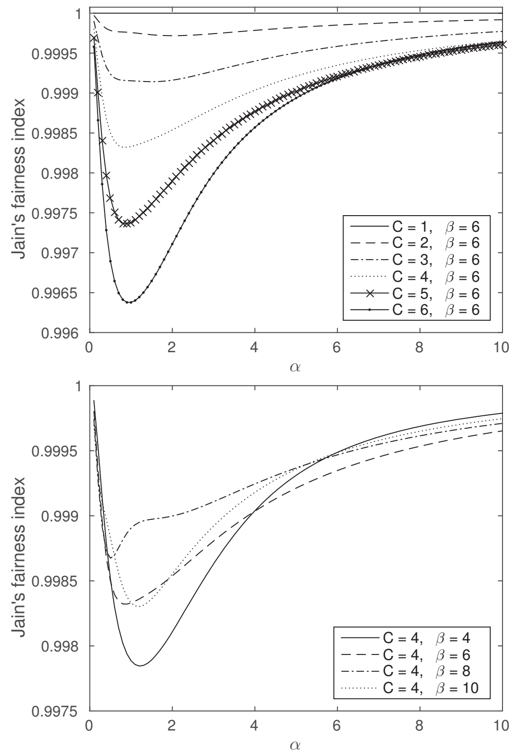

Figure 4: Without repacking: fairness index as a function of α with the activation rate $\nu_i = \alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$ with $n = 40$ and either C or β fixed.

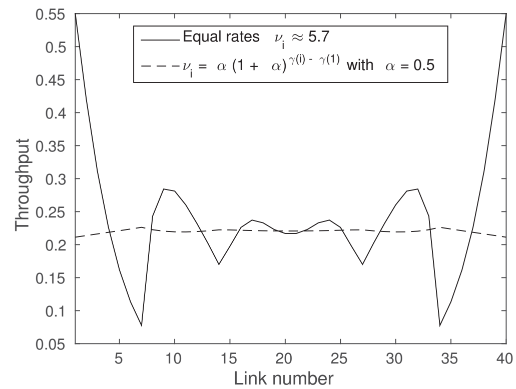

Figure 5: Without repacking: comparison of per-link throughputs $\theta_j(\nu)$ between equal activation rates and setting $\nu_i = \alpha(1+\alpha)^{\gamma(i)-\gamma(1)}$, when $\alpha = 0.5$, $n = 40$, $C = 2$ and $\beta = 6$.

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