Estimation of the parameters by using non-Bayesian in Modified Weibull Extension Distribution

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Abstract. In this paper, studying the modified Weibull extension model, and derive the estimators of three parameters by using maximum likelihood method, and ordinary least squares estimator method. Then by utilizing Monte Carlo method in simulation procedure for generate many various of samples sizes with many various of replicate the sample size. Finally, finding that the maximum likelihood method are the best values from the ordinary least squares method.

Keywords: Modified Weibull extension, Classical distribution methods, Newton Raphson method.

1 Introduction

There are number of application of Weibul distributions and modified Weibull distribution to modeling the test data in scientific research and life experiments. Also, number of research dealing with these models to estimate the parameters.

In 2002, Xie et al. [6], generalized the Weibull distribution to a modified Weibull extension, model with three parameters, and in 2005 [4], Nadarajah driving explicit algebraic formulas for the kth moment of the distribution. Furthermore , in 2012, Peng et al. [5], proposed comparative analysis to prove that the modified Weibull extension distribution is more accurate and flexible in modeling the satellite reliability than the classical Weibull distribution. The accuracy and convenience of the modified Weibull extension distribution indicates its practical application in health of living organisms and engineering designs. In 2018, Al Kanani and Kalt [3] using simulation to studied classical distribution methods of the modified Weibull distribution. In 2019, Al Kanani and Jalil [2] discussed some estimation method for new distribution (mixture distribution) to estimate the parameters. In 2021, Al Kanani and Ali [1] dealing with exponentiated Weibull as a special case of Weibul distributions for estimate the parameters. By the way, the aim of this research is estimating all the three parameters of the modified Weibull extension model, by using maximum likelihood estimator method, and ordinary least squares estimator method to show the best values of parameters.

This paper is organized as follows. In section 2 recalling the information of modified Weibull extension, such as probability density function, Cumulated distribution, the reliability function, and hazard function. Section 2, and section 3 are devoted to the parameter estimation, Maximum Likelihood Estimator, and Ordinary Least Squares Estimator, respectively. Some sample sizes are generating for the simulation procedure in section 4, to illustrate the application of the modified Weibull extension model. Finally, comparative studies by using mean square error are used in section 5.

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2 Modified Weibull Extension Model

The probability density function of this model is:

$$f(x) = \theta\,\gamma\left(\frac{x}{\delta}\right)^{\gamma-1} e^{\left(\frac{x}{\delta}\right)^{\gamma} + \theta\delta\left(1-e^{\left(\frac{x}{\delta}\right)^{\gamma}}\right)}; \text{ for } x,\delta,\theta,\gamma\ > 0 \tag{1}$$

Where the shape parameter is γ, the scale parameters are $\delta,\ and\ \theta$. And, the Cumulated distribution function is:

$$F(\,x) = \int_{0}^{x} f(t)dt = 1 - e^{\,\theta\delta\left(1-e^{\left(\frac{x}{\delta}\right)^{\gamma}}\right)},\ x > 0 \tag{2}$$

Where, the reliability function is:

$$\mathrm{R}(x) = \int_{x}^{\infty} f(t)dt = e^{\,\theta\delta\left(1-e^{\left(\frac{x}{\delta}\right)^{\gamma}}\right)},\ x > 0 \tag{3}$$

And, the hazard function:

$$h(x) = \frac{f(x)}{R(x)} = \theta\gamma\left(\frac{x}{\delta}\right)^{\gamma-1} e^{\left(\frac{x}{\delta}\right)^{\gamma}},\ x > 0 \tag{4}$$

3 Maximum Likelihood Estimator Method (MLE)

This method is one of the most widely utilizing in statistical estimation. It used to estimate the parameters of discrete and continuous distribution. The idea of this method is to find an estimate to parameter , say $\boldsymbol{\hat{\theta}(X)}$, that maximize the likelihood (joint) function.
Taking the likelihood function of three parameters modified Weibull extension distribution to get:

$$L(\delta,\theta,\gamma;\ x_i) = \prod_{i=1}^{n} f(x_i\ ;\ \delta,\theta,\gamma) \tag{5}$$

That is:

$$L(\delta,\theta,\gamma;\ x_i) = \theta^n\,\gamma^n \prod_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{\gamma-1} e^{\sum_{i=1}^{n}\left[\left(\frac{x_i}{\delta}\right)^{\gamma} + \theta\delta\left(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\right)\right]} \tag{6}$$

Taking the natural logarithm for the above function, to obtain:

$$LnL = n\,Ln\theta + n\,Ln\gamma + (\gamma-1)\sum_{i=1}^{n} Ln\left(\frac{x_i}{\delta}\right) + \sum_{i=1}^{n}\left[\left(\frac{x_i}{\delta}\right)^{\gamma} + \theta\delta\left(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\right)\right] \tag{7}$$

The partial derivatives for the logarithm likelihood function with respect to parameters δ , θ and γ are :

$$f_1(\delta) = \frac{\partial Ln\,L}{\partial\delta}$$

$$f_1(\delta) = \frac{n}{\delta} - \frac{n\gamma}{\delta} + \sum_{i=1}^{n}\left[-\frac{\gamma}{\delta}\left(\frac{x_i}{\delta}\right)^{\gamma} + \theta\left(1 - e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\right) + \theta\gamma\left(\frac{x_i}{\delta}\right)^{\gamma} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\right] \tag{8}$$

$$g_1(\theta) = \frac{\partial Ln\ L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\left[\delta\left(1 - e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\right)\right] \quad (9)$$

$$h_1(\gamma) = \frac{\partial Ln\ L}{\partial \gamma}$$

$$= \frac{n}{\gamma} + \sum_{i=1}^{n} Ln(\frac{x_i}{\delta}) + \sum_{i=1}^{n}\left[\left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta}) - \theta\delta e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta})\right] \quad (10)$$

The above equations (8), (9), and (10), are system of non-linear equations, can be solved by using Newton – Raphson method. Thus we need the Jacobian matrix J_{K1}, which is:

$$J_{K1} = \begin{bmatrix} \frac{\partial f_1(\delta)}{\partial \delta} & \frac{\partial f_1(\delta)}{\partial \theta} & \frac{\partial f_1(\delta)}{\partial \gamma} \\ \frac{\partial g_1(\theta)}{\partial \delta} & \frac{\partial g_1(\theta)}{\partial \theta} & \frac{\partial g_1(\theta)}{\partial \gamma} \\ \frac{\partial h_1(\gamma)}{\partial \delta} & \frac{\partial h_1(\gamma)}{\partial \theta} & \frac{\partial h_1(\gamma)}{\partial \gamma} \end{bmatrix}$$

Where the pratial derivative are getting as follows:

$$\frac{\partial f_1(\delta)}{\partial \delta} = \frac{\partial^2 Ln\ L}{\partial^2 \delta}$$

$$\frac{\partial f_1(\delta)}{\partial \delta} = \frac{-n}{\delta^2} + \frac{n\gamma}{\delta^2} + \frac{\gamma}{\delta^2}\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{\gamma} + \frac{\gamma^2}{\delta^2}\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{\gamma} + \frac{\theta\gamma}{\delta}\sum_{i=1}^{n} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\left(\frac{x_i}{\delta}\right)^{\gamma}$$
$$- \frac{\theta\gamma^2}{\delta}\sum_{i=1}^{n} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\left(\frac{x_i}{\delta}\right)^{\gamma} - \frac{\theta\gamma^2}{\delta}\sum_{i=1}^{n} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\left(\frac{x_i}{\delta}\right)^{2\gamma} \quad (11)$$

$$\frac{\partial f_1(\delta)}{\partial \theta} = \frac{\partial^2 Ln\ L}{\partial \theta \partial \delta}$$

$$\frac{\partial f_1(\delta)}{\partial \theta} = n - \sum_{i=1}^{n} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} + \gamma \sum_{i=1}^{n} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\left(\frac{x_i}{\delta}\right)^{\gamma} \quad (12)$$

$$\frac{\partial f_1(\delta)}{\partial \gamma} = \frac{\partial^2 Ln\ L}{\partial \gamma \partial \delta}$$

$$\frac{\partial f_1(\delta)}{\partial \gamma} = \frac{-n}{\delta} - \frac{1}{\delta}\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{\gamma} - \frac{\gamma}{\delta}\sum_{i=1}^{n}\left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta}) - \theta\sum_{i=1}^{n} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\left(\frac{x_i}{\delta}\right)^{\gamma}$$

$$Ln(\frac{x_i}{\delta}) + \theta \sum_{i=1}^{n} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} + \theta\gamma \sum_{i=1}^{n} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta}) +$$

$$\theta\gamma \sum_{i=1}^{n} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{2\gamma} Ln(\frac{x_i}{\delta}) \quad (13)$$

$$\frac{\partial g_1(\theta)}{\partial \delta} = \frac{\partial^2 Ln\, L}{\partial \delta \partial \theta} = \frac{\partial f_1(\delta)}{\partial \theta} \quad (14)$$

$$\frac{\partial g_1(\theta)}{\partial \theta} = \frac{\partial^2 Ln\, L}{\partial \theta^2} = \frac{-n}{\theta^2} \quad (15)$$

$$\frac{\partial g_1(\theta)}{\partial \gamma} = \frac{\partial^2 Ln\, L}{\partial \gamma \partial \theta} = -\delta \sum_{i=1}^{n} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta}) \quad (16)$$

$$\frac{\partial h_1(\gamma)}{\partial \delta} = \frac{\partial^2 Ln\, L}{\partial \delta \partial \gamma} = \frac{\partial f_1(\delta)}{\partial \gamma} \quad (17)$$

$$\frac{\partial h_1(\gamma)}{\partial \theta} = \frac{\partial^2 Ln\, L}{\partial \theta \partial \gamma} = \frac{\partial g_1(\theta)}{\partial \gamma} \quad (18)$$

$$\frac{\partial h_1(\gamma)}{\partial \gamma} = \frac{\partial^2 Ln\, L}{\partial \gamma^2}$$

$$\frac{\partial h_1(\gamma)}{\partial \gamma} = \frac{-n}{\gamma^2} + \sum_{i=1}^{n} \left(Ln(\frac{x_i}{\delta})\right)^2 \left(\frac{x_i}{\delta}\right)^{\gamma} - \theta\delta \sum_{i=1}^{n} \left(Ln(\frac{x_i}{\delta})\right)^2 \left(\frac{x_i}{\delta}\right)^{\gamma} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}$$

$$-\theta\delta \sum_{i=1}^{n} \left(Ln(\frac{x_i}{\delta})\right)^2 \left(\frac{x_i}{\delta}\right)^{2\gamma} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \quad (19)$$

Thus, all above equations in section 3, proved that the Jacobian matrix J_{K1} is non-singular symmetric matrix, therefore:

$$\begin{bmatrix} \delta_{k+1} \\ \theta_{k+1} \\ \gamma_{k+1} \end{bmatrix} = \begin{bmatrix} \delta_k \\ \theta_k \\ \gamma_k \end{bmatrix} - J_{K1}{}^{-1} \begin{bmatrix} f_1(\delta_k) \\ g_1(\theta_k) \\ h_1(\gamma_k) \end{bmatrix}$$

4 Ordinary Least Squares Estimator Method (OLS)

This method is one of the most popular technique, to estimate the parameters when the models is linear and nonlinear in variables. The idea of this method is to minimize the sum squared differences between observed sample and the expected sample, as follows:

$$\sum_{i=1}^{n} {\epsilon_i}^2 = \sum_{i=1}^{n} [Y_i - E(Y_i)]^2 \tag{20}$$

Now, applying this method to minimize the sum squared difference between the estimate of cumulative distribution function, and the empirical cumulative distribution function, thus:

$$\sum_{i=1}^{n} {\epsilon_i}^2 = \sum_{i=1}^{n} \left[F(x_i) - \hat{F}(x_i)\right]^2 \tag{21}$$

Where empirical cumulative distribution function is:

$$F(x_i) = \frac{i - 0.5}{n} \tag{22}$$

Thus:

$$\sum_{i=1}^{n} {\epsilon_i}^2 = \sum_{i=1}^{n} [\frac{n - i + 0.5}{n} - e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})}]^2 \tag{23}$$

The partial derivatives for the logarithm likelihood function with respect to parameters δ , θ and γ are :

$$f_2(\delta) = \frac{\partial \sum_{i=1}^{n} {\epsilon_i}^2}{\partial \delta}$$

$$f_2(\delta) = -2\theta \sum_{i=1}^{n} \left[\frac{n - i + 0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} - e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})}\right] \left[1 - e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} + \gamma e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma}\right] \tag{24}$$

$$g_2(\theta) = \frac{\partial \sum_{i=1}^{n} {\epsilon_i}^2}{\partial \theta}$$

$$g_2(\theta) = -2\delta \sum_{i=1}^{n} \left[\frac{n - i + 0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} - e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})}\right] (1 - e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}) \tag{25}$$

$$h_2(\gamma) = \frac{\partial \sum_{i=1}^{n} {\epsilon_i}^2}{\partial \gamma}$$

$$h_2(\gamma) \; = 2\theta\delta \sum_{i=1}^{n} \left[\frac{n - i + 0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})+\left(\frac{x_i}{\delta}\right)^{\gamma}} - e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})+\left(\frac{x_i}{\delta}\right)^{\gamma}}\right] \left(\frac{x_i}{\delta}\right)^{\gamma} Ln\left(\frac{x_i}{\delta}\right) \tag{26}$$

The above equations (24), (25), and (26) are system of non-linear equations, can be solved by using Newton – Raphson method. Thus we need the Jacobian matrix J_{K2}, which is:

$$J_{K2} = \begin{bmatrix} \frac{\partial f_2(\delta)}{\partial \delta} & \frac{\partial f_2(\delta)}{\partial \theta} & \frac{\partial f_2(\delta)}{\partial \gamma} \\ \frac{\partial g_2(\theta)}{\partial \delta} & \frac{\partial g_2(\theta)}{\partial \theta} & \frac{\partial g_2(\theta)}{\partial \gamma} \\ \frac{\partial h_2(\gamma)}{\partial \delta} & \frac{\partial h_2(\gamma)}{\partial \theta} & \frac{\partial h_2(\gamma)}{\partial \gamma} \end{bmatrix}$$

Where the pratial derivative are getting as follows:

$$\frac{\partial f_2(\delta)}{\partial \delta} = \frac{\partial^2 \sum_{i=1}^{n} \epsilon_i^2}{\partial^2 \delta}$$

$$\frac{\partial f_2(\delta)}{\partial \delta} = -2 \sum_{i=1}^{n} \Bigg[\Bigg\{ \frac{n-i+0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} \left(\theta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma}) + \theta\gamma\, e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma \right) - e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} \left(2\theta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma}) + 2\theta\gamma\, e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma \right) \Bigg\} \Bigg\{ \theta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma}) + \theta\gamma\, e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma \Bigg\} + \Bigg\{ \frac{n-i+0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} - e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} \Bigg\} \Bigg\{ \frac{\theta\gamma}{\delta} e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma - \frac{\theta\gamma^2}{\delta} e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^{2\gamma} - \frac{\theta\gamma^2}{\delta} e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma \Bigg\} \Bigg] \tag{27}$$

$$\frac{\partial f_2(\delta)}{\partial \theta} = \frac{\partial^2 \sum_{i=1}^{n} \epsilon_i^2}{\partial \theta \partial \delta}$$

$$\frac{\partial f_2(\delta)}{\partial \theta} = -2 \sum_{i=1}^{n} \Bigg[-e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} \delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma}) \left(\theta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma}) + \theta\gamma e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma \right) + \left(\frac{n-i+0.5}{n} - e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} \right) e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} \delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma}) \left(\theta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma}) + \theta\gamma e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma \right) + \left(\frac{n-i+0.5}{n} - e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} \right) e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} (1-e^{\left(\frac{x_i}{\delta}\right)^\gamma}) + + \left(\frac{n-i+0.5}{n} - e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} \right) e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} \gamma\, e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma \Bigg] \tag{28}$$

$$\frac{\partial f_2(\delta)}{\partial \gamma} = \frac{\partial^2 \sum_{i=1}^{n} \epsilon_i^2}{\partial \gamma \partial \delta}$$

$$\frac{\partial f_2(\delta)}{\partial \gamma} = 2\theta \sum_{i=1}^{n} \Big[\frac{n-i+0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} \theta\delta e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta})(1 - e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}) + \frac{n-i+0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} \theta\delta\gamma\, e^{2\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{2\gamma} Ln(\frac{x_i}{\delta}) - 2 e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} \theta\delta e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta})\,(1 - e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}) - 2\theta\delta\gamma e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} e^{2\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{2\gamma} Ln(\frac{x_i}{\delta}) + \frac{n-i+0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln\left(\frac{x_i}{\delta}\right) - \frac{n-i+0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} - \frac{n-i+0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} \gamma e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{2\gamma} Ln(\frac{x_i}{\delta}) - \frac{n-i+0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} \gamma e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta}) - e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta}) + e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} + \gamma e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{2\gamma} Ln(\frac{x_i}{\delta}) + \gamma e^{2\theta\delta\left(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}\right)} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta})\Big] \quad (29)$$

$$\frac{\partial g_2(\theta)}{\partial \delta} = \frac{\partial^2 \sum_{i=1}^{n} {\epsilon_i}^2}{\partial\delta\partial\theta} = \frac{\partial f_2(\delta)}{\partial \theta} \quad (30)$$

$$\frac{\partial g_2(\theta)}{\partial \theta} = \frac{\partial^2 \sum_{i=1}^{n} {\epsilon_i}^2}{\partial \theta^2}$$

$$\frac{\partial g_2(\theta)}{\partial \theta} = -2\delta^2 \sum_{i=1}^{n} (1 - e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})^2 e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} \left[\frac{n-i+0.5}{n} - 2e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})}\right] \quad (31)$$

$$\frac{\partial g_2(\theta)}{\partial \gamma} = \frac{\partial^2 \sum_{i=1}^{n} {\epsilon_i}^2}{\partial\gamma\partial\theta}$$

$$\frac{\partial g_2(\theta)}{\partial \gamma} = 2\delta \sum_{i=1}^{n} \Big[\frac{n-i+0.5}{n} \theta\delta\, e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta})(1 - e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}) e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} - 2\theta\delta e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta})(1 - e^{\left(\frac{x_i}{\delta}\right)^{\gamma}}) + \frac{n-i+0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta}) - e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^{\gamma}})} e^{\left(\frac{x_i}{\delta}\right)^{\gamma}} \left(\frac{x_i}{\delta}\right)^{\gamma} Ln(\frac{x_i}{\delta})\Big] \quad (32)$$

$$\frac{\partial h_2(\gamma)}{\partial \delta} = \frac{\partial^2 \sum_{i=1}^{n} \epsilon_i{}^2}{\partial \delta \partial \gamma} = \frac{\partial f_2(\delta)}{\partial \gamma} \tag{33}$$

$$\frac{\partial h_2(\gamma)}{\partial \theta} = \frac{\partial^2 \sum_{i=1}^{n} \epsilon_i{}^2}{\partial \theta \partial \gamma} = \frac{\partial g_2(\theta)}{\partial \gamma} \tag{34}$$

$$\frac{\partial h_2(\gamma)}{\partial \gamma} = \frac{\partial^2 \sum_{i=1}^{n} \epsilon_i{}^2}{\partial \gamma^2}$$

$$\frac{\partial h_2(\gamma)}{\partial \gamma} = 2\theta\delta \sum_{i=1}^{n} Ln(\frac{x_i}{\delta}) \Big[\Big\{-\theta\delta(\frac{n-i+0.5}{n})\, e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} e^{2\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma Ln(\frac{x_i}{\delta})$$
$$+\frac{n-i+0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma Ln(\frac{x_i}{\delta}) + 2\theta\delta e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})}$$
$$e^{2\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma Ln(\frac{x_i}{\delta}) - e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma Ln(\frac{x_i}{\delta})\Big\}\Big\{\left(\frac{x_i}{\delta}\right)^\gamma\Big\}$$
$$+\frac{n-i+0.5}{n} e^{\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} e^{\left(\frac{x_i}{\delta}\right)^\gamma} \left(\frac{x_i}{\delta}\right)^\gamma Ln\left(\frac{x_i}{\delta}\right) - e^{2\theta\delta(1-e^{\left(\frac{x_i}{\delta}\right)^\gamma})} e^{\left(\frac{x_i}{\delta}\right)^\gamma}$$
$$\left(\frac{x_i}{\delta}\right)^\gamma Ln(\frac{x_i}{\delta})\Big] \tag{35}$$

Thus, all above equations in section 4, proved that the Jacobian matrix J_{K2} is non-singular symmetric matrix, therefore:

$$\begin{bmatrix} \delta_{k+1} \\ \theta_{k+1} \\ \gamma_{k+1} \end{bmatrix} = \begin{bmatrix} \delta_k \\ \theta_k \\ \gamma_k \end{bmatrix} - J_{K2}{}^{-1} \begin{bmatrix} f_2(\delta_k) \\ g_2(\theta_k) \\ h_2(\gamma_k) \end{bmatrix}$$

5. Simulation Results

The Monte Carlo method is most popular method in simulation techniques using to generate the observations (samples) for any distribution. The simulation process is flexibility, and gave ability for the test and experimenting by frequency many time.

1- To apply Monte Carlo method for modified Weibull extension, we use the cumulative distribution function as follows:

$$F(x) = 1 - e^{\theta\delta(1-e^{\left(\frac{x}{\delta}\right)^\gamma})} \tag{36}$$

By substitute $F(x)$ as u, with a random number, then $u = F(x)$, that is $x = F^{-1}(u)$. Thus

$$e^{\theta\delta(1-e^{\left(\frac{x}{\delta}\right)^\gamma})} = 1-u$$

$$\theta\delta(1-e^{\left(\frac{x}{\delta}\right)^\gamma}) = Ln(1-u)$$

$$(1-e^{\left(\frac{x}{\delta}\right)^{\gamma}})=\frac{Ln(1-u)}{\theta\delta}$$

$$e^{\left(\frac{x}{\delta}\right)^{\gamma}}=1-\frac{Ln(1-u)}{\theta\delta}$$

$$\left(\frac{x}{\delta}\right)^{\gamma}=Ln(1-\frac{Ln(1-u)}{\theta\delta})$$

$$x^{\gamma}=\delta^{\gamma}Ln(1-\frac{Ln(1-u)}{\theta\delta})$$

$$x=\left(\delta^{\gamma}Ln(1-\frac{Ln(1-u)}{\theta\delta})\right)^{\frac{1}{\gamma}} \quad (37)$$

2- Selected many different sample size
(n = 10, 20, 30, 50, 100).
3- Replicate to each experiment as (m =1000, 2500, 5000).
4- Plot a system of six-nonlinear equations,
$\begin{bmatrix} f_1(\delta) \\ g_1(\theta) \\ h_1(\gamma) \\ f_2(\delta) \\ g_2(\theta) \\ h_2(\gamma) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, For one of above sample sizes, for example, taking samples sizes (n =100), and sample replications (m =5000), can assist in finding initial approximations for both estimator methods, to find the values of parameters in two estimator methods. The plot is depicted in **Fig.1.**

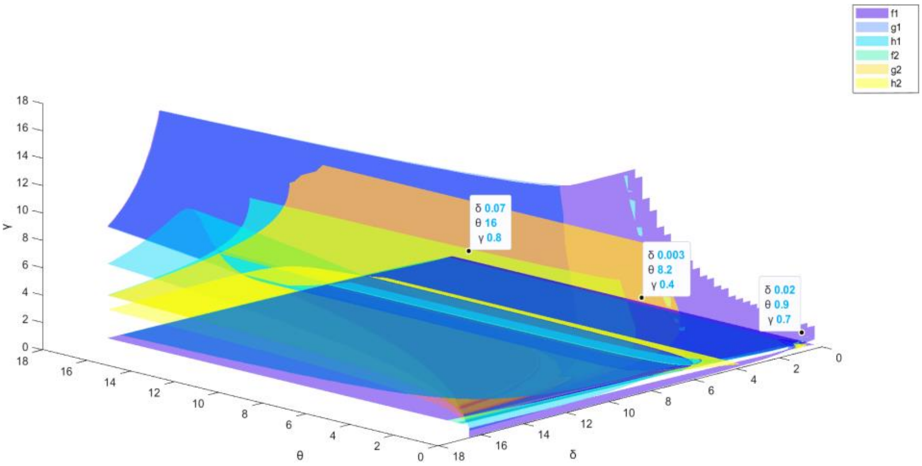

Fig. 1. Initial approximations for both estimator methods.

5-Selected the assumed values for the parameters
$\boldsymbol{\delta}$ = 0.07, 0.003, 0.02
$\boldsymbol{\theta}$ = 16.0, 8.2, 0.9
$\boldsymbol{\gamma}$ = 0.8, 0.4, 0.7.
6- To employ mesure of mean square error to compare between two methods of estimation given by the following formula $MSE(\hat{\theta})=\frac{1}{m}\sum_{i=1}^{m}(\hat{\theta}_i-\theta)^2$, where $\hat{\theta}$ is the estimate of δ,θ,γ

7- For illustrating the basics of the process in MLE, explaining the first iteration for finding the first estimating of $\boldsymbol{\delta_1}$, $\boldsymbol{\theta_1}$, $\boldsymbol{\gamma_1}$, at first initial guess in Simulink as follows, with m=5000, n=100

Finding J_{K1} at first initial guess $\boldsymbol{\delta_0 = 0.02}$, $\boldsymbol{\theta_0 = 0.9}$, $\boldsymbol{\gamma_0 = 0.7}$, And finding $\begin{bmatrix} f_1(\delta) \\ g_1(\theta) \\ h_1(\gamma) \end{bmatrix}$ at first initial guess $\boldsymbol{\delta_0}$, $\boldsymbol{\theta_0}$, $\boldsymbol{\gamma_0}$, And applying

$$J_{K1}\begin{bmatrix} \boldsymbol{\delta_1} \\ \boldsymbol{\theta_1} \\ \boldsymbol{\gamma_1} \end{bmatrix} - \begin{bmatrix} f_1(\delta) \\ g_1(\theta) \\ h_1(\gamma) \end{bmatrix} = 0, \text{ which determined as } f(z) = \begin{bmatrix} a_1\boldsymbol{\delta_1} + a_2\boldsymbol{\delta_1} + a_3\boldsymbol{\delta_1} + a_4 \\ a_5\boldsymbol{\theta_1} + a_6\boldsymbol{\theta_1} + a_7\boldsymbol{\theta_1} + a_8 \\ a_9\boldsymbol{\gamma_1} + a_{10}\boldsymbol{\gamma_1} + a_{11}\boldsymbol{\gamma_1} + a_{12} \end{bmatrix} = 0$$

After solving the above equation, adding the result with the first initial guess, to obtain the estimator for the first parameter $\boldsymbol{\delta_1}$, the second parameter $\boldsymbol{\theta_1}$, and the third parameter $\boldsymbol{\gamma_1}$, then finding the first iteration of mean square error for parameters, the process is depicted in **Fig.2.** By the same way for the process in OLS.

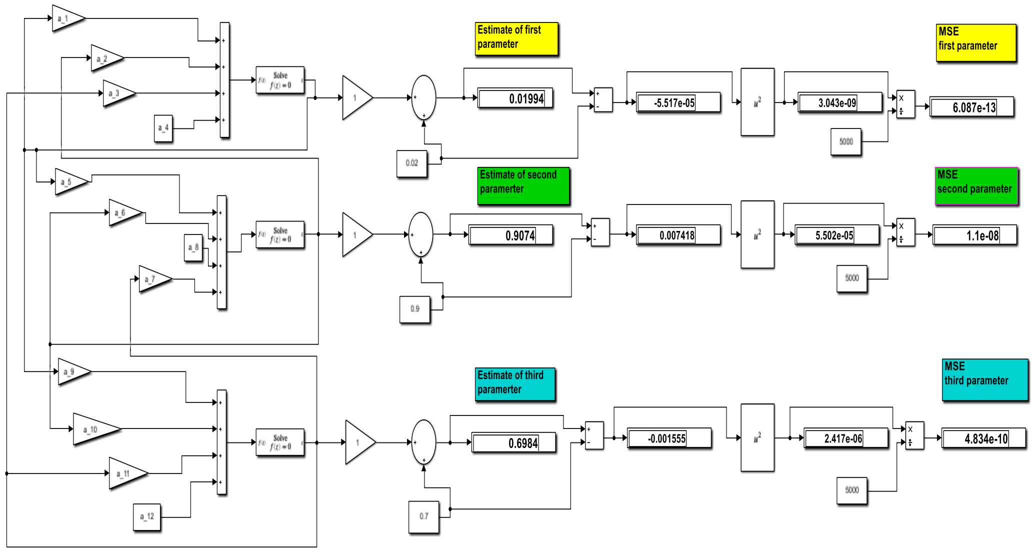

Fig. 2. The process in MLE, for the first iteration.

Table 1. MSE for the estimation of δ at **E1 {' δ = 0.02'} {' θ = 0.9'} {' γ = 0.7'}.**

m	n	MLE	OLS	best
1000	10	7.4699 e-09	1.5353 e-05	MLE
	20	7.7755 e-09	1.5102 e-05	MLE
	30	7.8831 e-09	1.4979 e-05	MLE
	50	7.6583 e-09	1.4895 e-05	MLE
	100	7.6147 e-09	1.4833 e-05	MLE
2500	10	7.5737 e-09	1.5424 e-05	MLE
	20	7.7216 e-09	1.5101 e-05	MLE
	30	7.593 e-09	1.4972 e-05	MLE
	50	7.579 e-09	1.4853 e-05	MLE
	100	7.7307 e-09	1.4796 e-05	MLE
5000	10	7.7353 e-09	1.5498 e-05	MLE
	20	7.6006 e-09	1.5054 e-05	MLE
	30	7.5871 e-09	1.4974 e-05	MLE
	50	7.7125 e-09	1.4878 e-05	MLE
	100	7.6811 e-09	1.4812 e-05	MLE

Table 2. MSE for the estimation of θ at **E1 {' δ = 0.02'} {' θ = 0.9'} {' γ = 0.7'}.**

m	n	MLE	OLS	best
1000	10	0.00010051	0.19623	MLE
	20	0.00010313	0.1969	MLE
	30	0.00010396	0.19737	MLE
	50	0.00010201	0.19756	MLE
	100	0.00010189	0.19773	MLE
2500	10	0.00010155	0.19588	MLE
	20	0.00010273	0.19692	MLE
	30	0.00010175	0.19724	MLE
	50	0.00010163	0.19761	MLE
	100	0.00010301	0.19788	MLE
5000	10	0.00010276	0.1958	MLE
	20	0.00010164	0.19709	MLE
	30	0.00010169	0.1973	MLE
	50	0.00010264	0.1963	MLE
	100	0.00010242	0.19783	MLE

Table 3. MSE for the estimation of γ at **E1 {' δ = 0.02'} {' θ = 0.9'} {' γ = 0.7'}.**

m	n	MLE	OLS	best
1000	10	5.5494 e-06	0.011528	MLE
	20	5.7568 e-06	0.011417	MLE
	30	5.8284 e-06	0.011367	MLE
	50	5.6752 e-06	0.011326	MLE
	100	5.6499 e-06	0.011296	MLE
2500	10	5.6224 e-06	0.011555	MLE
	20	5.7213 e-06	0.011418	MLE
	30	5.6361 e-06	0.011356	MLE
	50	5.6267 e-06	0.011302	MLE
	100	5.7311 e-06	0.011278	MLE
5000	10	5.7288 e-06	0.011596	MLE
	20	5.6381 e-06	0.011395	MLE
	30	5.6318 e-06	0.011359	MLE
	50	5.715 e-06	0.011317	MLE
	100	5.6945 e-06	0.011287	MLE

Table 4. MSE for the estimation of δ at **E2 {' δ = 0.003'} {' θ = 8.2'} {' γ = 0.4'}**

m	n	MLE	OLS	best
1000	10	4.151 e-09	9.7399 e-07	MLE
	20	4.265 e-09	9.6521 e-07	MLE
	30	4.2976 e-09	9.6219 e-07	MLE
	50	4.2103 e-09	9.5829 e-07	MLE
	100	4.2078 e-09	9.5573 e-07	MLE
2500	10	4.1923 e-09	9.7461 e-07	MLE
	20	4.2451 e-09	9.6549 e-07	MLE
	30	4.2011 e-09	9.598 e-07	MLE
	50	4.1966 e-09	9.558 e-07	MLE
	100	4.2524 e-09	9.5453 e-07	MLE
5000	10	4.2445 e-09	9.7905 e-07	MLE
	20	4.1962 e-09	9.637 e-07	MLE
	30	4.1972 e-09	9.6058 e-07	MLE
	50	4.2403 e-09	9.5771 e-07	MLE
	100	4.2296 e-09	9.5536 e-07	MLE

Table 5. MSE for the estimation of θ at **E2 {' δ = 0.003'} {' θ = 8.2'} {' γ = 0.4'}**

m	n	MLE	OLS	best
1000	10	0.084508	1.675	MLE
	20	0.08583	1.723	MLE
	30	0.086261	1.747	MLE
	50	0.085138	1.768	MLE
	100	0.08535	1.78	MLE
2500	10	0.085168	1.671	MLE
	20	0.085797	1.722	MLE
	30	0.085291	1.753	MLE
	50	0.085382	1.774	MLE
	100	0.085841	1.79	MLE
5000	10	0.08567	1.655	MLE
	20	0.085215	1.735	MLE
	30	0.08517	1.751	MLE
	50	0.085642	1.77	MLE
	100	0.085582	1.783	MLE

Table 6. MSE for the estimation of γ at **E2 {' δ = 0.003'} {' θ = 8.2'} {' γ = 0.4'}**

m	n	MLE	OLS	best
1000	10	3.0392 e-05	0.0063268	MLE
	20	3.1084 e-05	0.0063016	MLE
	30	3.1287 e-05	0.0062922	MLE
	50	3.0746 e-05	0.0062786	MLE
	100	3.0759 e-05	0.0062706	MLE
2500	10	3.0666 e-05	0.0063294	MLE
	20	3.0983 e-05	0.006303	MLE
	30	3.0723 e-05	0.0062827	MLE
	50	3.0705 e-05	0.0062693	MLE
	100	3.1034 e-05	0.0062654	MLE
5000	10	3.0968 e-05	0.006346	MLE
	20	3.0684 e-05	0.0062955	MLE
	30	3.0694 e-05	0.006286	MLE
	50	3.0947 e-05	0.0062767	MLE
	100	3.0888 e-05	0.0062691	MLE

Table 7. MSE for the estimation of $\boldsymbol{\delta}$ at E3 {' $\boldsymbol{\delta}$ = 0.07'} {' $\boldsymbol{\theta}$ = 16.0'} {' $\boldsymbol{\gamma}$ = 0.8'}.

m	n	MLE	OLS	best
1000	10	3.0821 e-07	0.00034378	MLE
	20	2.2765 e-07	0.000332	MLE
	30	1.9724 e-07	0.00032659	MLE
	50	1.7103 e-07	0.00032311	MLE
	100	1.468 e-07	0.00032067	MLE
2500	10	3.1315 e-07	0.00034643	MLE
	20	2.2851 e-07	0.0003319	MLE
	30	1.988 e-07	0.00032654	MLE
	50	1.7004 e-07	0.00032179	MLE
	100	1.4464 e-07	0.00031935	MLE
5000	10	3.2276 e-07	0.00034953	MLE
	20	2.2595 e-07	0.00033012	MLE
	30	1.9861 e-07	0.0003266	MLE
	50	1.7072 e-07	0.00032264	MLE
	100	1.452 e-07	0.00031988	MLE

Table 8. MSE for the estimation of $\boldsymbol{\theta}$ at E3 {' $\boldsymbol{\delta}$ = 0.07'} {' $\boldsymbol{\theta}$ = 16.0'} {' $\boldsymbol{\gamma}$ = 0.8'}.

m	n	MLE	OLS	best
1000	10	0.062166	1.756	MLE
	20	0.052586	2.113	MLE
	30	0.048369	2.346	MLE
	50	0.044588	2.399	MLE
	100	0.040646	2.484	MLE
2500	10	0.062834	1.571	MLE
	20	0.05279	2.098	MLE
	30	0.048731	2.236	MLE
	50	0.044351	2.402	MLE
	100	0.040172	2.572	MLE
5000	10	0.063675	1.561	MLE
	20	0.052417	2.214	MLE
	30	0.048696	2.282	MLE
	50	0.044472	2.458	MLE
	100	0.040279	2.542	MLE

Table 9. MSE for the estimation of γ at E3 {' δ = 0.07'} {' θ = 16.0'} {' γ = 0.8'}.

m	n	MLE	OLS	best
1000	10	2.1727 e-05	0.024765	MLE
	20	1.6545 e-05	0.024225	MLE
	30	1.4526 e-05	0.023989	MLE
	50	1.2769 e-05	0.023815	MLE
	100	1.1106 e-05	0.023693	MLE
2500	10	2.2051 e-05	0.024879	MLE
	20	1.6609 e-05	0.024225	MLE
	30	1.4642 e-05	0.023963	MLE
	50	1.2695 e-05	0.023737	MLE
	100	1.0948 e-05	0.023629	MLE
5000	10	2.2629 e-05	0.025043	MLE
	20	1.6438 e-05	0.024136	MLE
	30	1.4629 e-05	0.023971	MLE
	50	1.2742 e-05	0.023788	MLE
	100	1.0987 e-05	0.023658	MLE

6.Conclusion

Noting from all above tables that Mean Square Error values for Maximum likelihood estimator method is less than Mean Square Error values for ordinary least squares estimator method, thus we found that the best method for estimating the parameters is Maximum likelihood estimator method.

References

[1] Ali, A. H. and Al Kanani, I. H. "Classical methods to estimate the parameters of exponentiated weibull distribution,". Iraqi Academics Syndicate International Conference for Pure and Applied Sciences, Journal of Physics: Conference Series 2021; 1818 012078, doi:10.1088/1742-6596/1818/1/012078

[2] Jalil, M. and Al Kanani, I. H. " Some estimation methods for new mixture distribution with simulation and application," 2019 IOP conference , series: materials science and engineering 571(2019) 012014.

[3] Kalt, H G. and Al Kanani, I. H. " A comparison of MLE method and OLS for the estimation of modified Weibull distribution parameters by using the simulation ," ARPN Journal of Engineering and Applied Sciences, 2018; sept 13(17), ISSN 1819-6608.

[4] Nadarajah, S. "On the moments of the modified Weibull distribution," Reliability Engineering and System Safety, Elsevier, 2005; vol. 90(1), pages 114-117.

[5] Peng, W. Zhang, H. Huang, H. Gong, Z. and Liu,Y. "Satellite reliability modeling with modified Weibull extension distribution," 2012 International Conference on Quality, Reliability, Risk, Maintenance, and Safety Engineering, 2012; pp. 195-199, doi: 10.1109/ICQR2MSE.2012.6246218.

[6] Xie, M. Tang, Y. and Goh, T. N. "A modified Weibull extension with bathtub-shaped failure rate function," Reliability Engineering & System Safety, 2002; vol. 76, no. 3, pp. 279–285.