

A Positioning of Cooperative Differential Games

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ABSTRACT

In this paper we embed the theory of cooperative differential games within the theory of dynamic games and discuss a number of results of them.

Keywords

Differential games, Pareto optimality, Dynamic consistency, Coalitional games

1. THE GENERAL SETTING

Dynamic game theory brings together four features that are key to many situations in economy, ecology, and elsewhere: optimizing behavior, presence of multiple agents/players, enduring consequences of decisions and robustness with respect to variability in the environment.

To deal with problems bearing these four features the dynamic game theory methodology splits the modeling of the problem into three parts.

One part is the modeling of the environment in which the agents act. To obtain a mathematical model of the agents' environment, usually, a set of differential or difference equations is specified. These equations are assumed to capture the main (dynamical) features of the environment. A characteristic property of this specification is that these dynamic equations mostly contain a set of so-called "input" functions. These input functions model the effect of the actions taken by the agents on the environment during the course of the game. In particular by viewing "nature" as a separate player in the game who can choose an input functional that works against the other player(s) one can model worst case scenario's and, consequently, analyze robustness of the "undisturbed" game solution.

A second part is the modeling of the agents' objectives. Usually the agents' objectives are formalized as cost/utility functionals which have to be minimized. Since this minimization

has to be performed subject to the specified dynamic model of the environment, techniques developed in optimal control theory play an important role in solving dynamic games. In fact, from a historical perspective, the theory of dynamic games arose from a merge of static game theory and optimal control theory. However, this merge can not be done without further reflection.

This is exactly what the third modeling part is about. To understand this point it is good to summarize the rudiments of static games. Most research in the field of static game theory has been, and is being, concentrated on the normal form of a game. In this form all possible sequences of decisions of each player are set out against each other. So, e.g., for a two-player game this results in a matrix structure. The information agents have about the game is crucial for the outcome of the decision making. In static games a distinction is made between complete and incomplete information games. In a complete information game, agents know not only their own payoffs, but also the payoffs and strategies of all the other agents.

Characteristic for a static game is that it takes place in one moment of time: all players make their choice once and simultaneous and, dependent on the choices made, each player receives his payoff.

In such a formulation important issues like the order of play in the decision process, information available to the players at the time of their decisions, and the evolution of the game are suppressed, and this is the reason why this branch of game theory is usually classified as "static". In case the agent's act in a dynamic environment these issues are, however, crucial and need to be properly specified before one can infer what the outcome of the game will be. This specification is the third modeling part that characterizes the dynamic game theory methodology.

One branch of dynamic games that are extensively analyzed in literature are the static noncooperative games that are played repeatedly.

In this paper we review results obtained for another special class of dynamic games. We consider games where the environment can be modeled by a set of differential equations, the so-called differential games (*DG*), see Figure 1. Of course many other mathematical models exist to describe systems which change over time (or sequentially). Well-known models are, e.g., difference equations, partial differential equations, time-delay equations where either stochastic uncertainty is added or not. All of these give rise to different classes of dynamic games that have their own specific

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features.

Current applications of differential games range from economics, financial engineering, ecology, marketing to the military. See e.g. [31], [7], [17], [30]. Furthermore, the proceedings and the associated Annals of the International Symposia on Dynamic Games and Applications document the development of both theory and applications over the last 30 years.

Within this class of differential games we pay some specific attention to the popular case that the objectives are modeled as functionals containing just (affine) quadratic terms and the differential equations are linear (readers interested in more details on linear quadratic differential games are referred to [9]). The popularity of these so-called linear-quadratic differential games is on the one hand caused by practical considerations. To some extent this kind of differential games are analytically and numerically solvable. If one leaves this track, one easily gets involved in the problem of solving sets of nonlinear partial differential equations, and only few of such equations can be solved analytically. Even worse, when the number of state variables is more than two in these equations, a numerical solution is in general hard to obtain.

On the other hand this linear-quadratic problem setting naturally appears if the agents' objective is to minimize the effect of a small perturbation of their nonlinear optimally controlled environment. By solving a linear quadratic control problem, and using the by this problem implied optimal actions, players can avoid most of the additional cost incurred by this perturbation (see [9] for a treatment of linear quadratic differential games).

To realize his objective an agent can either cooperate with one or more agents in the game or not. In case all agents cooperate we speak about a cooperative game (C). In case none of the agents cooperates with someone else the game is called a noncooperative game (NC). The intermediate case that groups of agents cooperate in coalitions against each other in a noncooperative way is called a coalitional game (CG), see Figure 2.

We will assume here that if agents cooperate that the agreed solution is binding (so it can be enforced).

In some situations where agents cooperate it is possible that agents can transfer (part of) their revenues/cost to another agent. If this is the case the game is called a transferable utility (TU) game. Otherwise it is called a non-transferable utility (NTU) game, see Figure 3.

Another important issue that affects the outcome of the game is the organization of the decision making process. In case there is one agent who has a leading position in the decision making process the game is called a Hierarchical or Stackelberg game (after H. von Stackelberg [39]). So in this case there is a vertical structure in the decision making process. In case there does not exist such a dependency we talk about a horizontal decision making structure.

To capture information aspects in a static game one uses the so-called extensive form of the game. Basically this involves a tree structure with several nodes and branches, providing an explicit description of the order of play and the infor-

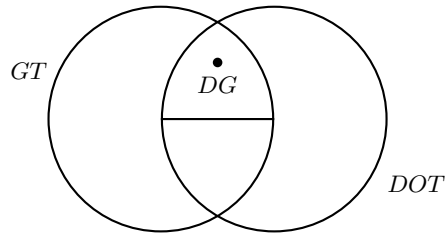


Figure 1: Differential Games ($\dot{D}G$).

mation available to each agent at the time of his decision. In case at least one of the agents has an infinite number of actions to choose from it is impossible to use the tree structure to describe the game. In those cases the extensive form involves the description of the evolution of the underlying decision process using a set of difference or differential equations. That is, introducing $u(k) := [u_1(k) \cdots u_N(k)]$, within a continuous time framework the extensive form is given by

$$\begin{aligned} \dot{x}(k) &= f_k(k, x(k), u(k)), \quad x(0) = x_0; \\ y_i(k) &= h_k(k, x(k), u(k)), \quad i = 1, \dots, N; \\ \dot{\eta}_i(k) &= I_k(k, \eta_i(k), y_i(k), u(k), z_i(k)), \quad \eta_i(0) = \eta_{i0}; \\ u_i(k) &= \gamma_i(k, \eta_i(k)), \end{aligned}$$

together with a cost functional $J_i : X \times U_1 \times \cdots \times U_N \rightarrow \mathbb{R}$ of agent i . Here $x(k) \in X$ represents the state of the game, $u_i(k) \in U_i$ the actions taken by agent i , $y_i(k)$ the observations of agent i , $\eta_i(k)$ the information available to agent i and $z_i(k)$ information that gets available exogenously, all at stage k . By considering different functions $I(\cdot)$ different information structures can be modeled. Some well-known information structures are the so-called open-loop and feedback (perfect state) information case. The open-loop information structure models the case that all agents know all future functions f_k , h_k and cost functions J_i at stage $k = 0$ together with the initial state x_0 . In this framework it is assumed that the agents determine their actions at $k = 0$ for the whole planning horizon of the game. Next submit these plans to some authority, who then enforces these plans as binding commitments for the whole planning period. The feedback information case assumes that separate from the fact that all agents know all future functions f_k , h_k and cost functions J_i at stage $k = 0$ they also know at any stage k the current state of the system $x(k)$. Clearly, this information structure fits much better reality in most cases.

2. CHOICE OF ACTIONS

From the previous section it will be clear that the actions played by the agents in a dynamic game depend on the coordination structure, organizational structure, information structure and decision rule (or strategy) followed by the agents. Assuming that every agent likes to minimize his cost the problem as stated so far, however, is not well defined. Depending on the coordination structure and organizational structure different solution concepts can be considered.

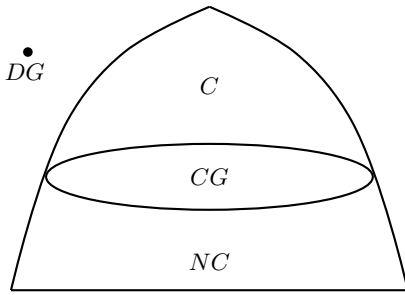


Figure 2: Classification Differential Games.

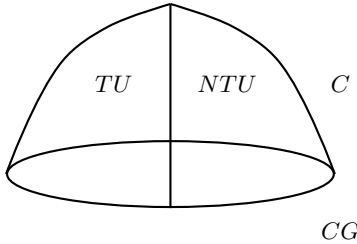


Figure 3: Classification Cooperative Differential Games.

If a static noncooperative game is played repeatedly, the notion of mixed strategies is often used. In a mixed strategy the agents choose their actions based on a probability distribution. The probability distribution chosen by the agents is assumed to be such that their average value of the game is optimized.

In a Stackelberg game (see e.g. [16] for a review of its use in supply chain and marketing models) it is assumed that the leader announces his decision u_L , which is subsequently made known to the other player, the follower. With this knowledge, the follower chooses his decision u_F by minimizing his cost for this choice of u_L . So, the optimal reaction of the follower u_F is a function of u_L . The leader takes this optimal reaction function of the follower into account to determine his action as the argument that minimizes his cost $J_L(u_L, u_F(u_L))$. Notice that in this solution concept it is assumed that the leader has a complete knowledge about the follower's preferences and strategy. Other solution concepts have been studied too, like e.g. the so-called inverse Stackelberg equilibrium, where the leader does not announce his action u_L , but his reaction function $\gamma_L(u_F)$. This concept can be used to enforce a by the leader desired behavior of the follower (see [28],[29]).

In a noncooperative game one of the most frequent used solutions is the Nash equilibrium. As the name suggest this is an equilibrium concept. It is assumed that ultimately those actions will be played by the agents that have the property that none of the agents can unilaterally improve by playing a different action. One of the main references

that documents the theoretical developments on this issue is the seminal book of [4]. Furthermore uncertainty can be dealt with within this framework by assuming that the player "nature" always selects a worst-case scenario (see e.g. [3], [22], [5] and [2]).

In a cooperative setting it seems reasonable to look for those combinations of control actions that have the property that the resulting cost incurred by the different players cannot be improved for all players simultaneously by choosing a different set of control actions. Formally, a set of control actions \hat{u} is called *Pareto efficient* if the set of inequalities $J_i(u) \leq J_i(\hat{u}, i = 1, \dots, N$, where at least one of the inequalities is strict, does not allow for any solution $u \in \mathcal{U}$. The corresponding point $(J_1(\hat{u}), \dots, J_N(\hat{u}))$ is called a *Pareto solution*. Usually there is more than one Pareto solution. The set of all Pareto solutions is called the *Pareto frontier*. In particular this implies that this Pareto efficiency concept in general does not suffice to conclude which action is optimal for an agent in a cooperative setting.

In case the cost can not be transferred between the agents the game is called a non-transferable utility game. In those cases, the cost of the agents are fixed once the actions of the agents are fixed. So, the question is then which point is reasonable to select on the Pareto frontier. Bargaining theory may help then to select a point on the Pareto frontier.

Bargaining theory has its origin in two papers by Nash [26] and [27]. In these papers a bargaining problem is defined as a situation in which two (or more) individuals or organizations have to agree on the choice of one specific alternative from a set of alternatives available to them, while having conflicting interests over this set of alternatives. Nash proposes in [27] two different approaches to the bargaining problem, namely the *axiomatic* and the *strategic* approach. The axiomatic approach lists a number of desirable properties the solution must have, called the *axioms*. The strategic approach on the other hand, sets out a particular bargaining procedure and asks what outcomes would result from rational behavior by the individual players.

So, bargaining theory deals with the situation in which players can realize -through cooperation- other (and better) outcomes than the one which becomes effective when they do not cooperate. This non-cooperative outcome is called the *threatpoint*. The question is to which outcome the players may possibly agree.

In Figure 4 a typical bargaining game is sketched. The ellipse marks out the set of possible outcomes, the *feasible set* S , of the game. The point d is the threatpoint. The edge P is the Pareto frontier.

Three well-known solutions are the Nash bargaining solution, the Kalai-Smorodinsky solution and the Egalitarian solution. The *Nash bargaining solution*, selects the point of S at which the product of utility gains from d is maximal. The Kalai-Smorodinsky solution divides utility gains from the threatpoint proportional to the player's most optimistic expectations. For each agent, the most optimistic expectation is defined as the lowest cost he can attain in the feasible set subject to the constraint that no agent incurs a cost higher

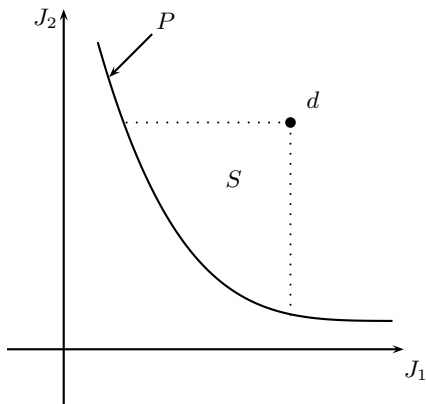


Figure 4: The bargaining game.

than his coordinate of the threatpoint. Finally, the *Egalitarian solution*, represents the idea that gains should be equal divided between the players. For more background and other axiomatic bargaining solutions we refer to Thomson [40].

In transferable utility games it may happen that it is less clear-cut how the gains of cooperation should be divided. Consider, e.g., the case that agents are involved in a joint project and the joint benefits of this cooperation have to be shared. In those cases an agreement in the cooperative dynamic game, or *solution* of the cooperative dynamic game, involves both an agreement on the allocation rule and an agreement on the set of strategies/controls. Of course also in this case the allocation rule should be individually rational in the sense that no agent should be worse off than before his decision to cooperate.

In differential games an important issue is then at what point in time the "payments" occur. Is this at the beginning of planning horizon of the game, at the end of the planning horizon of the game, at some a priori determined fixed points in time of the game or is every agent paid continuously during the length of the game. Particularly in the last case it seems reasonable to demand from the allocation rule that it is consistent over time. That is, the allocation rule is such that the allocation at any point in time is optimal for the remaining part of the game along the optimal state trajectory. So in particular at any point in time the payment should be individually rational for every player. An allocation rule that has this property is called subgame-consistent. Of course in a dynamic cooperative game not only the payment allocation rule is important but, like for all dynamic games, also the time-consistency of the strategies is important from a robustness point of view. A solution is called *subgame-consistent* if the allocation rule is subgame-consistent and the cooperative strategies are strongly time consistent. [42] Yeung and Petrosyan give a rigorous framework for the study of subgame-consistent solutions in cooperative stochastic differential games (see also [43] for an extension of this theory).

3. THE PARETO FRONTIER

As outlined above, in cooperative (differential) games a solution is chosen on the Pareto frontier if all agents have their own cost function and they decide to cooperate in order to improve their performance. So the question is how one can determine in those cases the Pareto frontier.

Assume that agent i , $i = 1, \dots, N$, likes to minimize the performance criterium:

$$J_i := \int_{t_0}^T g_i(t, x(t), u_1(t), \dots, u_N(t)) dt + h_i(x(T)), \quad (1)$$

w.r.t. u_i , where $x(t)$ is the solution of the differential equation

$$\dot{x}(t) = f(t, x(t), u_1(t), \dots, u_N(t)), \quad x(t_0) = x_0. \quad (2)$$

Without going into details we make, of course, the assumption that the above integrals and differential equations are well-defined in the sense that they have a (unique) solution for every considered control function.

One way to find Pareto solutions is as follows.

LEMMA 3.1. (Weighting Method)

Let $\alpha_i \in (0, 1)$, with $\sum_{i=1}^N \alpha_i = 1$. Assume $\hat{u} \in \mathcal{U}$ is such that

$$\hat{u} \in \arg \min_{u \in \mathcal{U}} \left\{ \sum_{i=1}^N \alpha_i J_i(u) \right\}. \quad (3)$$

Then \hat{u} is Pareto efficient. \square

The above Lemma, which is called the weighting method in static nonlinear optimization problems (see e.g. [21]), gives us an easy way to find Pareto efficient controls. It is, however, unclear whether we obtain all Pareto efficient controls in this way. In fact the above procedure may yield no Pareto efficient controls, while an infinite number of Pareto solutions exist. The next example illustrates this point.

EXAMPLE 3.2. Consider

$$\dot{x}(t) = u_1(t) - u_2(t); \quad x(0) = 0, \quad (4)$$

together with the cost functions

$$J_1 = \int_0^1 (u_1(t) - u_2(t)) dt \quad \text{and} \quad J_2 = \int_0^1 x^2(t)(u_2(t) - u_1(t)) dt.$$

Then, by construction, $J_2 = -\frac{1}{3}x^3(1) = -\frac{1}{3}J_1^3$ for all (u_1, u_2) . Obviously by choosing different values for the control functions $u_i(\cdot)$, every point in the (J_1, J_2) plain satisfying $J_2 = -J_1^3$ can be attained. Furthermore it is clear that every point on this curve is Pareto optimal.

Next consider the minimization of $J(\alpha) := \alpha J_1(u_1, u_2) + (1 - \alpha) J_2(u_1, u_2)$ subject to (4), where $\alpha \in (0, 1)$. If we choose $u_1(t) = 0$ and $u_2(t) = c$ straightforward calculations yield that $J(\alpha) = -\alpha c + \frac{1-\alpha}{3}c^3$. So by choosing c arbitrarily negative $J(\alpha)$ can be made arbitrarily small, i.e. $J(\alpha)$ does not have a minimum. \square

In [11] a maximum principle is derived to find Pareto efficient controls for problem (1,2) if the planning horizon T is finite. We recall this result here below in Theorem 3.3. We use the notation $\mathcal{A} := \{\alpha = (\alpha_1, \dots, \alpha_N) \mid \alpha_i \geq 0 \text{ and } \sum_{i=1}^N \alpha_i = 1\}$ to denote the "unit-simplex".

THEOREM 3.3. (Maximum Principle)

Assume $(J_1(\hat{u}), \dots, J_N(\hat{u}))$ is a Pareto solution for problem (1,2). Then, there exists an $\alpha \in \mathcal{A}$, a costate function $\lambda^T(t) : [0, T] \rightarrow \mathbb{R}^n$ (which is continuous and piecewise continuously differentiable) such that, with $H(t, x, u, \lambda) := \sum_{i=1}^N \alpha_i g_i(t, x, u) + \lambda f(t, x, u)$, \hat{u} satisfies

$$H(t, \hat{x}(t), \hat{u}(t), \lambda(t)) \leq H(t, \hat{x}(t), u(t), \lambda(t)),$$

at each $t \in [0, T]$,

$$\dot{\lambda}(t) = -\left[\sum_{i=1}^N \alpha_i \frac{\partial g_i}{\partial x} + \lambda(t) \frac{\partial f}{\partial x}\right]; \quad \lambda(T) = \frac{\partial(\sum_{i=1}^N \alpha_i h_i)}{\partial x},$$

$$\dot{\hat{x}}(t) = f(t, \hat{x}(t), \hat{u}_1(t), \dots, \hat{u}_N(t)), \quad \hat{x}(0) = x_0.$$

□

The necessary conditions from Theorem 3.3 are closely related to the minimization of $\sum_{i=1}^N \alpha_i J_i$ subject to (2). By considering the Hamiltonian for this problem $H := \sum_{i=1}^N \alpha_i g_i + \lambda f$, we obtain from the maximum principle the conditions stated in Theorem 3.3. Unfortunately the maximum principle conditions just provide necessary conditions. So, in case a solution satisfies all conditions from Theorem 3.3, we still can not conclude that it will give us a Pareto solution. On the other hand, as demonstrated in Example 3.2, the weighting method in Lemma 3.1 may not have a solution. This, although in this example the set of necessary conditions has an infinite number of solutions which all provide a Pareto solution.

In [11] also sufficient conditions are presented under which one can conclude that a solution satisfying the maximum principle conditions will be a Pareto solution.

It is also good to take notice of the fact that in some problems the interpretation of the weights to reflect the relative importance of the objective functions may be misleading. In, e.g., [11, Example 2.15] the Pareto frontier is derived for a two-player game using Theorem 3.3. In particular it is shown that the choice of $\alpha_i = 1, \alpha_j = 0, i \neq j$, yield Pareto solutions. On the other hand it is shown in that example that as well the minimization of the cost of player one and two has no solution. So the Pareto solution corresponding with $\alpha_1 = 1, \alpha_2 = 0$ cannot be interpreted as being the solution obtained by considering the minimization problem in which the interests of player two are ignored. Furthermore it is well-known (see e.g. [24] for the static multi-objective case) that an evenly distributed set of parameters usually does not produce an evenly distributed set of points in the Pareto surface.

The infinite-planning horizon case is dealt with in [34] (see also [35]). In that paper analogues of the results for the finite-planning horizon case [11] are derived. Like in ordinary optimal control problems the lack of a clear specification of transversality conditions cause additional technical problems. However, under some weak conditions, related

to a natural extension of finite horizon transversality conditions, it is shown that like in the finite planning horizon case the necessary conditions for Pareto optimality are the same as those of a weighted sum optimal control problem. Furthermore, conditions are presented under which the necessary conditions are also sufficient to conclude that a solution is Pareto optimal.

We conclude this section by considering the linear quadratic differential game. That is, if

$$J_i = \int_0^T [x^T(t), u_1^T(t), u_2^T(t)] M_i \begin{bmatrix} x(t) \\ u_1(t) \\ u_2(t) \end{bmatrix} dt, \quad (5)$$

where $M_i = \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix}$ is symmetric, $\begin{bmatrix} R_{1i} & N_i \\ N_i^T & R_{2i} \end{bmatrix}$ is positive definite, (> 0), and $x(t)$ is the solution of the linear differential equation

$$\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t) + c(t), \quad x(0) = x_0. \quad (6)$$

Here $x \in \mathbb{R}^n$ is the state of the system ; $u_i \in \mathbb{R}^{m_i}$ is the set of control variables of player i ; $Q_i \in \mathbb{R}^{n \times n}$, $V_i \in \mathbb{R}^{n \times m_1}$, $W_i \in \mathbb{R}^{n \times m_2}$, $R_{ki} \in \mathbb{R}^{m_k \times m_k}$, $k = 1, 2$, and $N_i \in \mathbb{R}^{m_1 \times m_2}$. The variable $c(\cdot) \in L_2^n$ is some given function of time. No definiteness assumptions are made w.r.t. matrix Q_i . So, in particular, the state objectives of the players might be either conflicting or both negative valued. These kind of problems often naturally occur, e.g. in economics, where players like to maximize a utility function using costly controls. Since the matrices $\begin{bmatrix} R_{1i} & N_i \\ N_i^T & R_{2i} \end{bmatrix}$, $i = 1, 2$, are assumed to be positive definite the problem is called regular.

This regular linear quadratic differential game is studied in [10]. In that paper both necessary and sufficient conditions are given under which the individual optimization problem, i.e. the minimization of J_i subject to (6) w.r.t. $[u_1 \ u_2]$, is a convex optimal control problem. It is well-known that if the cost functions are convex, that all Pareto solutions can be obtained using the weighting method. This gives, e.g., rise to the next procedure to calculate all Pareto efficient outcomes for the next game.

THEOREM 3.4. (Regular Convex LQ Game)

Consider the cooperative game (5,6) with $T = \infty$, $\mathcal{U} = L_{2,e,s}^+$, $B := [B_1 \ B_2]$ and (A, B) stabilizable.

For $\alpha \in \mathcal{A}$ let $M(\alpha) := \alpha_1 M_1 + \alpha_2 M_2 =: \begin{bmatrix} \tilde{Q} & \tilde{V} \\ \tilde{V}^T & \tilde{R} \end{bmatrix}$,

where

$$\tilde{Q} = \alpha_1 Q_1 + \alpha_2 Q_2, \quad \tilde{V} = [\alpha_1 V_1 + \alpha_2 V_2, \alpha_1 W_1 + \alpha_2 W_2],$$

$$\text{and } \tilde{R} = \alpha_1 \begin{bmatrix} R_{11} & N_1 \\ N_1^T & R_{21} \end{bmatrix} + \alpha_2 \begin{bmatrix} R_{12} & N_2 \\ N_2^T & R_{22} \end{bmatrix}.$$

Furthermore, let $\tilde{S} := B \tilde{R}^{-1} B^T$.

Assume that (7) below has a stabilizing solution X_1 for $(\alpha_1, \alpha_2) = (1, 0)$ and X_2 for $(\alpha_1, \alpha_2) = (0, 1)$, respectively.

$$A^T X + XA - (XB + \tilde{V}) \tilde{R}^{-1} (B^T X + \tilde{V}^T) + \tilde{Q} = 0. \quad (7)$$

Then the set of all cooperative Pareto solutions is given by $\{(J_1(u^*(\alpha)), J_2(u^*(\alpha))) \mid \alpha \in A\}$. Here

$$u^*(t) = -\tilde{R}^{-1}(B^T X_s + \tilde{V}^T)x(t) - \tilde{R}^{-1}B^T m(t),$$

$$\text{where } m(t) := \int_t^\infty e^{-A_{cl}^T(t-s)} X_s c(s) ds,$$

X_s is the stabilizing solution of (7) and, with $A_{cl} := A - B\tilde{R}^{-1}\tilde{V}^T - \tilde{S}X_s$, the closed-loop system is $\dot{x}(t) = A_{cl}x(t) - \tilde{S}m(t) + c(t)$, $x(0) = x_0$. In case $c(\cdot) = 0$ the corresponding cost are $J_i(x_0, u^*) = x_0^T \tilde{M}_i x_0$, where \tilde{M}_i is the unique solution of the Lyapunov equation

$$A_{cl}^T \tilde{M}_i + \tilde{M}_i A_{cl} = -[I, -(X_s B + \tilde{V})\tilde{R}^{-1}] M_i \begin{bmatrix} I \\ -\tilde{R}^{-1}(B^T X_s + \tilde{V}^T) \end{bmatrix}. \quad \square$$

Notice, that in [23, Section 11.3], it is shown that if the parameters appearing in an algebraic Riccati equation are, e.g., differentiable functions of some parameter α (or, more general, depend analytically on a parameter α), and the maximal solution exists for all α in some open set V , then this maximal solution of the Riccati equation will be a differentiable function of this parameter α too on V (c.q., depend analytically on this parameter α too). Since in Theorem 3.4 the parameters depend linearly on α the Pareto frontier will in this case be a smooth function of α .

4. COALITIONAL GAMES

The bargaining approach presented in the previous section does not consider the formation of coalitions. In the presence of non binding agreements, even if the players agree upon a cooperative outcome, situations arise where the grand coalition could break down. Classical coalitional games are casted in characteristic function form. When the utilities are transferable a *characteristic function* $v(\cdot)$ assigns to every coalition a real number (*worth*), representing the total payoff of this coalition of players when they cooperate. Stated differently, it denotes the power structure of the game i.e., the players in a coalition collectively demand a payoff $v(S)$ to stay in the grand coalition. In the bargaining problem the coordinates of the threat point d_i represent the payoff each player receives by acting on their own. Similarly $v(S)$ represents the collective payoff that a coalition $S \subset N$ can receive when the left out players in the coalition $N \setminus S$ act against S . In a non-transferable utility setting, however, two distinct set valued characteristic functions have been proposed, see [1], as the α and β characteristic functions. The main difference originates from the functional rules used in deriving them from the normal form game.

Under α notion, the characteristic function indicates those payoffs that coalition S can secure for its members even if the left out players in $N \setminus S$ strive to act against S . Here, players in S first announce their joint correlated strategy before the joint correlated strategy of the players in $N \setminus S$ is chosen. So, this is an *assurable* representation. Under β notion, the characteristic function indicates those payoffs that the left out players in $N \setminus S$ cannot prevent S from getting. Here, players in S choose their joint correlated strategy after the joint correlated strategy of the players in $N \setminus S$ is announced. So, this is an *unpreventable* representation.

An *imputation* is a set of allocations which are individually rational, i.e., every allocation is such that it guarantees the involved player a payoff more than what he could achieve on his own. A set of allocations is in the *core* when it is coalitionally rational. That is, the core consists of those imputations for which no coalition would be better off if it would separate itself and get its coalitional worth. Or, stated differently, a set of allocations belongs to the core if there is no incentive to any coalition to break off from the grand coalition. Clearly, the core is a subset of the Pareto frontier. The core is obtained by solving a linear programming problem. It can be empty. There are other solution concepts based on axioms such as Shapley value and nucleolus.

The cooperative solutions mentioned above are static concepts. Introducing dynamics in a coalitional setting raises new conceptual questions. It is not straightforward as to how one can extend the classical definition of core in a dynamic setting since there exist many notions of a profitable deviation. As a result, an unifying theory of dynamic coalitional games, at present, seems too ambitious. However, intuitively, in this context one expects the definition of core should capture those situations in which at each stage the grand coalition is formed no coalition has a profitable deviation, i.e., dynamic stability, taking into account the possibility of future profitable deviations of sub-coalitions. In an environment with non binding agreements only self-enforcing allocations are deemed to be stable. The main difference between static and dynamic setting is the credibility [33] of a deviation. A deviation of a coalition S is *credible*, if there is no incentive for a sub-coalition $T \subset S$ to deviate from S . The set of deviations and credible deviations coincide for a static game but differ in a dynamic setting. Kranich et al. [19] suggest new formulations of the core in dynamic cooperative games using credible deviations. For instance, if one makes an assumption that once a coalition deviates players cannot return to cooperation later, results in a core concept called strong sequential core. This allows for further splitting of the deviating coalition in the future. They also introduce a notion of weak sequential core which is a set of allocations for the grand coalition from which no coalition has ever a credible deviation. See, [13] for more details.

We review some work done towards extending the idea of a core in a differential game setting. Haurie [14] constructs an α characteristic function assuming the behavior of left out players is modeled as unknown bounded disturbances. Using this construction he introduces in [15] collectively rational Pareto optimal trajectories with an intent to extend the concept of core to dynamic multi stage games. Analogously, a Pareto equilibrium is called collectively optimal (C-optimal) when, at any stage, no coalition of a subset of the decision-makers can assure each of its members a higher gain than what he can get by full cooperation with all the other decision-makers. It is shown that if the game evolves on these trajectories any coalition does not have an incentive to deviate from the grand coalition in the later stages.

Time consistency, as introduced by Petrosjan et al. [32], in a dynamic cooperative game means that when the game evolves along the cooperative trajectory generated by a solution concept (which can be any solution concept such as

core, Shapley value and nucleolus) then no player has an incentive to deviate from the actions prescribed by that solution. The notion of strong sequential core introduced in Kranich et. al [19] is the same as time consistency. Zaccour [44] studies the computational aspects of characteristic functions for linear state differential games. Evaluation of characteristic function involves $2^N - 2$ equilibrium problems and one optimization problem (for the grand coalition) which is computationally expensive with large number of players. Therefore, instead, they propose an approach by optimizing the joint strategies of the coalition players while the left out players use their Nash equilibrium strategies. This modification involves solving one equilibrium problem and $2^N - 2$ optimization problems. Further, they characterize a class of games where this modified approach provides the same characteristic function values.

Assuming that players at each period/instant of time consider alternatives 'cooperate' and 'do not cooperate', Klompstra [20] studies a linear quadratic differential game. It is shown that for a 3 player game, there exists time dependent switching between different modes namely the grand coalition, formation of sub-coalitions and total non cooperation. Assuming similar behavior of players, i.e., to 'cooperate' or 'do not cooperate' at each time instant, Germain et. al [12] introduce a rational expectations core concept. They use γ characteristic function [6] where the left out players act individually against the coalition instead of forming a counter coalition. They show, using an environmental pollution game, that if each period of time players show interest in continued cooperation then, based on the rational expectations criterion, there exists a transfer scheme that induces core-theoretic cooperation at each period of time. Recently, Jorgensen [18] studies a differential game of waste management and proposes a transfer scheme that sustains inter-temporal core-theoretic cooperation.

5. DECENTRALIZATION

In a cooperative setting, agents coordinate their strategies and it is not always feasible to maintain communication to implement their coordinated actions. Further, problems can arise due to lack of stability in the cooperative agreement. Threats and deterrence are some of the common stability inducing mechanisms used to enforce cooperation like, for instance, trigger strategies where a player using a trigger strategy initially cooperates but punishes the opponent if a certain level of defection (i.e., the trigger) is observed. In the context of differential games, see section 6.2 of [7] for more details on such strategies. In his seminal paper, Rosen [36] introduces a concept of normalized equilibrium that deals with a class of noncooperative games in which the constraints as well as the payoffs may depend on the strategies of all players. Using this approach Tidball et.al [38] show in a static game that a cooperative solution can be attained by a suitable choice of the normalized equilibrium. Further, they show, in a dynamic context, that only by introducing a tax mechanism it is possible to attain cooperation in a decentralized manner. Rosenthal [37] introduced a class of games which admit pure strategy Nash equilibria, which were latter studied in a more general setting by Monderer et al. [25] as potential games. A strategic game is a potential game if it admits a potential function. A potential function is a real valued function, defined globally on the strategies

of the players, such that its local optimizers are precisely the equilibria of game. So, these games enable the use of optimization methods to find equilibria in a game instead of fixed point techniques. If, the social objective of the game coincides with the potential function then we see that the social optimum can be implemented in a noncooperative manner. Dragone et al. [8] present some preliminary work towards the extension of potential games in a differential game setting and study games that arise in advertising.

6. REFERENCES

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