Signal-to-Noise Ratio Penalties for Continuous-Time Phase Noise Channels

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***Abstract*—Signal-to-noise ratio (SNR) penalties are studied for continuous-time additive white Gaussian noise channels with white and Wiener phase noise. For white phase noise, recent results showing an unavoidable SNR penalty are extended to include uncorrelated phase-noise samples. For Wiener phase noise, bounds on the SNR penalty are developed for integrate-and-dump receivers that have limited time resolution.**

I. Introduction

Phase noise occurs in most communication systems and its impact is often minor. However, if symbols are long or are processed by filters, e.g., when using orthogonal frequency division multiplexing (OFDM) or other long equalizers, then phase noise can become a serious impairment. In fact, recent results [1], [2] for Wiener phase noise and integrate-and-dump receivers show that even weak phase noise becomes the limiting factor at high signal-to-noise ratio (SNR). Related results [3], [4] for white phase noise show that even the best projection receiver suffers an SNR penalty that is interpreted as a *spectral loss*. The motivation for considering white phase noise is that high-power signals in optical fiber can expand signal bandwidth much beyond that of the receive filters [5].

The goal of this document is to develop further results on phase noise channels. We first consider the white phase noise models studied in [3], [4]. We show that the SNR penalty (or spectral loss) extends to uncorrelated, rather than independent, phase noise samples. We next consider whether Wiener phase noise causes similar SNR penalties. We study the integrate-and-dump receivers used in [1], [2] and develop simple bounds on an SNR penalty. The paper is organized as follows. The system models are introduced in Sec. II, and the SNR penalties are derived and discussed in Sec. III and IV.

II. Phase Noise Models

The output of a continuous-time additive white Gaussian noise (AWGN) channel with phase noise can be written as

$$Y(t) = X(t)e^{j\Theta(t)} + W(t), \qquad 0 \le t \le T \tag{1}$$

where $j = \sqrt{-1}$, $X(t)$ is the input waveform, $\Theta(t)$ is a phase noise process, and $W(t)$ is a complex-valued circularly symmetric white Gaussian noise process with two-sided power spectral density $N_0/2$. We consider two types of phase noise, namely white and Wiener phase noise. We describe these processes in more detail below.

A. Signals and Signal Space

Suppose $X(t)$ is in the set $\mathcal{L}^2[0,T]$ of finite-energy signals in the interval $[0,T]$. Let $\{\phi_m(t)\}_{m=1}^{\infty}$ be an orthonormal basis of $\mathcal{L}^2[0,T]$. We may write

$$X(t) = \sum_{m=1}^{\infty} X_m\, \phi_m(t), \quad W(t) = \sum_{m=1}^{\infty} W_m\, \phi_m(t) \tag{2}$$

where

$$X_m = \langle X(t), \phi_m(t) \rangle = \int_0^T X(t)\, \phi_m(t)^\star \, \mathrm{d}t \tag{3}$$

$x^\star$ is the complex conjugate of x, and the $\{W_m\}_{m=1}^{\infty}$ are independent and identically distributed (i.i.d.), complex-valued, circularly symmetric, Gaussian random variables with zero mean and variance N_0.

B. White Phase Noise

White phase noise was defined in [4] in terms of a generalized stationary random process with mean $\mu_\Theta = \mathsf{E}\left[e^{j\Theta(t)}\right]$ for all $0 \le t \le T$. The definition of the process is based on projecting $\phi_n(t)e^{j\Theta(t)}$ onto $\phi_m(t)$ to obtain the variables

$$\begin{aligned}\Phi_{n,m} &= \left\langle \phi_n(t)e^{j\Theta(t)}, \phi_m(t) \right\rangle \\ &\triangleq \lim_{l\to\infty} \frac{1}{l} \sum_{i=1}^{l} \phi_n\left(t_i^{(l)}\right) e^{j\Theta\left(t_i^{(l)}\right)} \phi_m\left(t_i^{(l)}\right)^\star \end{aligned} \tag{4}$$

where $t_i^{(l)} = (i-1)T/l$.

C. Wiener Phase Noise

Wiener phase noise is the process

$$\Theta(t) = \Theta(0) + \gamma B(t), \qquad 0 \le t \le T \tag{5}$$

where γ is a positive constant and $B(\cdot)$ is a standard Brownian process, i.e., a process characterized by the properties:

- $B(0) = 0$,
- for any $t > s \ge 0$, $B(t) - B(s) \sim \mathcal{N}(0, t-s)$ is independent of the sigma algebra generated by $\{B(u) : u \le s\}$,
- B has continuous sample paths.

One can think of $\Theta(t)$ as an accumulation of white noise:

$$\Theta(t) = \Theta(0) + \gamma \int_0^t N(t)\, \mathrm{d}t, \quad 0 \le t \le T \tag{6}$$

where $N(t)$ is a standard white Gaussian noise process.

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III. SNR Penalty For White Phase Noise

The paper [4] shows that white phase noise causes the projection of $Y(t)$ onto $\phi_m(t)$ to take on the form

$$Y_m = \mu_\Theta X_m + W_m. \tag{7}$$

The channel is thus statistically equivalent to an AWGN channel with an SNR penalty $|\mu_\Theta|^2$ where $|\mu_\Theta|^2 \le 1$. The proof in [4] requires that the sampled values of the process $e^{j\Theta(t)}$ in (4) are *statistically independent*. For example, this would be the case if the sampled phase values $\Theta(t_i)$ are statistically independent. The purpose of this section is to extend these results to *uncorrelated* sampled values of $e^{j\Theta(t)}$.

The projection of the channel output onto $\phi_m(t)$ is

$$\begin{aligned} Y_m &= \left\langle X(t)e^{j\Theta(t)}, \phi_m(t) \right\rangle + W_m \\ &= \left\langle \sum_{n=1}^{\infty} X_n \phi_n(t) e^{j\Theta(t)}, \phi_m(t) \right\rangle + W_m \\ &= \left[\sum_{n=1}^{\infty} X_n \Phi_{n,m} \right] + W_m. \end{aligned} \tag{8}$$

Let $1(\cdot)$ be the indicator function that takes on the value 1 if its argument is true and is zero otherwise. The following Lemma characterizes the distribution of $\Phi_{n,m}$.

Lemma. *If* $|\phi_n(t)\phi_m(t)^*|$ *is uniformly bounded in* t*, then* $\Phi_{n,m}$ *converges almost surely to*

$$\Phi_{n,m} \overset{a.s.}{=} \mu_\Theta \int_0^T \phi_n(t)\phi_m(t)^\star \, \mathrm{d}t = \mu_\Theta \cdot 1(n=m) \tag{9}$$

Proof. We write the proof for the real part of $\Phi_{n,m}$; the procedure for the imaginary part is analogous. For a fixed l and $i = 1, \ldots, l$, define the uncorrelated random variables

$$Z_i^{(l)} = \Re\left\{ \phi_n\left(t_i^{(l)}\right) \phi_m\left(t_i^{(l)}\right)^\star e^{j\Theta\left(t_i^{(l)}\right)} \right\} \tag{10}$$

where $\Re\{\cdot\}$ denotes the real part of a complex number. Consider the sequence $\{S_l\} = S_1, S_2, S_3, \ldots$ of partial sums

$$S_l = \sum_{i=1}^{l} Z_i^{(l)} \tag{11}$$

and define $\mu_\Re^{(l)} = \mathsf{E}[S_l]/l$ and $\mu_\Re = \Re\{\mu_\Theta\}$. We compute

$$\begin{aligned} \lim_{l\to\infty} \mu_\Re^{(l)} &= \lim_{l\to\infty} \frac{1}{l} \sum_{i=1}^{l} \mathsf{E}\left[Z_i^{(l)} \right] \\ &= \lim_{l\to\infty} \Re\left\{ \mu_\Theta \frac{1}{l} \sum_{i=1}^{l} \phi_n\left(t_i^{(l)}\right) \phi_m\left(t_i^{(l)}\right)^\star \right\} \\ &= \Re\left\{ \mu_\Theta \int_0^T \phi_n(t)\phi_m(t)^\star \, \mathrm{d}t \right\} \\ &= \mu_\Re \cdot 1(n=m). \end{aligned} \tag{12}$$

We show that the subsequence $\{S_{q^2}/q^2\}$ where $q = 1, 2, 3, \ldots$, converges almost surely to $\mu_\Re \cdot 1(n=m)$. Since the $Z_i^{(l)}$ are bounded, we can invoke Chebyshev's inequality to write

$$\begin{aligned} p_q &= \Pr\left[\frac{1}{q^2} \left| S_{q^2} - q^2 \mu_\Re^{(q^2)} \right| > \varepsilon \right] \\ &\le \frac{\mathsf{Var}\left[S_{q^2}\right]}{q^4 \varepsilon^2} \\ &\overset{(a)}{=} \frac{\sum_{i=1}^{q^2} \mathsf{Var}\left[Z_i^{(q^2)} \right]}{q^4 \varepsilon^2} \\ &\le \frac{\sup_{i\in\{1,\ldots,q^2\}} \mathsf{Var}\left[Z_i^{(q^2)} \right]}{q^2 \varepsilon^2} \end{aligned} \tag{13}$$

where (a) follows because the $Z_i^{(l)}$ are uncorrelated. From (13), we have $\sum_{q=1}^{\infty} p_q < \infty$ and therefore

$$\Pr\left[\{ |S_{q^2}/q^2 - \mu_\Re^{(q^2)}| > \varepsilon \} \text{ infinitely often } \right] = 0 \tag{14}$$

by the Borel-Cantelli lemma. This identity is valid for all $\varepsilon > 0$, therefore we have

$$\lim_{q\to\infty} S_{q^2}/q^2 \overset{a.s.}{=} \lim_{q\to\infty} \mu_\Re^{(q^2)} = \mu_\Re \cdot 1(n=m) \tag{15}$$

where the last step follows by (12).

Finally, we show convergence for l with $q^2 \le l \le (q+1)^2$. Suppose the boundedness condition is $|\phi_n(t)\phi_m(t)^*| \le B$ for some positive B. The sequence $\{S_l + lB\}$ is therefore non-decreasing and we have

$$S_{q^2} + q^2 B \le S_l + lB \le S_{(q+1)^2} + (q+1)^2 B. \tag{16}$$

Dividing by l and using $q^2 \le l \le (q+1)^2$ we find that $\{S_l/l\}$ converges almost surely $\mu_\Re \cdot 1(n=m)$. Repeating the steps for the imaginary part of $\Phi_{n,m}$, we obtain (9). □

Using the Lemma we arrive at the model (7). We make a few remarks. First, the boundedness condition required by the Lemma is mild and can be easily met. Second, the proof of the Lemma requires that the phase noise samples $e^{j\Theta(t)}$ are uncorrelated. As pointed out in [4], this condition is not unrealistic, as it is met when phase noise samples $\{\Theta(t)\}$ are uncorrelated and Gaussian, and thus independent. This can be the case, e.g., with phase noise generated by cross-phase modulation caused by many neighboring channels in a multichannel fiber-optic communication system [3], [5]. Finally, one may wish to choose the transmitter and receiver bases signals differently. One can then use similar steps as above to show that the SNR penalty does not change.

IV. SNR Penalty For Wiener Phase Noise

The SNR penalty for white phase noise motivates asking whether Wiener phase noise also exhibits such behavior, and whether the answer gives insight into this more difficult model. We show that an SNR penalty indeed exists, although our results are limited. One limitation is that we consider only a particular type of receiver. Another is that we are able to provide only bounds on the penalty. The bounds seem poor for weak (small γ) phase noise, but they show the general principle. For strong (large γ) phase noise, the bounds give a penalty that we relate to the white phase noise penalty.

Consider the same setup as in Sec. II-A. Consider the M transmitter symbols $X^M = X_1X_2 \dots X_M$. We project $Y(t)$ onto the corresponding M basis functions to obtain Y^M, and the mutual information

$$I(X^M; Y^M) = h(Y^M) - h(Y^M|X^M) \tag{17}$$

bounds the rate of reliable communication. Here $h(A)$ and $h(A|B)$ are the differential entropies of the continuous random variable A when not conditioned and conditioned on the random variable B, respectively. Standard information-theoretic arguments give

$$h(Y^M) \le \sum_{m=1}^{M} h(Y_m) \le \sum_{m=1}^{M} \log(\pi e \,\mathsf{Var}\,[Y_m]) \tag{18}$$

$$h(Y^M|X^M) \ge h(W^M) = M \log(\pi e N_0) \tag{19}$$

where $\mathsf{Var}\,[Y_m]$ is the variance of Y_m.

We proceed to bound $\mathsf{Var}\,[Y_m]$ to determine an SNR penalty. Consider X_m with zero mean and variance $E_m = \mathsf{E}\left[|X_m|^2\right]$ and compute

$$\mathsf{Var}\,[Y_m] = N_0 + \mathsf{E}\left[\left|\int_0^T \left(\sum_{n=1}^{M} X_n \phi_n(t)\right) e^{j\Theta(t)} \phi_m(t)\,\mathrm{d}t\right|^2\right]$$

$$= N_0 + \int_0^T \int_0^T \left(\sum_{n=1}^{M} \sum_{n'=1}^{M} \mathsf{E}\left[X_n X_{n'}^*\right] \phi_n(t) \phi_{n'}(\tau)^\star\right)$$

$$\phi_m(t)\phi_m(\tau)^\star \mathsf{E}\left[e^{j(\Theta(t)-\Theta(\tau))}\right] \mathrm{d}\tau\,\mathrm{d}t. \tag{20}$$

The expectation simplifies to (see [3, App. A.B])

$$\mathsf{E}\left[e^{j(\Theta(t)-\Theta(\tau))}\right] = e^{-(\gamma^2/2)|t-\tau|}. \tag{21}$$

The next step seems difficult without further assumptions. Perhaps the simplest approach is the following. Consider a receiver whose time resolution is limited to T_s seconds in the sense that every projection must include at least a T_s-second interval. More precisely, set $T = MT_s$ and consider an integrate-and-dump receiver with

$$\phi_m(t) = \begin{cases} 1/\sqrt{T_s}, & t \in [(m-1)T_s, mT_s) \\ 0, & \text{else.} \end{cases} \tag{22}$$

The simplification (22) gives

$$\mathsf{Var}\,[Y_m] = N_0 + \frac{E_m}{T_s^2} \int_0^{T_s} \int_0^{T_s} e^{-(\gamma^2/2)|t-\tau|}\,\mathrm{d}\tau\,\mathrm{d}t$$

$$= N_0 + E_m \cdot \frac{4}{\gamma^2 T_s}\left[1 - \frac{2}{\gamma^2 T_s}\left(1 - e^{-\gamma^2 T_s/2}\right)\right]. \tag{23}$$

We remark that this analysis permits dependent X_m, i.e., it includes oversampling. We also remark that one could think of performing the same type of analysis in the frequency domain. In this case, one would have a band-limited rather than time-limited receiver.

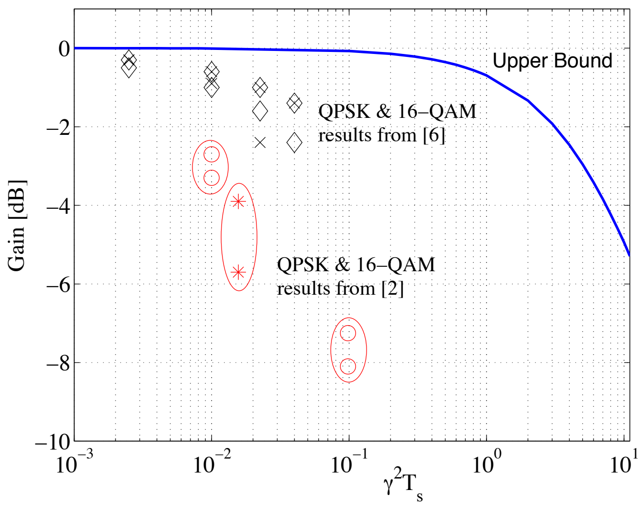

Fig. 1. SNR gains from [2], [6] and upper bound based on (23). Crosses and asterisks are for QPSK; diamonds and circles are for 16-QAM.

Suppose the phase noise is weak or T_s is small, i.e., $\gamma^2 T_s$ is small. We use (23) to compute

$$\mathsf{Var}\,[Y_m] \approx N_0 + E_m \left(1 - \frac{\gamma^2 T_s}{6}\right). \tag{24}$$

For example, with $\gamma^2 T_s = 0.25$ we find that the multiplicative SNR penalty is less than 0.96; alternatively the SNR penalty is at least $-10\log_{10}(0.96) \approx 0.18$ dB. But for many oscillators this level of phase noise is considered rather strong (see [2, Sec. V.C]). On the other hand, for oscillators that exhibit very strong phase noise or for receivers with very large T_s, e.g., long OFDM symbols or long filters, the value (23) becomes

$$\mathsf{Var}\,[Y_m] \approx N_0 + E_m \cdot \frac{4}{\gamma^2 T_s}. \tag{25}$$

The SNR penalty is thus at least $10\log_{10}(\gamma^2 T_s/4)$ dB. We find that we may decrease this lower bound on the penalty (or the penalty itself) by making T_s small, e.g., by using an oversampling receiver [1], [2]. Finally, note that Wiener phase noise becomes "like" white phase noise as γ increases, and we have $\mu_\Theta = \mathsf{E}\left[e^{j\Theta(t)}\right] \to 0$ as $\gamma \to \infty$.

Figure 1 compares the bound based on (23) to penalties given in the literature. The six circled points are taken from [2, Fig. 3 and 6]; they are for continuous-time Wiener phase noise channels with rectangular pulses and an oversampling factor of 16. The points at $\gamma^2 T_s \approx 0.016$ are for QPSK at an SNR of 0 dB and 10 dB; the points at $\gamma^2 T_s = 0.01$ and $\gamma^2 T_s = 0.1$ are for 16-QAM at an SNR of 0 dB and 10 dB. The upper bound is clearly loose but it is valid for any modulation. The other points (black crosses and diamonds) are taken from [6, Fig. 3] and are for QPSK and 16-QAM at an SNR of 0 dB and 10 dB. However, these points are for discrete-time Wiener phase noise channels and are therefore not directly comparable to the bound (23) or the results of [2].

V. Conclusion

SNR penalties were studied for white and Wiener phase-noise channels. The penalties for white phase noise are exact, while those for Wiener phase noise are loose. It is interesting to consider how one might improve the latter bounds to account not only for power loss due to filtering, but also for other mechanisms of mutual information loss.

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