The Double Butterfly Theorem on Hyperbolas

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Abstract. The butterfly theorem and the double butterfly theorem are fundamentally applied to circles. Subsequently, several authors have extended these theorems to ellipses and parabolas. As a hyperbola is another type of conic section, the author develops the double butterfly theorem for the hyperbola. In the proof process, the butterfly and double butterfly theorems for circles, ellipses, and parabolas rely on lemma of Haruki, which is applicable only to circles. Therefore, in this paper, the author first develops a version of lemma of Haruki specifically for the hyperbola. Based on this lemma, the proof of the double butterfly theorem for the hyperbola is then established. In other words, the double butterfly theorem can be extended to the hyperbola.

Keywords: Butterfly theorem, double butterfly theorem, lemma of Haruki, hyperbola.

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1 Introduction

The history of the butterfly theorem was introduced in a letter sent by the astronomer Sir William Herschel to William Wallace in 1805 [1]. The content of the letter is as follows: given two chords AB and CD in a circle that intersect at point K. Then, there are two other chords, EF and HG, which also intersect at K. Next, the chords HF and EG intersect the chord CD at points L and M, respectively. If K is the midpoint of both CD and ML, then $MK=LK$. The illustration of the historical butterfly theorem is shown in **Figure 1**.

Several solutions to the contents of the letter in the history of the butterfly theorem have been presented [2–7]. The butterfly theorem was generalized using complex coordinates [8]. The development of a generalization of the butterfly theorem, which demonstrates the symmetry of distances between intersection points on a chord passing through the midpoint, was discussed using an analytical approach based on complex numbers [9]. In addition, the origin, development, and various proofs of the butterfly theorem, together with the introduction of its general versions and extensions, including the double butterfly theorem, have been discussed [10]. The butterfly theorem has been generalized to conic sections [11]. Another development of the butterfly theorem, which originally applied to circles, extending it to convex

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quadrilaterals, has been presented [12]. The butterfly theorem has been further developed for hyperbolas [13].

In addition, a further development of the butterfly theorem is the double butterfly theorem [14]. A proof of the double butterfly theorem has been provided [15]. Furthermore, the double butterfly theorem was developed for other conic sections, namely ellipses and parabolas, using a proof based on Haruki's lemma [16,17]. Haruki's lemma was introduced by considering two non-intersecting chords of a circle [18].

Currently, the development of the double butterfly theorem on hyperbolas has not been discussed by anyone. Therefore, the author is interested in discussing the double butterfly theorem that applies to hyperbolas. In proving the double butterfly theorem on hyperbolas, Haruki's lemma is used, which has been developed into Haruki's lemma on hyperbolas. Broadly speaking, this research focuses on the double butterfly theorem developed on hyperbolas with Haruki's lemma as its proof.

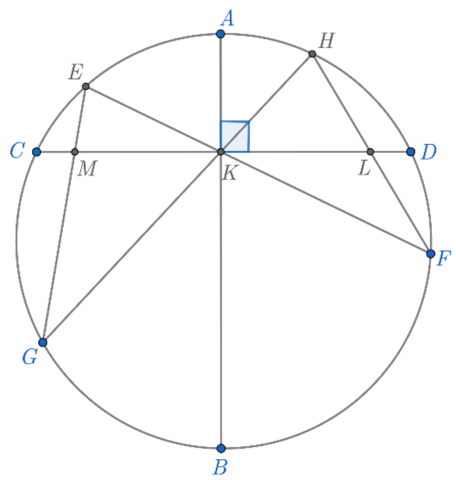

Fig. 1. The illustration of the historical butterfly theorem found in the letter

2 Review of Literature

In this section, several supporting theories related to the discussion in the subsequent section are outlined. The discussion begins with an introduction to Haruki's lemma, the butterfly theorem, the butterfly theorem on hyperbolas, and finally the double butterfly theorem.

2.1 Haruki's Lemma

The Japanese mathematician named Hirosi Haruki discovered Haruki's lemma and published it in Forum Geometricorum by Yaroslav Bezverkhnyev. Haruki's lemma for circles has been described [18]. The following provides Haruki's lemma that applies to circles along with its proof.

Lemma 1. (Haruki) Given two chords AB and CD that do not intersect within a circle, and a point P on the arc AB that does not contain points C and D. Let E and F be the intersection points of chord PC with AB and chord PD with AB, respectively. The value of $(AE \cdot BF)/EF$ is independent of the position of P.

Proof. The illustration of Haruki's Lemma can be seen in **Figure 2**. Furthermore, the proof of Haruki's lemma has been provided [18].

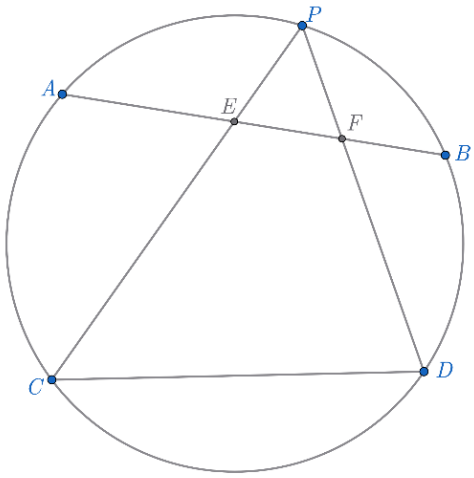

Fig. 2. Illustration of Haruki's lemma

2.2 Butterfly theorem

In this section, the butterfly theorem that applies to circles is discussed along with its proof. The proof of the butterfly theorem uses Lemma 1. The butterfly theorem for circles has been explained in various studies [1–3,13,19,20]. The following is an explanation of the butterfly theorem that applies to circles.

Theorem 2. (Butterfly) Suppose P is the midpoint of chord AB in a circle. There are also other chords JK and LM. If chord JM intersects AB at R and chord LK intersects AB at S, then P is the midpoint of RS.

Proof. The illustration of the butterfly theorem can be seen in **Figure 3**. Furthermore, proofs of the butterfly theorem have been presented [2–4].

2.3 Butterfly theorem on hyperbolas

he butterfly theorem has been developed for hyperbolas [13,19]. The following is an explanation of the butterfly theorem on hyperbolas along with its proof.

Theorem 3. (Butterfly on Hyperbolas) Suppose M is the midpoint of chord AB. Two chords CD and EF intersect at point M. If chord DE intersects AB at point P and chord FC intersects AB at point Q, then M is the midpoint of PQ.

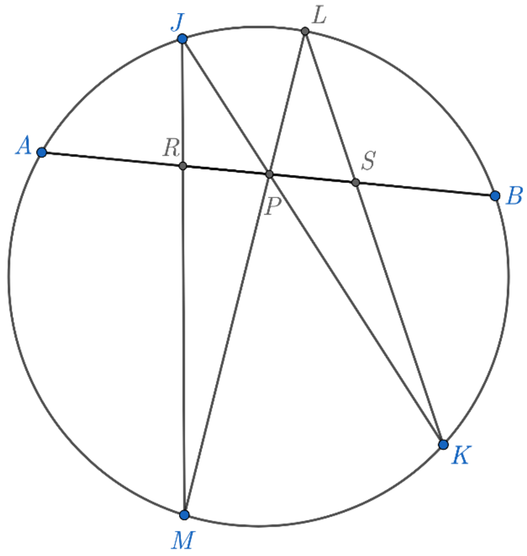

Fig. 3. Illustration of the butterfly theorem

Proof. The illustration of the butterfly theorem on hyperbolas can be seen in **Figure 4**. Furthermore, proofs of the butterfly theorem on hyperbolas have been presented [9,13].

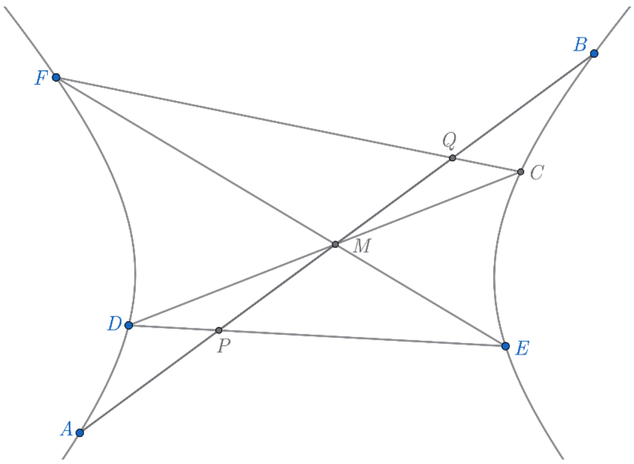

Fig. 4. Illustration of the butterfly theorem on hyperbolas

2.4 Double butterfly theorem

Researchers have developed the butterfly theorem by adding one more butterfly to a circle. The double butterfly theorem has been discussed in several studies with the same understanding [14–17]. The following is an explanation of the double butterfly theorem.

Theorem 4. Suppose *AB* is any chord in a circle that intersects the butterflies *JKML* and *XYWZ* in the circle, (in order from left to right) at points R_4, R_3, R_2, R_1 and S_1, S_2, S_3, S_4. If $AR_1 = BS_1$, $AR_2 = BS_2$, and $AR_3 = BS_3$, then $AR_4 = BS_4$.

Proof. The illustration of the double butterfly theorem is shown in **Figure 5**. Furthermore, proofs of the double butterfly theorem have been presented [14,15].

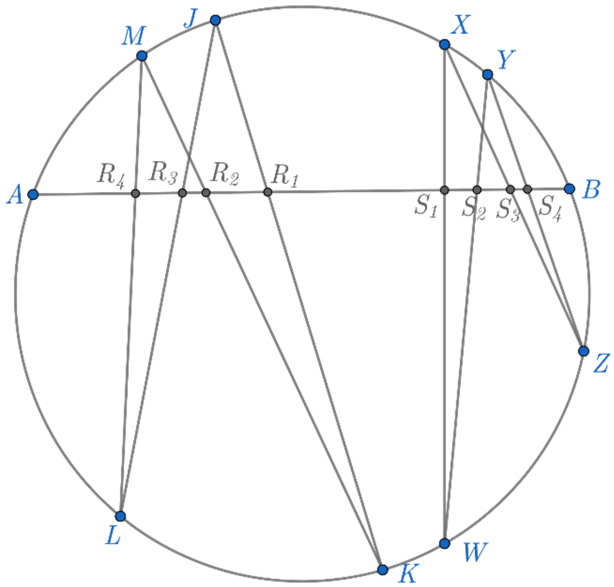

Fig. 5. Illustration of the double butterfly theorem

3 Results

This section discusses that the double butterfly theorem, which holds for circles, also applies to other conic sections, namely hyperbolas; therefore, the theorem is referred to as the double butterfly theorem on hyperbolas. Subsequently, the proof of the double butterfly theorem on hyperbolas is presented. The proof uses Haruki's lemma on hyperbolas. Furthermore, this paper considers a hyperbola defined by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with $a,b > 0$ [13,19]. Section 3.1 explains Haruki's lemma on hyperbolas, which is used to prove the double butterfly theorem on hyperbolas. Subsequently, the proof of Haruki's lemma on hyperbolas is provided. Finally, Section 3.2 discusses the double butterfly theorem on hyperbolas along with its proof.

3.1 Haruki's Lemma on hyperbolas

Haruki's lemma is a lemma that applies to circles. If the lemma is applied to hyperbolas, it remains valid; therefore, the lemma is referred to as Haruki's lemma on hyperbolas.

Subsequently, Haruki's lemma on hyperbolas is proved using Lemma 1. The following presents Haruki's lemma on hyperbolas along with its proof.

Lemma 5. (Haruki on Hyperbolas) Given two chords DE and FG (respectively on the left and right of the hyperbola) and a point P on the chord DE. If I and J are respectively the intersections of the extension of PF with DE and the extension of PG with DE, then the value of $(JD \cdot EI)/IJ$ does not depend on the position of P.

Proof. The illustration of Haruki's lemma on the hyperbola can be seen in **Figure 6**.

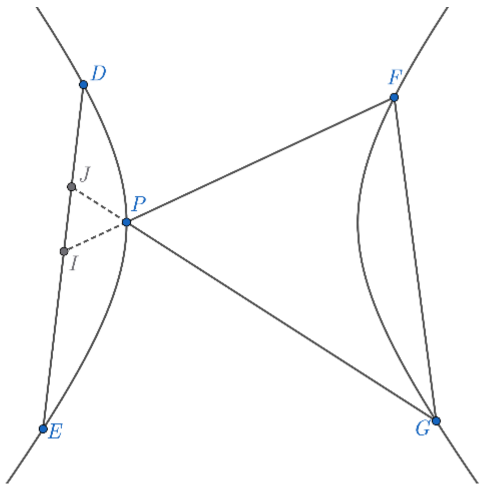

Fig. 6. The illustration of Haruki's lemma on the hyperbola

A circle is constructed passing through points E, P, and D, and let the extensions of PF and PG intersect this circle at points K and L respectively, as shown in **Figure 7**.

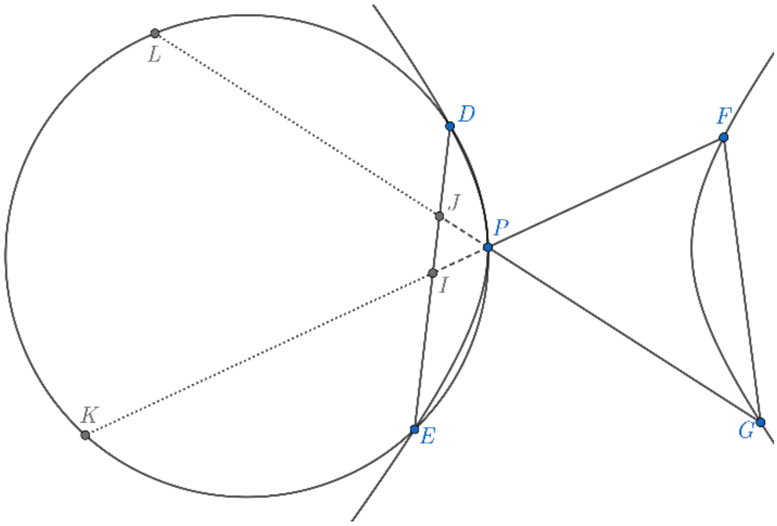

Fig. 7. The circle passing through points E, P, and D

Then, the circumcircle of ΔPIL is also constructed, and let the extension of DE intersect the circumcircle of ΔPIL at point H, as shown in **Figure 8**.

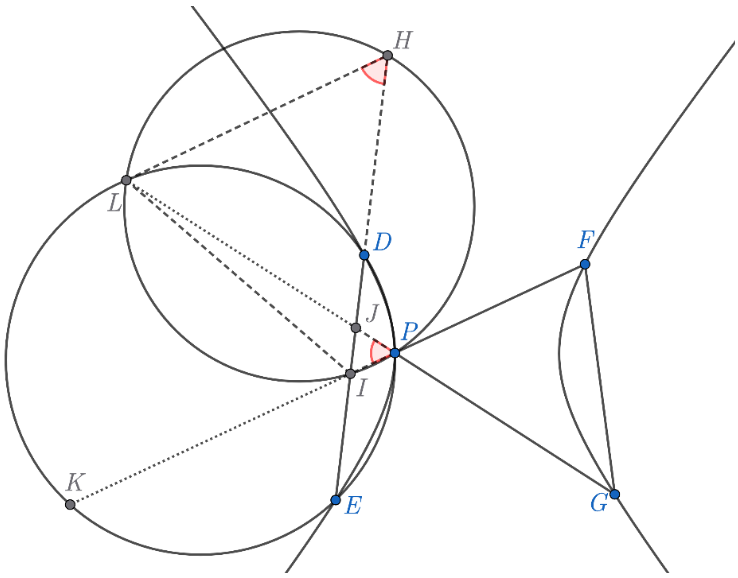

Fig. 8. The circumcircle of ΔPIL

The chord PL intersecting with the chord IH results in $\angle LPI = \angle LHI$ because they subtend the same arc, namely arc IL, and the measure of $\angle LHI$ remains constant for various positions of P on the chord DE. Thus, point H remains fixed on the extension of DE, making DH constant.

Furthermore, by applying lemma 1 to the chords PL and DE as well as the chords PL and IH, we obtain

$$LJ \cdot JP = JD \cdot EJ. \tag{1}$$

$$LJ \cdot JP = JH \cdot IJ. \tag{2}$$

By considering equations (1) and (2), the left-hand sides of both equations are the same, so we obtain

$$\frac{JD \cdot EI}{IJ} = DH. \tag{3}$$

Note that the length of DH is constant, so the left-hand side of equation (3) is constant. Therefore, it is proven that $(JD \cdot EI)/IJ$ does not depend on the position of P.

3.2 The Double butterfly theorem on hyperbolas

The double butterfly theorem is a development of the butterfly theorem that was introduced in 1976 [14]. The developed theorem applies to circles. If the theorem is applied to hyperbolas, it remains valid; therefore, it is referred to as the double butterfly theorem on hyperbolas. In an educational context, the double butterfly theorem on hyperbolas can be used as a concrete learning example to demonstrate how a classical theorem on circles can be extended to hyperbolas, as well as to train students to understand the differences in geometric approaches between closed conic sections and open conic sections. In this section, the double butterfly theorem on hyperbolas and its proof are presented. The proof uses Lemma 5. The following presents the double butterfly theorem on hyperbolas along with its proof.

Theorem 6. (The Double Butterfly Theorem on Hyperbolas) Let DE be a transversal chord of a hyperbola, with butterflies $FGIH$ and $JKML$ on the hyperbola such that their wings intersect DE (in order from left to right) at X_4, X_3, X_2, X_1 and Y_1, Y_2, Y_3, Y_4. If $DX_1 = EY_1$, $DX_2 = EY_2$, and $DX_3 = EY_3$, then $DX_4 = EY_4$.

Proof. The construction of the double butterfly on a hyperbola is carried out separately and intersects with each other. The points of the first butterfly (left part) are F, G, I, and H, while the points of the second butterfly (right part) are J, K, M, and L, as shown in **Figure 9**.

A chord DE is constructed with point D located on the left side of the hyperbola and point E located on the right side of the hyperbola, so that the chord DE intersects the double butterfly construction. Sequentially from left to right, the intersection points of the chord DE with the double butterfly are X_4, X_3, X_2, X_1 and Y_1, Y_2, Y_3, Y_4 thereby forming an illustration of Theorem 6, as shown in **Figure 10**.

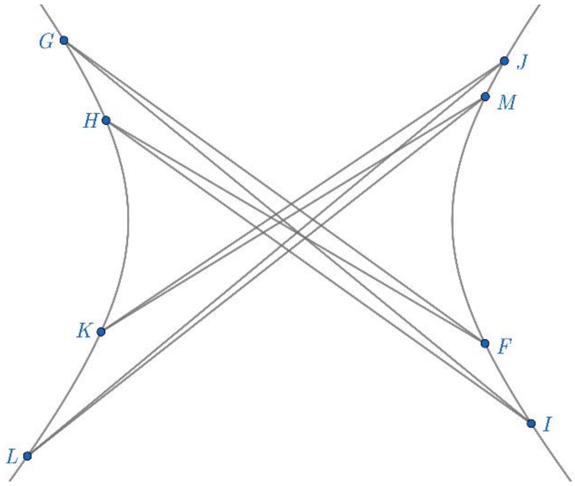

Fig. 9. The construction of the double butterfly on a hyperbola

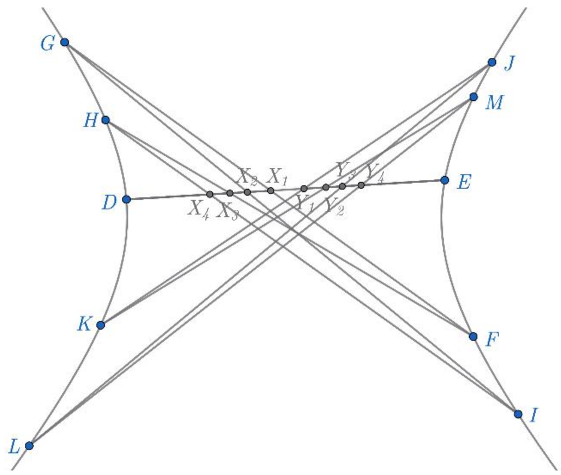

Fig. 10. The chord DE intersects the double butterfly

The double butterfly in **Figure 10** can be decomposed into the forms shown in **Figure 11** and **Figure 12**.

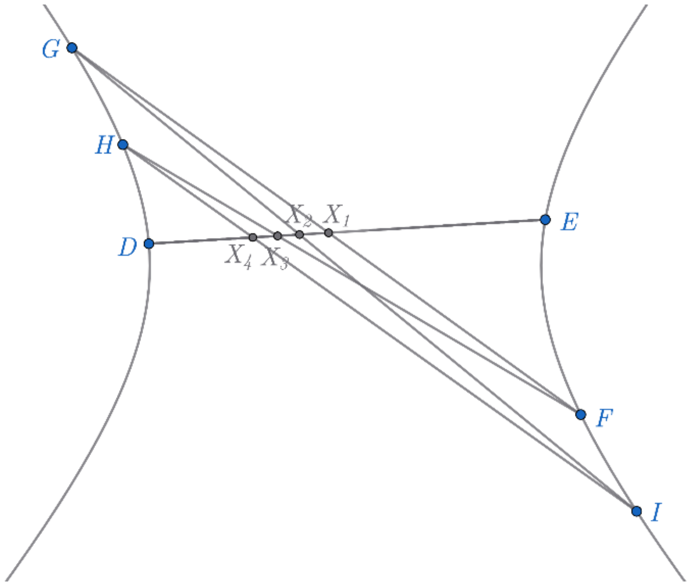

Fig. 11. Butterfly FGIH

By noting that point G in **Figure 10** is the result of the displacement of point H along the hyperbolic arc, then based on Lemma 5, it follows that

$$\frac{DX_4 \cdot EX_3}{X_3X_4} = \frac{DX_2 \cdot EX_1}{X_1X_2}. \tag{4}$$

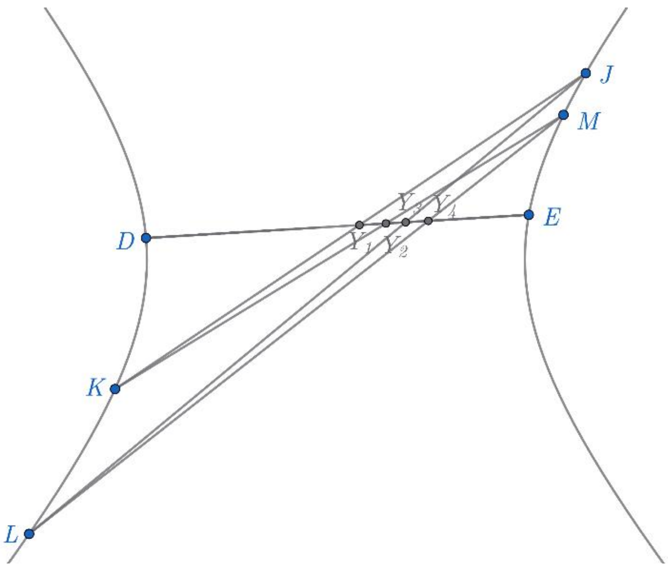

Fig. 12. Butterfly *JKML*

Furthermore, with the same argument applied to **Figure 11**, it follows that

$$\frac{DY_4 \cdot EY_3}{Y_3Y_4} = \frac{EY_2 \cdot DY_1}{Y_1Y_2}. \tag{5}$$

If the double butterfly theorem on a hyperbola is proven, it will establish DX_4=EY_4. Given in the double butterfly theorem on a hyperbola are DX_1=EY_1, DX_2=EY_2, and DX_3= EY_3. From the equation DX_1=EY_1 it follows that

$$X_1X_2 = Y_1Y_2. \tag{6}$$

By considering the butterfly *FGIH* in **Figure 11**, it is obtained

$$EX_3 = DX_3+2\cdot X_3X_2+2\cdot X_1X_2+X_1Y_1. \tag{7}$$

$$X_3X_4 = DX_3 - DX_4. \tag{8}$$

$$EX_1 = DX_3+X_3X_2+X_1X_2+X_1Y_1. \tag{9}$$

$$DX_2 = DX_3+X_3X_2. \tag{10}$$

Substitute equations (6), (7), (8), (9), and (10) into equation (4), thereby obtaining

$$\frac{DX_4 \cdot (DX_3+2\cdot X_3X_2+2\cdot X_1X_2+X_1Y_1)}{(DX_3-DX_4)} = \frac{(DX_3+X_3X_2)\cdot(DX_3+X_3X_2+X_1X_2+X_1Y_1)}{X_1X_2}. \tag{11}$$

By considering the butterfly *JKML* in **Figure 12**, it is obtained

$$DY_3 = DX_3+2\cdot X_3X_2+2\cdot X_1X_2+X_1Y_1. \tag{12}$$

$$Y_3Y_4 = DY_3 - EY_4. \tag{13}$$

$$DY_1 = DX_3+X_3X_2+X_1X_2+X_1Y_1. \tag{14}$$

$$EY_2 = DX_3+X_3X_2. \tag{15}$$

Then, substitute equations (6), (12), (13), (14), and (15) into equation (5), thereby obtaining

$$\frac{EY_4\cdot(DX_3+2\cdot X_3X_2+2\cdot X_1X_2+X_1Y_1)}{(DX_3 - EY_4)} = \frac{(DX_3+X_3X_2)\cdot(DX_3+X_3X_2+X_1X_2+X_1Y_1)}{X_1X_2}. \tag{16}$$

By considering equations (11) and (16), the right-hand sides of both equations are equal, thereby obtaining

$$DX_4 \quad = \quad EY_4. \tag{17}$$

From equation (17), the double butterfly theorem on the hyperbola is thus proven.

4 Conclusion

The double butterfly theorem is a development of the butterfly theorem. The double butterfly theorem, which originally applies to circles, also remains valid for hyperbolas. The development of the double butterfly theorem on hyperbolas is similar to that on circles. However, there is a fundamental difference: in circles, all chords lie inside the circle and the property of inscribed angles can be used without additional constructions. In contrast, for hyperbolas, the chords must be extended and may involve two branches of the curve; therefore, an auxiliary circle construction is required. Subsequently, the proof of the double butterfly theorem on hyperbolas is carried out using a modified version of Haruki's lemma on hyperbolas. Finally, the double butterfly theorem on hyperbolas has the potential to be used in the development of other plane figures, for example, hexagons.

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