HJB approach for motion planning and reachabilty analysis

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ABSTRACT
Because of their importance in many applications, questions of path planning and reachability analysis for nonlinear dynamical systems have been studied extensively in the control theory. Here we focus on the cases when the controlled systems are constrained to evolve in a certain known set (e.g avoidance of obstacles). We study general framework based on an optimal control approach and on solving Hamilton-Jacobi (HJ) equations. This approach provides a very efficient tool for treating many cases encountered in real applications and can be extended to general situations including moving targets and/or obstacles problems, dynamical systems under uncertainties, or differential games. The relevance of the method will be shown on some numerical examples (motion planning with obstacle avoidance).

Categories and Subject Descriptors
G.1 [**Mathematics of Computing**]: Numerical analysis; G.4 [**Mathematics of Computing**]: Mathematical software; H.4 [**Information Systems Applications**]: Miscellaneous

General Terms
Optimal control theory, numerical analysis, scientific computing

Keywords
Minimal time problem, moving targets, time-dependant state constraints, motion planning, obstacle avoidance, Hamilton-Jacobi-Bellman equations, level set method, reachablity analysis

1. INTRODUCTION
Consider a dynamic system whose state $y(t) \in \mathbb{R}^d$ obeys a system of differential equations:

$$\dot{y}(s) = f(y(s), s, \alpha(s)), \text{ for a.e. } s \in [0, t], \quad (1a)$$
$$y(0) = x, \quad (1b)$$

where f is a given function. The vector $\alpha(s) \in \mathbb{R}^m$ is the control input of the dynamic system. For $t \geq 0$, we assume that $\alpha \in \mathbb{A}_t := \{\alpha : (0, t) \to \mathcal{A}, \text{ measurable}\}$ is a measurable function, with $\mathcal{A}$ is a nonempty compact subset of $\mathbb{R}^m$. The trajectory starting at initial state $x \in \mathbb{R}^n$ associated with a control policy $\alpha \in \mathbb{A}_t$, is defined as the solution $y = y_x^\alpha$ of (9).

We are interested in path planning and reachable sets (or capture basins) analysis for dynamic systems, by using an optimal control approach based on solving equations of HJ type. This approach provides a general framework for treating many cases encountered in applications: safety of ground transportation, air traffic management, flight control, and many other applications.

The reachability analysis may be stated in two equivalent ways, depending on the applications:

Reachable sets. For a given initial set $X \subset (\mathcal{K}_0)$, determine the set of states that the system may reach before a given, fixed time $T > 0$:

$$\text{Reach}_{X,\mathcal{K}}^f(T) := \Big\{ y \in \mathbb{R}^d, \ \exists \alpha \in \mathbb{A}_t, \ \exists x \in X, \quad (2)$$
$$\exists \theta \in [0, T] \mid y_x^\theta(s) = y \Big\}$$

Capture basin. Determine the set of initial states from which the set $\mathcal{C}_t$ is reachable before $t < T$.

$$\text{Cap}_{\mathcal{C},\mathcal{K}}^f(T) := \Big\{ x \in \mathbb{R}^d, \ \exists \alpha \in \mathbb{A}_t \mid y_x^\alpha(t) \in \mathcal{C}_t, \Big\}$$

The figure 1 shows an example of a reachable set and capture basin. It is important to emphasis that a state x is in the capture basin if at least one trajectory starting at x reach the target set $\mathcal{C}$. The figure 1 (b) shows some possible trajectories starting from different points of the capture

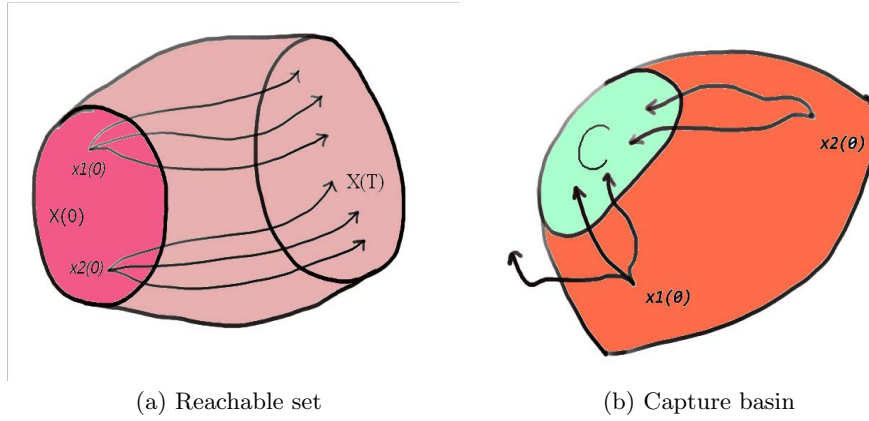|	
(a) Reachable set	(b) Capture basin

Figure 1: Reachable set and capture basin

bassin. Note that it is easy to see that the capture basin is the backwardly reachable set starting from the target $\mathcal{C}$.

Determining reachable sets or prohibited zones is important for many applications. Knowledge of these sets allows a qualitative analysis of the system before designing quantitative solutions to different problems (e.g path planning). This qualitative analysis can for example show the feasibility of a mission, or isolate hazardous areas to avoid. The simplest example of a navigation problem is a ground vehicle that has the objective of reaching a target in a given, finite time. The state of the system is characterized by the state vector (x, y, θ) where (x, y) are the 2D-coordinates of the center of mass of the vehicle in a given reference frame, and θ is the angle between the velocity vector and the x-axis. The state space is $\mathbb{R} \times \mathbb{R} \times [0, 2\pi]$n and the motion of the vehicle obeys the following :

$$\begin{cases} x' = u\cos(\theta) \\ y' = u\sin(\theta) \\ \theta' = \omega \end{cases} \quad (3)$$

where the first control input $u \in [u_{min}, u_{max}]$ is the velocity modulus and the second control input $\omega \in [\omega_{min}, \omega_{max}]$ is the angular speed. We may assume that the target set is a cylinder of radius $R > 0$ centered at $(0,0)$:

$$C = B(0, R) \times [0, 2\pi]$$

If the vehicle must reach the target in a finite time less than $T > 0$, we may firstly determine all initial positions from where this mission is possible. To do so we may compute the corresponding capture basin. If the initial position of the vehicle lies inside the capture basin, we may want to determine what trajectory to choose to reach the target, based on a given criterion. Problems of exhibiting minimal time trajectories are therefore connected to this reachability problem.

Direct characterization of reachability concepts is one of the topics addressed by viability theory [1]. An indirect approach to reachability questions consists on using optimal control theory. In this case, the reachable sets can be characterized as level-sets of the value function of an appropriate optimal control problem. Using the dynamic programming principle, the value function can be characterized as the solu-

tion of a Hamilton-Jacobi equation. Here, we study such an optimal control approach for a very general setting including problems with uncertainties, differential games, moving target problems.

The main advantage of the approach developed here is that the resulting partial differential equations can be solved by several efficient and fast numerical methods developed in the last decades. The approach does not require any controllability assumption. Moreover, our optimisation-based method allows reconstruction of the optimal control law, for any initial position given in the capture basin. On robots, this control law may be used as a guidance policy.

1.1 State-constrained control systems

In real control applications, such as in robotics for instance, the motion of the vehicle may be limited due to environment constraints or obstacles. We formalise this by introducing à subset $\mathcal{K} \subset \mathbb{R}^n$ which represents the set of admissible states. We require that the state of the considered dynamical system verify $y(t) \in \mathcal{K}$, $\forall t \geq 0$. In this case we define the reachable sets and capture basins in the following manner.

Reachable set for a state-constrained system. For an initial set $X \subset (\mathcal{K}_0)$, determine the set of states that the system may reach before a given, fixed time $T > 0$:

$$\text{Reach}_{X,\mathcal{K}}^f(T) := \Big\{ y \in \mathbb{R}^d, \ \exists \alpha \in \mathbb{A}_t, \ \exists x \in X, \quad (4)$$
$$\exists \theta \in [0, T] \mid y_x^\theta(\theta) = y \text{ and } y_x^\alpha(\tau) \in \mathcal{K}, \ \forall \tau \in [0, \theta] \Big\}$$

Capture basin for a state-constrained system. Determine the set of initial states from which the set $\mathcal{C}$ is reachable before time sT.

$$\text{Cap}_{\mathcal{C},\mathcal{K}}^f(T) := \Big\{ x \in \mathbb{R}^d, \ \exists \alpha \in \mathbb{A}_t, \quad (5)$$
$$y_x^\alpha(t) \in \mathcal{C}, \ \text{ and } y_x^\alpha(\theta) \in \mathcal{K}, \ \forall \theta \in [0, t] \Big\}$$

We illustrate this concept with the figure 2. Here the evolution of the system is constrained by the set K. In some cases the reachability analysis allows to find some forbidden zones around the obstacles (in blue on the figure below). These
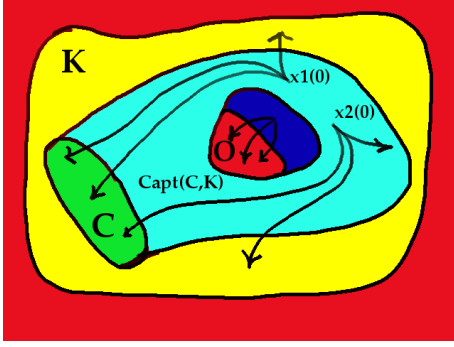
Figure 2: Example of a capture basin with constraints and obstacles

areas represent the initial states for which it is not possible to find any *admissible* evolution: all trajectories starting in such a zone leave the set $\mathcal{K}$.

Problems related to navigation. No matter what the specific context, the capture basin may be determined when a vehicle has to reach a target in a limited or unlimited time. The system dynamics and the controls are determined by the nature of the vehicle (car, robot, surface ship, ...); the constraint sets are determined by the environment of the vehicle (obstacles, control limits, ...). Other forms of constraints can be considered in applications related to navigation. We can assume for example that in the environment of the vehicle one ou several observers are present, located at known positions. One can then search for paths minimizing the time spent outside the areas of visibility of observers (or, conversely, in those areas).

Safety related problems. This class of problems is related to collision avoidance, and includes air traffic Control, with the subset of "Sense and Avoid" problems. In this case, the reachable sets are crucial elements of trajectory control policies. Indeed these sets describe all possible relative positions of an aircraft and an intruder at a given instant, therefore they allow forecast of all possible collisions. Avoidance maneuvers may only be safely determined based on this data.

Finally let us point out that in our approach (based on optimal control theory), the reachability analysis and the path planning can be studied for general controlled systems, including systems lacking controllability properties. On the other hand, let us also mention that in our setting, mixed constraints (state-control) can be also considered.

1.2 Nonlinear controlled systems under uncertainties

In numbers of applications, it is crucial to be able to take environment uncertainty into account. For instance some obstacles may move in an unknown manner, and even do so in an agressive fashion against the vehicle, so as to preclude it to reach the target. One of a popular approach is to study this problem in the framework of differential games, where the vehicle's controls are actions of one player. The control player plays against environment uncertainties, considered as a second player with an opposite goal : preclude the opponent from reaching the target. In this case we need

to determine a guaranteed capture basin defining all initial positions from which the target can be reached, no matter what the environment is able to play against the vehicle. this set is defined as follows. The dynamic system in this cas is of the form

$$\dot{y}(s) = f(y(s), s, \alpha(s), \beta(s)), \text{ for a.e. } s \in [0, t], \quad (6a)$$
$$y(0) = x, \quad (6b)$$

where $\alpha(s)$ is the control input of the first player and $\beta(s)$ is the control input of the second player. The guarantee capture basin for the first player is the following set:

$$\text{Cap}_{\mathcal{C},\mathcal{K}}^{f}(t) := \left\{ x \in \mathbb{R}^d, \ \exists a \in \Gamma_t, \ \forall \beta \in \mathbb{B}_t, \quad (7) \right.$$
$$\left. \left(y_x^{a[\beta],\beta}(t) \in \mathcal{C}_t, \ \text{ and } y_x^{a[\beta],\beta}(\theta) \in \mathcal{K}, \ \forall \theta \in [0, t] \right) \right\}$$

To show how guaranteed capture basins are used in safety related problems, we shall now present a simplified version of the collision avoidance problem for UAVs. In order to do so, we choose a local reference frame centered at the center of mass of the aircraft, and such that the x-axis is parallel to the velocity. The relative coordinates of an intruder in this reference frame are subjected to the following differential system :

$$F : \begin{cases} x'(t) = -V_d + V_a \cos(\psi(t)) + \omega(t)y(t), \\ y'(t) = V_a \sin(\psi(t)) - \omega(t)x(t), \\ \psi'(t) = v(t) - \omega(t). \end{cases} \quad (8)$$

Here V_a and V_d are modulus of the velocity of the intruder and the UAV respectively. It is assumed that both aircraft are flying at constant speed. The UAV is is supposed to be controllable, through the single control input of rotation rate, $\omega \in [\omega_{min}, \omega_{max}]$. We assume also that the intruder can change his behavior in a unexpected manner: he can change his route with the rotation rate $v \in [v_{min}, v_{max}]$. The collision zone around the UAV is defined as :

$$C = B_2(0, R) \times [0, 2\pi]$$

where $B_2(0, R) \subset \mathbb{R}^2$ is a ball in the (x, y) plan with radius R and centered at 0. Any intrusion in this zone is considered as collision. We can consider this problem in the differential game framework. The first player il the UAV. His goal is to avoid the set C. The second player is the intruder and his goal is to reach the set C. From the UAV's point of view, it can be useful to determine the risk zone containing all the initial relative positions of two aircrafts such that the UAV can't guarantee the avoidance of the collision. This zone is precisely the guarantee capture basin for the second player, the intruder.

2. GENERAL PROBLEM POSITION

In what follows we consider a very general problem formulation that includes both differential game and control problem cases, state constraints, obstacles (eventually moving with known trajectories).

Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty compact subset of $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively. For $t \geq 0$, let $\mathbb{A}_t := \{\alpha : (0, t) \to \mathcal{A}, \text{ measurable}\}$ and $\mathbb{B}_t := \{\alpha : (0, t) \to \mathcal{B}, \text{ measurable}\}$. We consider a dynamics $f : \mathbb{R}^d \times [0, +\infty[\times \mathcal{A} \times \mathcal{B} \to \mathbb{R}^d$ such that

(H1) f is Lipschitz continuous and with linear growth with respect to its first argument.

For every $x \in \mathbb{R}^d$ and $(\alpha, \beta) \in \mathbb{A}_t \times \mathbb{B}_t$, we consider the trajectory $y = y_x^{\alpha,\beta}$ defined as the (absolutely continuous) solution of

$$\dot{y}(s) = f(y(s), s, \alpha(s), \beta(s)), \text{ for a.e. } s \in [0, t], \quad (9a)$$
$$y(0) = x. \quad (9b)$$

Let $(\mathcal{K}_t)_{t \geq 0}$ and $(\mathcal{C}_t)_{t \geq 0}$ be two families of closed sets of $\mathbb{R}^d$. The first one represents the set of "constraints", while the second one, the "target" sets.

We consider a game involving two players, starting at time $t = 0$. The first player wants to steer the system (initially at point x) to the target $\mathcal{C}_t$ in some minimal time $t \geq 0$, and by staying in $\mathcal{K}_t$ (and using her input α), while the second player tries to steer the system away from $\mathcal{C}_t$ or from $\mathcal{K}_t$ (with her input β). More precisely we will say that the trajectory is "admissible on $[0, t]$" if it satisfies the constraints on the time interval $[0, t]$:

$$y_x^{\alpha,\beta}(\theta) \in \mathcal{K}_\theta, \quad \forall \theta \in [0, t].$$

We define the set of non-anticipative strategies for the first player, as follows:

$$\Gamma_t := \left\{ a : \mathbb{B}_t \to \mathbb{A}_t, \ \forall (\beta, \tilde{\beta}) \in \mathbb{B}_t \text{ and } \forall s \in [0, t], \right.$$
$$\left(\beta(\theta) = \tilde{\beta}(\theta) \ a.e. \ \theta \in [0, s] \right) \Rightarrow$$
$$\left. \left(a[\beta](\theta) = a[\tilde{\beta}](\theta) \ a.e. \ \theta \in [0, s] \right) \right\}.$$

Then we are interested to characterize the following capture basin for the first player:

$$\text{Cap}_{\mathcal{C},\mathcal{K}}^f(t) := \left\{ x \in \mathbb{R}^d, \ \exists a \in \Gamma_t, \ \forall \beta \in \mathbb{B}_t, \quad (10) \right.$$
$$\left. \left(y_x^{a[\beta],\beta}(t) \in \mathcal{C}_t, \text{ and } y_x^{a[\beta],\beta}(\theta) \in \mathcal{K}_\theta, \ \forall \theta \in [0, t] \right) \right\}$$

Thus $x \in \text{Cap}_{\mathcal{C},\mathcal{K}}^f(t)$ means that there exists a non anticipative strategy $a \in \Gamma_t$ such that for any adverse strategy $\beta \in \mathbb{B}_t$, we have $y_x^{a[\beta],\beta}(t) \in \mathcal{C}_t$ (we reach the target $\mathcal{C}_t$ at time t).

This setting includes the case of a fixed target $\mathcal{C}_t \equiv \mathcal{C}$ or of a fixed constraint $\mathcal{K}_t \equiv \mathcal{K}$. It also contains the particular case of a one-player game (it suffices to take $B = \{b_0\}$ a fixed value, $\mathbb{B}_t = \{\beta\}$ a fixed constant function and then any $\alpha \in \mathbb{A}_t$ represents an admissible non-anticipative strategy).

We are also interested in computing the minimal time function $\mathcal{T}(x)$ defined by

$$\mathcal{T}(x) := \inf \left\{ t \geq 0, \ \exists a \in \Gamma_t, \forall b \in \mathcal{B}_t, \right.$$
$$y_x^{a[\beta],\beta}(t) \in \mathcal{C}_t \text{ and }$$
$$\left. y_x^{a[\beta],\beta}(\theta) \in \mathcal{K}_\theta, \ \forall \theta \in [0, t] \right\}$$

Since we have $\mathcal{T}(x) = \inf\{t \geq 0, \ x \in \text{Cap}_{\mathcal{C},\mathcal{K}}^f(t)\}$, it is sufficient to characterize the sets $\text{Cap}_{\mathcal{C},\mathcal{K}}^f(t)$.

It is well known that the set $\text{Cap}_{\mathcal{C},\mathcal{K}}^f(t)$ is linked to a control problem. We consider the associated optimal control problem:

$$u(x, t) := \inf_{a \in \Gamma} \max_{\beta \in \mathbb{B}_t} \left\{ \vartheta_0(y_x^{a[\beta],\beta}(t), t), \quad (11) \right.$$
$$\left. \beta \text{ s.t. } y_x^{a[\beta],\beta} \text{ admissible on } [0, t] \right\} \quad (12)$$

(with value $u(x, t) = +\infty$ if for any strategy a there is no admissible trajectory).

One can show (with some additional assumptions later) that the capture basin is related to the negative region of $u(., t)$:

$$\text{Cap}_{\mathcal{C},\mathcal{K}}^f(t) = \{x \in \mathbb{R}^d, u(x, t) \leq 0\}. \quad (13)$$

The characterization of u by means of an HJB equation is not easy because of the state constraints, unless some strong assumptions are satisfied. Our aim in this paper is to give a simple way to characterize the capture basin.

For the unconstrained case, several works have been devoted to the characterization of the value function u as a continuous viscosity solution of a Hamilton-Jacobi equation, see [13, 2]. In presence of state constraints (and when $\mathcal{K}_t \equiv \mathcal{K}$ and is different from $\mathbb{R}^d$), the continuity of this value function is no longer satisfied, unless the dynamics satisfy a special controllability assumption on the boundary of the state constraints. This assumption called "inward pointing qualification (IPQ)" condition was first introduced by [20]. It asks that at each point of $\mathcal{K}$ there exists a field of the system pointing inward $\mathcal{K}$. Clearly this condition ensures the viability of $\mathcal{K}$ (from any initial condition in $\mathcal{K}$, there exists an admissible trajectory which could stay for ever in $\mathcal{K}$). Under the (IPQ) condition, the value function u is the unique continuous *constrained viscosity solution* of a HJB equation with a suitable new boundary condition.

Unfortunately, in many control problems, the (IPQ) condition is not satisfied and the value function u could be discontinuous. In this framework, Frankowska introduced in [14] another controllability assumption, called "outward pointing condition." Under this assumption it is still possible to characterize the value function as the unique lower semi-continuous (l.s.c. for short) solution of an HJB equation.

In absence of any controllability assumption, the function u is discontinuous and its characterization becomes more complicate, see for instance [21, 7] and the references therein. We refer also to [11, 12] for a characterization based on viability theory.

Several papers in the literature deal with the link between reachability and HJB equations. In the case when $\mathcal{K} = \mathbb{R}^d$, we refer to [17] and the references therein. The case when $\mathcal{K}$ is an *open set* in $\mathbb{R}^d$ is investigated in [16]. We also refer to [15] for a short discussion linking the reachability sets under state constraints to HJB equations. The treatment in this reference assumed a C^1 value function.

In a recent work [8], the case $\mathcal{K}_t \equiv \mathcal{K}$, $\mathcal{C}_t \equiv \mathcal{C}$ and with no

time dependency in the dynamics, has been investigated. It was shown that the capture basin $\mathrm{Cap}_{\mathcal{C},\mathcal{K}}^f$ can be characterized by means of a control problem whose value function is continuous (even Lipschitz continuous). Let us recall here the main idea. For simplicity we consider also the one-player game $(f(x,t,a,b) \equiv f(x,a))$. We first consider continuous functions $g : \mathbb{R}^d \to \mathbb{R}$ and $\vartheta_0 : \mathbb{R}^d \to \mathbb{R}$ such that

$$g(x) \leq 0 \Longleftrightarrow x \in \mathcal{K} \quad \text{and} \quad \vartheta_0(x) \leq 0 \Longleftrightarrow x \in \mathcal{C}.$$

Then we introduce the new control problem:

$$\vartheta(x,t) \quad := \quad \inf_{\alpha \in \mathbb{A}_t} \left\{ \max \left(\vartheta_0(y_x^\alpha(t)), \max_{\theta \in [0,t]} g(y_x^\alpha(\theta)) \right) \right\}.$$

It is proved in [8] that the value function ϑ is the unique continuous viscosity solution of the equation:

$$\min \left(\partial_t \vartheta(x,t) + H(x, D_x \vartheta(x,t)), \vartheta(x,t) - g(x) \right) = 0,$$
$$\text{for } t \in [0, +\infty[, \ x \in \mathbb{R}^d, \quad (14a)$$
$$\vartheta(x,0) = \max(\vartheta_0(x), g(x)), \quad (14b)$$

where $H(x,p) := \max_{a \in \mathcal{A}} \left(-f(x,a) \cdot p \right)$, and furthermore we have

$$\mathrm{Cap}_{\mathcal{C},\mathcal{K}}^f(t) = \{x, \ \vartheta(x,t) \leq 0\}.$$

The main feature of (14) is to use a modelization with a supremum cost, in order to handle easily the state constraints and to determine the corresponding capture basins. This idea generalizes the known level-set approach usually used for unconstrained problems. Moreover, the continuous setting opens a large class of numerical schemes to be used for such problems (such as Semi-Lagrangian or finite differences schemes). We refer to [8] for numerical results and comparison of various approaches for state-constrained problems.

We shall now consider the general problem of moving (or time-dependent) targets as well as moving obstacles.

*N*otations. Throughout the paper $|\cdot|$ is a given norm on $\mathbb{R}^d$ (for $d \geq 1$). For any closed set $K \subset \mathbb{R}^d$ and any $x \in K$, we denote by $d(x,K)$ the distance from x to K: $d(x,K) := \inf\{|x - y|, \ y \in K\}$. We shall also denote by $d_K(x)$ the signed distance function to K, i.e., with $d_K(x) := d(x,K)$ for $x \notin K$, and $d_K(x) := -d(x, \mathbb{R}^d \backslash K)$ for $x \in K$.

3. MAIN RESULTS

We assume that

(H3) the set-valued applications $\theta \rightsquigarrow \mathcal{K}_\theta$ and $\theta \rightsquigarrow \mathcal{C}_\theta$ are upper semi-continuous.

We recall that if $(Q_t)_{t \geq 0}$ denotes a family of subsets of $\mathbb{R}^{d+1}$, then the set valued map $t \rightsquigarrow Q_t$ is said to be "upper semi-continuous" if

$$\forall \epsilon > 0, \ \exists \alpha > 0, \ \forall \theta \in [t - \alpha, t + \alpha], \quad Q_\theta \subset Q_t + \epsilon B(0,1).$$

REMARK 1. *For every $t \geq 0$, the set $Cap_{\mathcal{C},\mathcal{K}}^f(t)$ contains the initial positions which can be steered to the target (exactly) at time t. Of course, we can also define the "backward*

reachable set", which is the set of points from which one can reach the target $\mathcal{C}_t$ before time t. This set is also a capture basin for the dynamics $\tilde{f}$ where

$$\tilde{f}(x,t,(a,\lambda),b) := \lambda f(x,t,a,b), \quad \lambda \in [0,1]$$

(see [17, 8]).

We start by embedding the position $y(t)$ and time t into a "space time" space. To do so, we set for every $z = (y,t) \in \mathbb{R}^d \times \mathbb{R}$, the set-valued map $F : \mathbb{R}^{d+1} \times [0, +\infty) \times \mathcal{A} \times \mathcal{B} \rightsquigarrow \mathbb{R}^{d+1}$ such that :

$$F(z,a,b) := \{f(y,t,a,b)\} \times \{1\},$$

and we remark that F satisfies similar Lipschitz continuity and linearity assumptions as in (H1) and (H2). For a given $\xi \in \mathbb{R}^d \times \mathbb{R}$ and $(\alpha,\beta) \in \mathbb{A}_t \times \mathbb{B}_t$, we can then consider $z = z_\xi^{\alpha,\beta}$, the absolutely continuous solution of

$$\dot{z}(s) = F(z(s), \alpha(s), \beta(s)), \text{a.e. } s \in [0,t], \quad z(0) = \xi.$$

(we shall simply denote $z = z_\xi$ if there is no ambiguity). Any solution $z_\xi(s) = (y(s), \eta(s))$ of the previous system satisfies equivalently, if $\xi = (x, t_0)$,

$$\begin{cases} \dot{y}(s) = f(y(s), t_0 + s, \alpha(s), \beta(s)), \ s \geq 0, \quad y(0) = x, \\ \eta(s) = t_0 + s, \quad s \geq 0 \end{cases}$$

Moreover, let also introduce two subsets of $\mathbb{R}^{d+1}$:

$$\mathbb{C} := \bigcup_{t \geq 0} \mathcal{C}_t \times \{t\} \quad \text{and} \quad \mathbb{K} := \bigcup_{t \geq 0} \mathcal{K}_t \times \{t\}.$$

We have the following elementary result :

LEMMA 1. *Under (H3), the sets $\mathbb{C}$ and $\mathbb{K}$ are closed subsets of $\mathbb{R}^{d+1}$.*

Hence there exists Lipschitz continuous functions $\vartheta_0 : \mathbb{R}^{d+1} \to \mathbb{R}$ and $g : \mathbb{R}^{d+1} \to \mathbb{R}$ such that

$$\vartheta_0(\xi) \leq 0 \Leftrightarrow \xi \in \mathbb{C}. \quad (15)$$

and

$$g(\xi) \leq 0 \Leftrightarrow \xi \in \mathbb{K} \quad (16)$$

(for instance we may choose $\vartheta_0(\xi) := d_{\mathbb{C}}(\xi)$ and $g(\xi) := d_{\mathbb{K}}(\xi)$). In particular, for any $t < 0$ we have $\vartheta_0((x,t)) > 0$ and $g((x,t)) > 0$.

We can then define a capture basin associated to the new dynamics F:

$$\mathrm{Cap}_{\mathbb{C},\mathbb{K}}^F(\tau) \quad := \quad \left\{ x \in \mathbb{R}^{d+1}, \ \exists a \in \Gamma_\tau, \ \forall \beta \in \mathbb{B}_\tau, \quad (17) \right.$$
$$\left. \left(z_\xi^{a[\beta],\beta}(\tau) \in \mathbb{C}, \ \text{and } z_\xi^{a[\beta],\beta}(\theta) \in \mathbb{K}, \ \forall \theta \in [0,\tau] \right) \right\}$$

Notice that in the case $\xi = (x,0)$, we have $z_\xi(t) = (y_x(t), t)$ where y_x is a trajectory for the dynamics f. Hence

$$y_x(t) \in \mathcal{C}_t \quad \Leftrightarrow \quad z_\xi(t) \in \mathbb{C},$$

and in the same way,

$$y_x(\theta) \in \mathcal{K}_\theta \quad \Leftrightarrow \quad z_\xi(\theta) \in \mathbb{K}.$$

Therefore we can easily deduce the following result:

PROPOSITION 1. *For all $t \geq 0$, we have*
$$x \in Cap_{\mathcal{C},\mathcal{K}}^{f}(t) \quad \Leftrightarrow \quad (x,0) \in Cap_{\mathbb{C},\mathbb{K}}^{F}(t). \tag{18}$$

Since $Cap_{\mathbb{C},\mathbb{K}}^{F}(s)$ has a fixed state constraint $\mathbb{C}$ and fixed target $\mathbb{K}$ and an autonomous dynamics F, we can use the results of [8].

We consider the control problem, for $\xi \in \mathbb{R}^{d+1}$ and $\tau \geq 0$:
$$\vartheta(\xi, \tau) \quad :=$$
$$\min_{a \in \Gamma_{\tau}} \max_{\beta \in \mathbb{B}_{\tau}} \left\{ \max \left(\vartheta_0(z_{\xi}^{a[\beta],\beta}(\tau)), \max_{\theta \in [0,\tau]} g(z_{\xi}^{a[\beta],\beta}(\theta)) \right) \right\} \tag{19}$$

where we recall that the Lipschitz function g is related to the obstacle $\mathbb{K}$ by (16) (we note that for $t < 0$ and $\tau \geq 0$, we have $\vartheta(x, t, \tau) > 0$).

It is the use of the supremum norm that will enable us to deal with the controllability problem, because now (19) has no "explicit" state constraint. In fact, in this new setting, the term $\max_{\theta \in [0,\tau]} g(z_{\xi}(\theta))$ plays a role of a penalization that a trajectory z_{ξ} would pay if it violates the state-constraints. Theorem 2 will show the advantage of considering (19), because ϑ will be characterized as the unique continuous solution of an HJB equation.

THEOREM 1. *Assume (H1)-(H3). Let ϑ_0 (resp. g) be Lipschitz continuous functions satisfying (15) (resp. (16)). Let ϑ be the value function defined by (19). For every $\tau \geq 0$, we have:*
$$Cap_{\mathbb{C},\mathbb{K}}^{F}(\tau) = \left\{ \xi \in \mathbb{R}^{d+1}, \ \vartheta(\xi, \tau) \leq 0 \right\}.$$

In particular, we have,
$$Cap_{\mathcal{C},\mathcal{K}}^{f}(\tau) = \left\{ x \in \mathbb{R}^{d}, \ \vartheta(x, 0, \tau) \leq 0 \right\}.$$

Now, the function ϑ can be characterized as the unique solution of a Hamilton-Jacobi equation. More precisely, considering the Hamiltonian
$$H^F(\xi, P) := \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} \left(-F(\xi, a, b) \cdot P \right), \tag{20}$$

by using [4] (see also [18, 8]), we have

THEOREM 2. *Assume (H1), and that ϑ_0 and g are Lipschitz continuous. Then ϑ is the unique continuous viscosity solution of the variational inequation (or "obstacle" problem)*
$$\min(\partial_{\tau}\vartheta + H^F(\xi, D_{\xi}\vartheta), \ \vartheta - g(\xi)) = 0,$$
$$\tau > 0, \ \xi \in \mathbb{R}^{d+1}, \quad (21a)$$
$$\vartheta(\xi, 0) = \max(\vartheta_0(\xi), g(\xi)), \quad \xi \in \mathbb{R}^{d+1}. \quad (21b)$$

For sake of completeness the notion of viscosity solution is recalled in the appendix (see Definition 1).

APPLICATION. We are thus able to compute capture basin using regular functions. There are numerous schemes that can approximate the value function ϑ of the previous HJB or HJI equations. This gives a way to compute the set $Cap_{\mathbb{C},\mathbb{K}}^{F}(t)$. Then in view of Theorem 1 we can find the points x that belong to $Cap_{\mathcal{C},\mathcal{K}}^{f}(t)$.

REMARK 2. *From a theoretical point of view, the choice of g is not important, and g can be any Lipschitz function satisfying (16). Of course, the value function ϑ is dependent on g, while the set $\{\xi \in \mathbb{R}^{d+1}, \vartheta(\xi, t) \leq 0\}$ does not depend on g.*

There are other informations in the function ϑ. For instance, let $y_{x,t}^{\alpha,\beta}$ denotes the solution of the differential equation (9a) and such that $y(t) = x$ (instead of $y(0) = x$). For $s \geq t$, we define
$$Cap_{\mathcal{C},\mathcal{K}}^{f}(t;s) := \Big\{ x \in \mathbb{R}^{d}, \ \exists a \in \Gamma, \ \forall \beta \in \mathbb{B}, \tag{22}$$
$$\left(y_{x}^{a[\beta],\beta}(s) \in \mathcal{C}_s, \ \text{and} \ y_{x}^{a[\beta],\beta}(\theta) \in \mathcal{K}_{\theta}, \ \forall \theta \in [t,s] \right) \Big\}$$

Then we have

PROPOSITION 2. *For any $\tau \geq 0$,*
$$Cap_{\mathcal{C},\mathcal{K}}^{f}(t; t+\tau) = \{x \in \mathbb{R}^{d}, \ \vartheta(x, t, \tau) \leq 0\}.$$

Indeed this comes from the fact that $y_{x}^{\alpha,\beta}(s) = y_{x,t}^{\bar{\alpha},\bar{\beta}}(s+t)$ (where $\bar{\alpha}(s) := \alpha(s-t)$ and $\bar{\beta}(s) := \beta(s-t)$), and thus for $\xi = (x, t)$ we can deduce that
$$z_{\xi}(\tau) \in \mathbb{C} \quad \Leftrightarrow \quad y_{x,t}(\tau) \in \mathcal{C}_{\tau}$$
and
$$z_{\xi}(\theta) \in \mathbb{K} \quad \Leftrightarrow \quad y_{x,t}(\theta) \in \mathcal{K}_{\theta}.$$

In other words, $\vartheta(x, t, \tau) \leq 0$ (for some $\tau \geq 0$) is equivalent to say that there exists some non-anticipative strategy $a[\cdot]$ such that (for any adverse strategy β) we can reach the target $\mathcal{C}_{t+\tau}$ at time $t+\tau$ starting from x at time t.

4. NUMERICAL EXAMPLES
An important consequence of the study developed in the last section, is that the reachability analysis can be carried out by solving appropriate Hamilton-Jacobi equations. For these equations, several efficient numerical schemes have been developed and analyzed in the last decades.

Typical schemes are based on Finite Differences (ENO schemes, see [19]), or on Semi-Lagrangian approximations which are general methods used for solving Partial Diffential Equation. Let us also mention the anti-diffusive schemes developped recently to approximate discontinuous solutions of the Hamilton-Jacobi equations. These schemes can have very stable long-time numerical behavior (we refer to [10] or [9] for illustrations and implementation details of such approachs).

In this section, the numerical experiments are performed by using the C++ software *HJB-Ref*, see [6, 5].
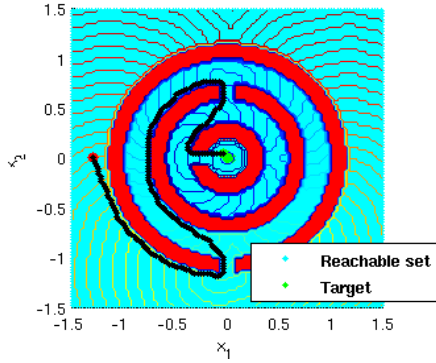
(a) Optimal trajectory

Figure 3: A player (starting point in red) that aims to reach a target inside a labyrinth.

4.1 Path planning

Figure 3 shows a first example of a single player that have to reach the central point inside a simplified labyrinth. In that case the player moves with a direct control θ, which is the angle of its speed, as follows:

$$\begin{cases} x' = v\cos(\theta) \\ y' = v\sin(\theta) \end{cases} \tag{23}$$

We can consider that the speed modulus v is fixed and θ takes any value in the range $[0, 2\pi]$ (we could also consider only a finite number of allowed directions in $[0, 2\pi]$). In Figure 3, the small ball in green centered at 0 represents the target set, red rings correspond to the obstacles that should be avoided, and the contours shown on the graph correspond to the curves of the minimum time function. We show also in black line an optimal trajectory starting from point $(-1.2, 0)$ and that reaches the target in minimum time and without crossing the obstacles.

In the next example, we consider the car navigation problem, defined by (3). We show in Figure 4 an example of an optimal trajectory obtained by our numerical algorithm. Figure 4(a) presents the car at its initial position, the target set (in green) and some obstacles (in red). Remark that the state of the vehicle here is defined by tree components: (x, y) are his coordinates in the plane and θ is the direction of his velocity vector (it is represented by an arrow on the figure). One of the input controls is the angular velocity. In the example on the figure the vehicle's isn't initially oriented in the direction of the target. Then, the optimal trajectory shown on Figure 4(b) takes into account the bounds on the angular velocity. Indeed, it is impossible in this problem to change the direction instantaneously. Therefore, the vehicle must spend some time to maneuver.

4.2 Collision avoidance

We consider now the collision avoidance problem for UAVs presented in section 1.2. In Fig. 5, we show the guarantee capture basin, that is the set containing all initial relative positions (in (x, y, ψ) coordinates) of two aircrafts such that there is a non zero risk of collision. We present two cases:
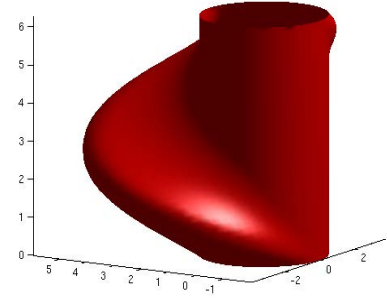
(a) no disturbance ($v = 0$)
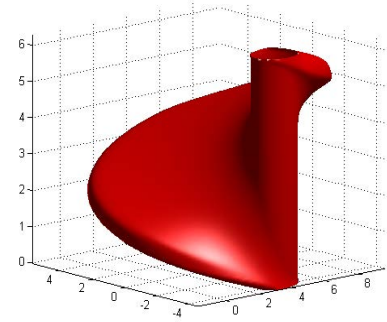
(b) with disturbance v ($v \in [v_{\min}, v_{\max})$

Figure 5: collision avoidance problem

in Figure 5(a), no disturbance is considered, while in in Figure 5(b), some disturbance is allowed.

APPENDIX

A. PROOFS OF THE MAIN RESULTS

For sake of simplicity these proofs are given in the one-player game setting.

Proof of Theorem 1. We reproduce here the idea of the proof that can be found in [8]. Assume that $\xi \in \mathrm{Cap}_{\mathbb{C}, \mathbb{K}}^F(\tau)$. Then there exists an admissible trajectory z_ξ such that

$$\vartheta_0(z_\xi(t)) \leq 0, \text{ and } z_\xi(\theta) \in \mathbb{K} \text{ for every } \theta \in [0, \tau].$$
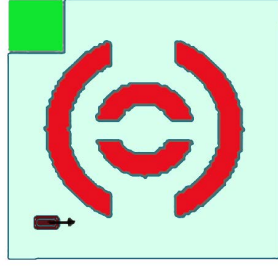
Hence, $\max_{\theta \in [0, \tau]} g(z_\xi(\theta)) \leq 0$, and we have:

$$\vartheta(\xi, \tau) \leq \max(\vartheta_0(z_\xi(\tau)), \max_{\theta \in [0, \tau]} g(z_\xi(\theta))) \leq 0.$$
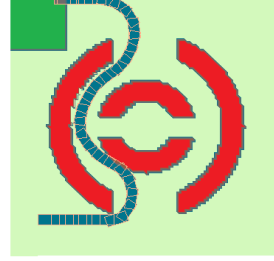
Conversely, assume that $\vartheta(\xi, \tau) \leq 0$. Then there exists a trajectory z_ξ for the dynamics F, such that

$$0 \geq \vartheta(\xi, \tau) = \max(\vartheta_0(z_\xi(\tau)), \max_{\theta \in [0, \tau]} g(z_\xi(\theta))).$$

Thus, for all $\theta \in [0, \tau]$, $g(z_\xi(\theta)) \leq 0$, i.e. $z_\xi(\theta) \in \mathbb{K}$, and z_ξ is an admissible trajectory. Moreover, we have $\vartheta_0(z_\xi(\tau)) \leq 0$, hence $z_\xi(\tau) \in \mathbb{C}$ and we can conclude that $\xi \in \mathrm{Cap}_{\mathbb{C}, \mathbb{K}}^F(\tau)$. □

(a) Initial position   (b) Optimal trajectory

Figure 4: Car navigation problem with obstacles

Proof of Theorem 2. It is based on a dynamic programming principle (DPP) for ϑ that we shall not reproduce here (see for instance [4, Proposition 3.1]).

We recall here the definition of viscosity solution for (14) (the definition in the case of (21) is similar).

DEFINITION 1 (VISCOSITY SOLUTION). *An upper semi-continuous (resp. lower semi-continuous) function $\vartheta : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of* (14) *if $\vartheta(x,0) \leq \vartheta_0(x)$ in $\mathbb{R}^d$ (resp. $\vartheta(x,0) \geq \vartheta_0(x)$) and for any $(x,t) \in \mathbb{R}^d \times (0,\infty)$ and any test function $\phi \in C^1(\mathbb{R}^d \times \mathbb{R}^+)$ such that $\vartheta - \phi$ attains a maximum (resp. a minimum) at the point $(x,t) \in \mathbb{R}^d \times (0,\infty)$, then we have*

$$\min(\partial_t \phi + H(x, \nabla\phi), \; \vartheta - g(x)) \leq 0$$

$$(\text{resp.} \quad \min(\partial_t \phi + H(x, \nabla\phi), \; \vartheta - g(x)) \geq 0) .$$

A continuous function ϑ is a viscosity solution of (14) *if ϑ is a viscosity subsolution and a viscosity supersolution of* (14).

The fact that ϑ is the unique solution of (14) follows from the comparison principle for (14) (which is classical, see for instance [3]), and the fact that the Hamiltonian function H satisfies

$$|H(x_2, p) - H(x_1, p)| \leq C(1 + |p|) \, |x_2 - x_1|,$$
$$|H(x, p_2) - H(x, p_1)| \leq C|p_2 - p_1|,$$

for some constant $C \geq 0$ and for all x_i, p_i, x and p in $\mathbb{R}^d$. $\square$

B. REFERENCES

[1] J.-P. Aubin. *Viability Theory.* Birkhäuser, Boston, 1991.

[2] M. Bardi and I. Cappuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, with appendices by M. Falcone and P. Soravia.* Systems and Control: Foundations and Applications. Birkhäuser, Boston, 1997.

[3] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*, volume 17 of *Mathématiques & Applications (Berlin).* Springer-Verlag, Paris, 1994.

[4] E. N. Barron and H. Ishii. The Bellman equation for minimizing the maximum cost. *Nonlinear Nanalysis T.M.A*, 13:1067–1090, 1989.

[5] O. Bokanowski, N. Forcadel, J.-Y. Zhao, and H. Zidani. Hjb-parallel software, developed jointly with hpc project. *see http://commands.saclay.inria.fr,* 2011.

[6] O. Bokanowski, N. Forcadel, J.-Y. Zhao, and H. Zidani. Hjb-ref software. *see http://commands.saclay.inria.fr,* 2011.

[7] O. Bokanowski, N. Forcadel, and H. Zidani. Deterministic state constrained optimal control problems without controllability assumptions. *To appear in ESAIM: COCV*, page DOI: 10.1051/cocv/2010030, 2010.

[8] O. Bokanowski, N. Forcadel, and H. Zidani. Reachability and minimal times for state constrained nonlinear problems without any controllability assumption. *SIAM J. Control Optim.,* 48(7):4292–4316, 2010.

[9] O. Bokanowski, S. Martin, R. Munos, and H. Zidani. An anti-diffusive scheme for viability problems. *Applied Numerical Mathematics*, 56(Issue 9, in Numerical Methods for Viscosity Solutions and Applications):1135–1254, 2006.

[10] O. Bokanowski and H. Zidani. Anti-diffusive schemes for linear advection and application to Hamilton-Jacobi-Bellman equations. *J. Sci. Computing*, 30(1):1–33, 2007.

[11] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre. Optimal times for constrained nonlinear control problems without local controllability. *Appl. Math. Optim*, 36:21–42, 1997.

[12] P. Cardaliaguet, M. Quincampoix, and P. Saint-Pierre. Pursuit differential games with state constraints. *SIAM Journal on Control and Optimization*, 39(5):1615–1632, 2000.

[13] M. Falcone, T. Giorgi, and P. Loreti. Level sets of viscosity solutions : some applications to fronts and rendez-vous problems. *SIAM J. Applied Mathematics*, 54(5):1335–1354, 1994.

[14] H. Frankowska and S. Plaskacz. Semicontinuous solutions of Hamilton-Jacobi-Bellman equations with

degenerate state constraints. *J. Math. Anal. Appl.*, 251(2):818–838, 2000.

[15] A. Kurzhanski and P. Varaiya. Ellipsoidal techniques for reachability under state constraints. *SIAM J. Control Optim.*, 45:1369–1394, 2006.

[16] J. Lygeros. On reachability and minimum cost optimal control. *Automatica*, 40:917–927, 2004.

[17] I. Mitchell, A. Bayen, and C. Tomlin. A time-dependent Hamiliton-Jacobi formulation of reachable sets for continuous dynamic games. *IEEE Transactions on automatic control*, 50(7):947–957, 2005.

[18] M. Quincampoix and O. Serea. A viability approach for optimal control with infimum cost. *Annals. Stiint. Univ. Al. I. Cuza Iasi,s.I a, Mat*, XLVIII:113–132, 2002.

[19] C.-W. Shu. High order numerical methods for time dependent Hamilton-Jacobi equations. *in Mathematics and Computation in Imaging Science and Information Processing, S.S. Goh, A. Ron and Z. Shen, Editors, Lecture Notes Series*, 11:47–91., 2007.

[20] H. M. Soner. Optimal control with state-space constraint. I. *SIAM J. Control Optim.*, 24(3):552–561, 1986.

[21] P. Soravia. Optimality principles and representation formulas for viscosity solutions of Hamilton-Jacobi equations.ii. equations of control problems with state constraints. *Diff. and Int. Equations*, 12(2):275–293, 1999.