Outage Probability for MIMO MAC Channels

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***Abstract*—In this paper, we investigate the outage probability for the mutual information of the sum rate for the MIMO (multiple input multiple output) MAC (multiple access channel) channel. We derive exact results for the "two users" case with two antennas and a general result for K users with an arbitrary number of antennas and different signal-to-noise ratios. As a secondary result, we derive an exact expression for the single user outage probability for the MIMO channel with two antennas.**

***Index Terms*—MIMO, mutual information, outage probability, Wishart matrices**

I. Introduction

Wireless systems have uninterrupted interest for higher data rates even after all of the evolution that has happened in the telecommunications area during the last ten years. A very well-known triad for the design of wireless devices keeps the focus on power, bandwidth, and complexity. Pioneering work by [1], Foschini [2], and Telatar [3] led to a huge interest in the research community by predicting very high spectral efficiencies for wireless systems that fed an enthusiasm by researchers to pursue more and more throughput. From this motivation, this work studies the outage probability for the mutual information of the sum rate for the MIMO MAC channel.

We focus on MIMO channel capacity in the sense of the Shannon theory. The Shannon capacity of a single-user time-invariant channel is defined as the maximum mutual information between the channel input and output. This maximum mutual information is shown by Shannon's capacity theorem to be the maximum data rate that can be transmitted over the channel with an arbitrarily small error probability. The capacity definitions, intended for this work, deal with channels that are minimum-rate capacity. These capacities require a fixed data rate in all nonoutage channel states.

It is also important to consider the transmission strategy. In this case we assume a strategy based on the channel distribution instead of the instantaneous channel state. The channel coefficients are typically assumed to be jointly Gaussian, so the channel distribution is specified by the channel mean and covariance matrices. In scenarios were the users do not move fast, the slow-fading dominates the channel. In this case, the time duration in which the channel behaves in a correlated way is long compared with the time duration of a transmission symbol [4]. Thus, one can expect the channel state to virtually remain unchanged during the time in which a symbol is transmitted (block fading). The primary degradation in a slow-fading channel is a loss in signal-to-noise ratio (SNR). Because the receiver needs at least a sufficient signal to noise ratio (SNR) level in order to correctly decode the transmitted signal, it is of extreme importance to estimate the probability of the received signal which will be below an arbitrary threshold. In other words, for each realization of the random MIMO channel, there is an associated mutual information for the sum rate, denoted here by ($\mathcal{I}_{MAC}$), between transmitted and received signals. The outage probability P_{out} is the probability that conditional $\mathcal{I}_{MAC}$ is less than a given rate R. Therefore, P_{out} is a function of R.

The MIMO outage probability is still an open subject due to the intricate nature of the problem. There are some results for the outage probability in the high SNR regime [5], but unfortunately they are not all valid for the SNR region.

In this paper, we wish to exploit the Wishart matrix properties and calculate the outage probability for the sum rate of the MIMO MAC channel. According to reference [6], it is possible to derive analytical expressions for P_{out} in terms of the moment generating function (MGF) of the random variable $\mathcal{I}$ for single user case. In [5], it is shown how to compute the joint probability density for the eigenvalues of a Wishart matrix. In [6], it is shown that the PDF of $\mathcal{I}$ can be well approximated by a Gaussian distribution, where the mean of ($\mu_{\mathcal{I}}$) and variance of ($\sigma^2_{\mathcal{I}}$) are calculated by using the probability density distribution of these eigenvalues.

The extension of this idea can be applied for the multi-user case. First, we develop an analytical formulation for a generic case with (M, N) antennas. In the MAC case the sum rate random variable $\mathcal{I}_{MAC}$ is a function of the sum of Wishart matrices and some properties of Wishart matrix can be exploited. It is possible to find an exact analytical expression for the outage probability for the two-user 2x2 MIMO case when the users have the same SNR as described in the section II. For other combinations of (M, N) antennas, we have to use the Gaussian approximation approach because of the complexity associated with the number of integrals and other aspects related with the analytic expressions that makes the solution very intricate.

The contributions of this paper are: 1) a new exact expression for the MIMO MAC sum rate for two user with two antennas per user; 2) the distribution of $a_1W_1 + a_2W_2$, where W_1 and W_2 are Wishart matrices is known to be Wishart distributed only for the case $a_1 = a_2$, in this paper we propose an excellent approximation for arbitrary a_1 and a_2; 3) We extend the result of the Gaussian approximation for the mutual information for a single user MIMO case to the mutual information for the sum rate of the MAC case.

To the best of our knowledge, all the results mentioned above are not known in the literature.

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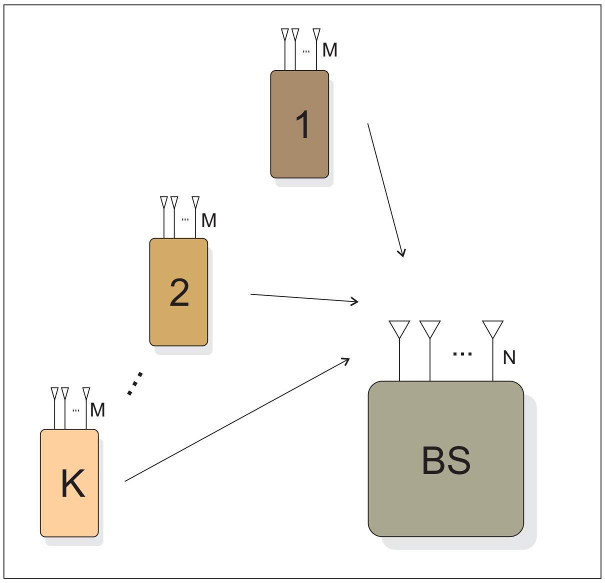

Fig. 1. MIMO MAC channel model

II. System Model

Consider a general MIMO MAC channel described by Fig. 1. In this case, we consider K users with (M x 1) antennas, one base station with (N x 1) antennas. The received signal at uplink node can be written as

$$\mathbf{Y}_d = \sum_{n=1}^{K} \sqrt{\eta_n}\mathbf{H}_n\mathbf{X}_n + \mathbf{Z}_d \tag{1}$$

where

- $\mathbf{X}_n$ are vectors (M_n x 1) of transmitted signals from the user nodes; the power constraints on the transmit signals are $\mathbb{E}$[1]$[\mathbf{X}_n^\dagger\mathbf{X}_n] \leq M_n$;
- $\mathbf{Y}_d$ is a vector (N x 1) of received signals at the destination node.
- $\mathbf{H}_n$ are matrices of channel gains (M_n x N). We consider the scenario where $\mathbf{H}_n$ are random and independent matrices, and the entries of each matrix are independent and identically distributed (i.i.d.) complex Gaussian variables with zero-mean, independent real and imaginary parts, each with variance σ^2, and they are available at the receiver node only (i.e. receiver CSI only).
- η_n are parameters related to the SNR [7]

  $$\eta_n = \frac{\mathrm{SNR}_n}{M_n} \tag{2}$$

  where SNR_n are the normalized power ratios of the $\mathbf{X}_n$ to the noise at each antenna of the destination node;
- $\mathbf{Z}_d$ is an independent (N x 1) circularly symmetric complex Gaussian noise vector with distribution $\mathcal{CN}(0, \mathbf{I}_N)$ and uncorrelated to $\mathbf{X}_n$.

Corollary 1: Define

$$\mathbf{W}_l = \begin{cases} \mathbf{H}_l^\dagger\mathbf{H}_l & N < M_l \\ \mathbf{H}_l\mathbf{H}_l^\dagger & N \geq M_l \end{cases} \tag{3}$$

where l is the index that identifies the user and the symbol $\dagger$ denotes the transposed conjugated matrix operator. Thus $\mathbf{W}_l$ is Wishart distributed according to

$$\mathbf{W}_l \sim \mathcal{W}_m(p, \Sigma_l). \tag{4}$$

where $p = \min(M, N)$ and Σ_l is the covariance matrix for each user.

III. Sum Rate Outage Probability

The mutual information for the sum rate for MIMO MAC case is given by [3]

$$\mathcal{I}_{MAC} = \log_2\left[\det\left(\mathbf{I} + \sum_{n=1}^{K}\eta_n\mathbf{W}_n\right)\right] \tag{5}$$

where $\mathbf{I}$, is the identity matrix. The MIMO sum rate outage probability for the $\mathcal{I}_{MAC}$ is defined as

$$\mathrm{P}_{out}(R) := \Pr\left[\mathcal{I}_{MAC} < R\right], \tag{6}$$

In this paper for the sake of complexity, we assume that all the users are equipped with the same number of antennas, i.e., $M_n = M$ for all n.

The joint probability density function of the complex Wishart $\mathbf{W}$, i.e. the multivariate density function of the real random variables $\mathbf{W} = W_{11}, \ldots, W_{mm}$, $\mathrm{Re}[W_{12}]$, $\mathrm{Im}[W_{12}], \ldots$, $\mathrm{Re}[W_{m-1,m}]$, $\mathrm{Im}[W_{m-1,m}]$ is defined by [8]

$$\mathrm{p}_W(\mathbf{W}) = \frac{\det(\mathbf{W})^{p-m}\exp(-\mathrm{tr}(\mathbf{\Sigma^{-1}W}))}{\pi^{\frac{m(m-1)}{2}}\det(\mathbf{\Sigma})^p\prod_{k=1}^{m}\mathbf{\Gamma}(\mathrm{p}-\mathrm{k}+1)} \tag{7}$$

where

$$\mathbf{\Gamma}(\mathrm{z}) = \int_0^\infty t^{z-1}e^t\,dt \tag{8}$$

is the Gamma function. As we can see from (7), the $\mathrm{p}_W(\mathbf{W})$ is a function of the $\mathbf{W}$ and $\mathbf{\Sigma}$. Since we are assuming the Rayleigh case channel, then $\mathbb{E}[\mathbf{H}_l] = 0$, which refers to the central Wishart case [9], [10].

Corollary 2: If $\mathbf{W_1}$ and $\mathbf{W_2}$ are distributed as $\mathbf{W} = \mathcal{W}_m(p_1, \Sigma)$ and $\mathbf{W} = \mathcal{W}_m(p_2, \Sigma)$, respectively, therefore from [11] the sum of $\mathbf{W_1}$ and $\mathbf{W_2}$ is distributed as $\mathbf{W_{1+2}} \sim \mathcal{W}_m(p_1 + p_2, \Sigma)$.

In the same way, the unordered joint density eigenvalue distribution is given by [12]

$$p_\lambda(\lambda_1, ..., \lambda_m) = \frac{1}{m!K_{m,n}}\prod_i e^{-\lambda_i}\lambda_i^{n-m}\prod_{i<j}(\lambda_i - \lambda_j)^2 \tag{9}$$

where $K_{m,n}$ is a normalizing factor.

IV. The two users case

For the two user case, (5) specializes to

$$\mathcal{I}_{MAC} = \log_2\left[\det\left(\mathbf{I}_N + \eta_1\mathbf{W}_1 + \eta_2\mathbf{W}_2\right)\right]. \tag{10}$$

When $\eta_1 = \eta_2$, then Corollary 2 can be applied. Using the property of the determinant, it is possible to write

$$\mathcal{I}_{MAC} = \log_2\left((1 + \eta\lambda_1)(1 + \eta\lambda_2)\right) \tag{11}$$

[1]$\mathbb{E}[\cdot]$ denotes the expectation operator.

where $\eta = \eta_1 = \eta_2$ and λ_1, λ_2 are the eigenvalues of the sum matrix $\mathbf{W}_1 + \mathbf{W_2}$. The sum rate outage probability can be found as

$$\Pr\left(\mathcal{I}_{MAC} < R\right) = \Pr\left(\left(1+\eta\lambda_1\right)\left(1+\eta\lambda_2\right) < 2^R\right)$$

$$\Pr\left(\mathcal{I}_{MAC} < R\right) = \int_0^{2^R}\int_0^{\frac{2^R}{1+\lambda_2}-1} p\left(\lambda_1,\lambda_2\right) d\lambda_1 d\lambda_2 \quad (12)$$

For the sake of simplicity, the value of η can be incorporated in the parameter σ of the function $p(\lambda_1, \lambda_2)$, (13).

Of course, it is clear that the joint probability density function of the eigenvalues is necessary in order to compute the outage probability.

For the case where $M = N = 2$, $\mathbf{W}_1 + \mathbf{W}_2$ is a Wishart matrix with four degrees of freedom and the joint eigenvalue distribution can be found as

$$p(\lambda_1,\lambda_2) = \frac{\lambda_1^2\left(\lambda_1-\lambda_2\right)^2\lambda_2^2 e^{-\frac{\lambda_1+\lambda_2}{\sigma^2}}}{24\sigma^{16}}, \quad (13)$$

where $\sigma = \sqrt{\eta}$.

Now, using (13) in (12), the outage probability can be found in an exact manner as given in (15).

A. Single User MIMO case with $M = N = 2$

The outage probability for a single MIMO user can also be calculated for the case $M = N = 2$, following the same rationale. In this case, the eigenvalue joint probability function for single user case with $M = N = 2$ can be written as,

$$p(\lambda_1,\lambda_2) = \frac{(\lambda 1-\lambda_2)^2 e^{-\frac{\lambda 1+\lambda_2}{\sigma^2}}}{2\sigma^8} \quad (14)$$

Now, using the joint eigenvalue distribution obtained in (14), it is possible to compute the exact outage probability for a single user mutual information, I_{SU}, given as the equation (16).

V. Gaussian Approximation for same η and arbitrary M,N

The sum rate outage probability for the MIMO MAC case for an arbitrary number of antennas is still an open problem due to its complexity. In order to circumvent this difficulty and based on the result for a single user MIMO channel [6], we propose to approximate the sum rate mutual information, $\mathcal{I}_{MAC}$, as a Gaussian random variate

$$p(\mathcal{I}_{MAC}, M, N) \approx \frac{1}{\sqrt{2\pi}\sigma_{I_{MAC}}}\exp\left(-\frac{(I-\mu_{I_{MAC}})^2}{2\sigma^2_{I_{MAC}}}\right) \quad (17)$$

where $\mu_{I_{MAC}}$ and $\sigma^2_{I_{MAC}}$ are the mean and variance of Gaussian variable and can written as

$$\mu_{I_{MAC}} = \int_0^\infty log(1+\tilde{\lambda}\rho/M)K(\tilde{\lambda},\tilde{\lambda})d\tilde{\lambda} \quad (18)$$

$$\sigma^2_{I_{MAC}} = \int_0^\infty \log^2(1+\tilde{\lambda}\rho/M)K(\tilde{\lambda},\tilde{\lambda}) \quad (19)$$

$$-\int_0^\infty\int_0^\infty \log(1+\tilde{\lambda}_1\rho/M) \quad (20)$$

$$\log(1+\tilde{\lambda}_2\rho/M)K^2(\tilde{\lambda},\tilde{\lambda})d\tilde{\lambda}_1 d\tilde{\lambda}_2 \quad (21)$$

where the auxiliary functions are given by [6],

$$K(x,y) := \sum_{i=0}^{k-1}\tilde{\phi}_i(x)\tilde{\phi}_i(y) \quad (22)$$

$$L_i^d(\tilde{\lambda}) := \frac{1}{i!}\exp(\tilde{\lambda})\tilde{\lambda}^{-d}\frac{d}{d\tilde{\lambda}^i}(e^{-\tilde{\lambda}}\tilde{\lambda}^{d+i}) \quad (23)$$

$$\tilde{\phi}_i(\tilde{\lambda}) := \left[i!/(i+d)!\right]^{\frac{1}{2}} L_d^i(\tilde{\lambda})\tilde{\lambda}^{\frac{d}{2}}e^{\frac{\tilde{\lambda}}{2}} \quad (24)$$

where $k = \min(M,N)$ and $d = \max(M,N) - k$. In order to find a Gaussian approximation that represents the sum of two Wishart distributions we state that $k = \min(M, 2N)$ and $d = \max(M, 2N)$. Thus, the outage probability can be easily be computed as a Gaussian cumulative density function (CDF) [13], given by

$$\Pr[\mathcal{I}_{MAC} < R] = \frac{1}{2}\left(1+\text{erf}\left(\frac{R-\mu}{\sqrt{2}\sigma}\right)\right) \quad (25)$$

where the function erf $(\cdot)$ denotes the Gaussian error function.

It is very important to note that the main reason to use the Gaussian approximation is to verify that the sum of Wishart matrices is still a Wishart matrix. Note that this condition will be true only for the same SNR (η) scenario.

A. Gaussian Approximation for different η and arbitrary M,N

In order to work with different η, lets state $\eta_1 \neq \eta_2$ and write it in the following way,

$$\eta_1\mathbf{W}_1 + \eta_2\mathbf{W}_2 = \eta_1\left(\mathbf{W}_1 + \eta\mathbf{W}_2\right) \quad (26)$$

where $\eta = \eta_2/\eta_1$. Unfortunately, linear combination of Wisharts with distinct coefficients **is not** Wishart distributed. In order to circumvent this problem, we propose a close approximation for the sum of Wisharts.

First, for values of η very close to one, if the situation discussed in section V holds, then the property of corollary 2 can be applied, that is, sum of two Wisharts is distributed as a Wishart with the sum of the degrees of freedom. For large values of η, we notice that the term $\eta\mathbf{W_2}$ dominates the sum and therefore the sum has a distribution closer to a Wishart with the same degrees of freedom of $\mathbf{W}_2$. Based on this, we propose that the number of degrees of freedom be a function of η in the following manner

$$\mathbf{W}_1 + \eta\mathbf{W}_2 \sim (1+\eta)\,\mathbf{W}_3 \quad (27)$$

where $\mathbf{W}_1 \sim \mathcal{W}_1(p_1,\Sigma)$, $\mathbf{W}_2 \sim \mathcal{W}_2(p_2,\Sigma)$, and $\mathbf{W}_3 \sim \mathcal{W}_3(p_3,\Sigma)$ with

$$p_3 = p_2 + \left\lfloor\frac{p_1}{\sqrt{\eta}}\right\rfloor \quad (28)$$

where $\lfloor\cdot\rfloor$ denotes the floor operator.

Note that for $\eta = 1$, this approximation leads to the same result of Corollary 2. On the other hand, for $\eta \to \infty$, $p_3 \to p_2$ since the $\eta\mathbf{W}_2$ term dominates the sum.

The validation of this approximation will be shown in the numerical results section.

$$\Pr\left(\mathcal{I}_{MAC} < R\right) = \int_0^{2^R} \frac{1}{24\sigma^{16}}\lambda_2^2 \left(2\sigma^6 e^{-\frac{\lambda_2}{\sigma^2}}\left(\lambda_2^2 - 6\lambda_2\sigma^2 + 12\sigma^4\right)\right.$$
$$- \sigma^2 e^{-\frac{\lambda_2 + \frac{2^R}{\lambda_2+1} - 1}{\sigma^2}} \left(2\sigma^4\left(\lambda_2^2 - 6\lambda_2\sigma^2 + 12\sigma^4\right) + \left(\lambda_2^2 - 6\lambda_2\sigma^2 + 12\sigma^4\right)\left(\frac{2^R}{\lambda_2+1} - 1\right)^2\right. \tag{15}$$
$$\left.\left. + 2\sigma^2\left(\lambda_2^2 - 6\lambda_2\sigma^2 + 12\sigma^4\right)\left(\frac{2^R}{\lambda_2+1} - 1\right) - 2\left(\lambda_2 - 2\sigma^2\right)\left(\frac{2^R}{\lambda_2+1} - 1\right)^3 + \left(\frac{2^R}{\lambda 2+1} - 1\right)^4\right)\right) d\lambda_2$$

$$\Pr\left(\mathcal{I}_{SU} < R\right) = \int_0^{2^R} \frac{1}{2\sigma^6} e^{-\frac{\lambda_2}{\sigma^2}} \left(\lambda_2^2 - 2\lambda_2\sigma^2 - e^{\frac{\lambda_2 - 2^R + 1}{(\lambda_2+1)\sigma^2}}\right.$$
$$\left.\left(\lambda_2^2 - 2\lambda_2\sigma^2 + 2\sigma^2\left(\frac{2^R}{\lambda_2\ +1} - 1\right) + \left(\frac{2^R}{\lambda_2+1} - 1\right)^2 - 2\lambda_2\left(\frac{2^R}{\lambda_2+1} - 1\right) + 2\sigma^4\right) + 2\sigma^4\right) d\lambda_2 \tag{16}$$

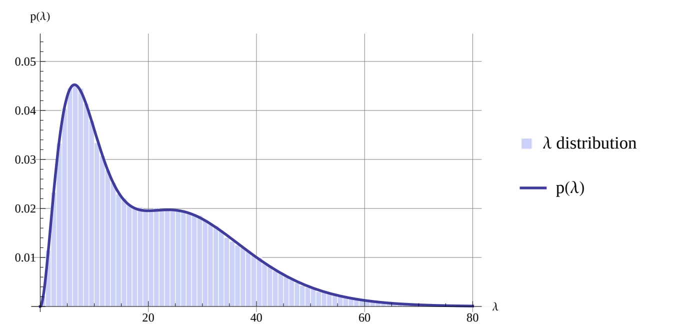

Fig. 2. Marginal Probability Density Function for $M = N = 2$.

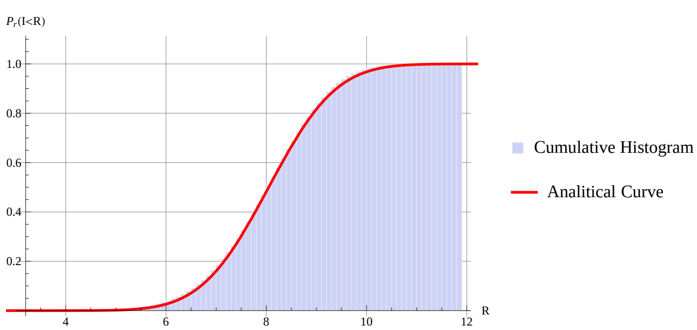

Fig. 3. Marginal Probability Density Function for $M = N = 2$.

VI. Numerical Results

In this section we will show some numerical results in order to validate our analytical framework.

A. *The $M = 2$, $N = 2$ case*

Fig. 2 shows the analytical result for the marginal probability density function of the eigenvalues obtained as the integration of (13) and also the computer simulation using the Mathematica® software. As can be seen, the agreement is perfect. The numerical simulation has been done with a SNR=10 dB.

In the same way, Fig. 3 shows the analytical (15) and simulated outage probability using the same parameters.

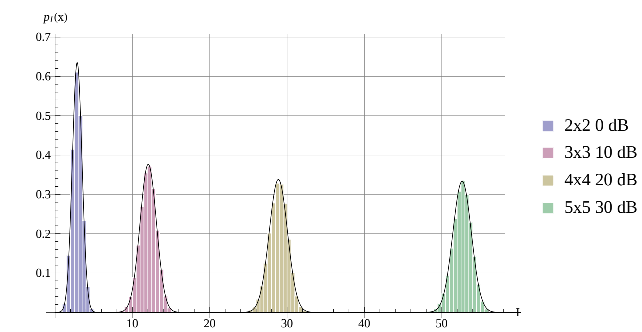

Fig. 4. Analytical Gaussian approximation and simulation for the marginal eigenvalue probability density function, $p_{I_{MAC}}(x)$, for several SNR values and M, N configurations.

B. *Gaussian Approximation for same SNR and arbitrary M,N*

Fig. 4 compares the analytical and simulated marginal eigenvalue probability density function for the "two users" case with the same SNR and several combinations of M and N. As it can be seen, in all the cases the approximation renders an excellent fit.

C. *Arbitrary number of users*

In order to show that the concept of the Wishart Matrices sum can be extended for more than two users, Fig. 5 shows the marginal probability density function for six users considering $M = N = 5$ and $SNR = 10$ dB. Fig. 6 shows the outage probability using the same parameters. Notice again, the excellent agreement between the proposed analytical approximation and the simulated curves.

D. *Gaussian Approximation for different SNR and arbitrary M,N*

In this subsection we want to validate our approximation given in (28) for two users using different values of η. Fig. 7 shows the outage probability for several values of η. In this figure, the dashed curve represents the simulation and the continuous curve represents the analytical Gaussian approximation for $M = N = 2$. Notice that for extreme values

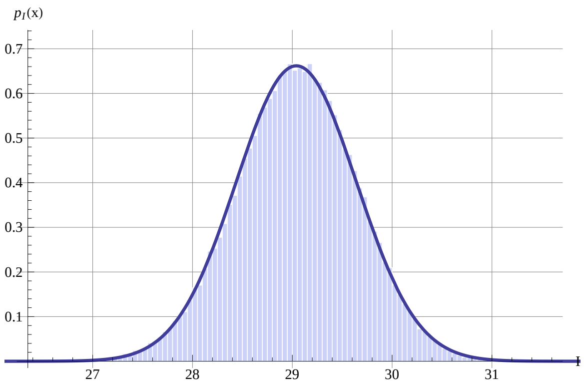

Fig. 5. Simulation and the Gaussian analytical approximation for $p_{I_{MAC}}(x)$ for 6 users and $M = N = 5$.

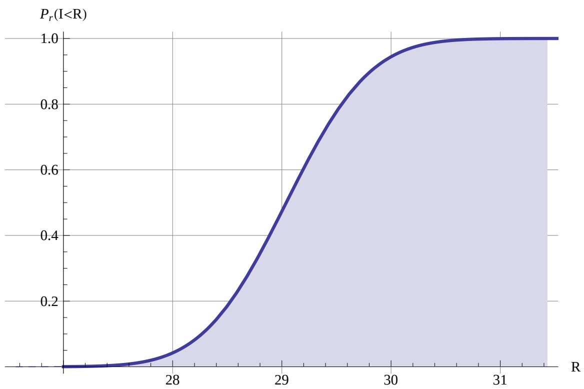

Fig. 6. Simulation and the Gaussian analytical approximation for the outage probability Pr $(I_{MAC} < R)$ for 6 users and $M = N = 5$.

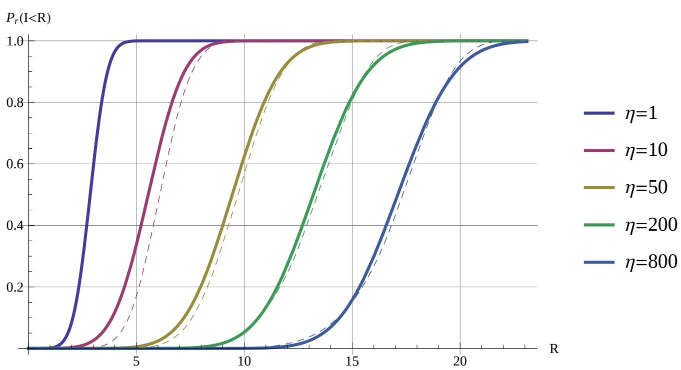

Fig. 7. Outage probability: comparison between analytic curves and simulation for several values of η and $M = N = 2$

of η (close to one or very large η) the Gaussian approximation is closer to the simulation curve, as expected. Fig. 8 shows the outage probability for a similar setup but for the case $M = N = 4$.

VII. Conclusions

This work studied the mutual information outage probability at multiple-input multiple-output (MIMO) nodes for MAC channels based on the eigenvalues' distribution of the sum of Wishart matrices. It was shown that the mutual information of the sum rate can be well approximated by a Gaussian

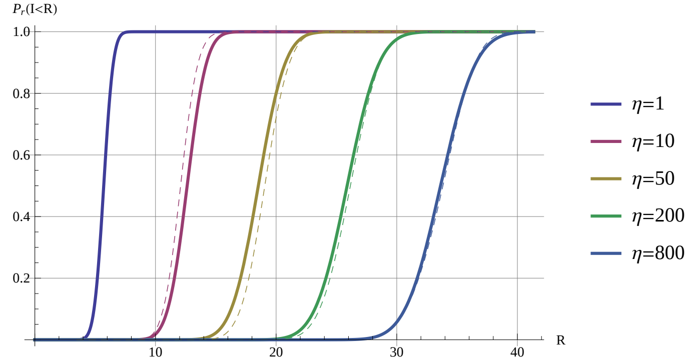

Fig. 8. Outage probability: Comparison between analytic curves and simulation for several values of η and $M = N = 4$

distribution. The analytical expression for the outage probability was calculated in an *exact* manner considering two users with $M = N = 2$ antennas. Furthermore, we proposed an approximation of the sum of two Wisharts matrices with different SNRs as being a Wishart matrix with a variable number of degrees of freedom. All the results were validated using computer simulation.

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