New Formula for Interval Numbers with Powers of Positive Rational Numbers and Its Properties

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Abstract. Development of interval numbers have shown various beliefs regarding the arithmetic and the formulas, one of them is regarding interval numbers with powers. However, there is a deficiency regarding further use for the previous formula of interval number $\tilde{a}$ with power k where k is a positive integer, that is the basic characteristic of exponents does not apply such as $\tilde{a}^k \otimes \tilde{a}^l \neq \tilde{a}^{k+l}$. There are some multiplication formulas for interval numbers, but the result of multiplication of an interval number with its invers is not equal to $\tilde{I} = [1,1]$. Therefore, this article will establish a new formula for interval numbers with powers of positive integers using the new formula of multiplication for interval numbers so that $\tilde{a}^k \otimes \tilde{a}^l = \tilde{a}^{k+l}$ apply. Based on this new formula of interval numbers with positive integer powers, formula for interval numbers with fractional powers will also be construct along with the properties that apply to this formula.

Keywords: Interval arithmetic, Powers of interval number

1 Introduction

There is a property of exponents in real numbers such as $a^m \otimes a^n = a^{m+n}$, this basic property underlies other properties therefore this property should also apply in interval numbers such that $\tilde{a}^m \otimes \tilde{a}^n = \tilde{a}^{m+n}$. In order to determine whether this property apply to interval numbers, we need to specify the multiplication formula we are going to use. There are many opinions regarding the multiplication operation that applies to interval numbers. In [1-4], minimum, maximum and some constants are used in the multiplication of two interval numbers. Meanwhile in [6-13], multiplying two interval numbers is sufficient by only using the maximum and minimum. The formulas for multiplication of two interval numbers given in [1-4] and [6-13] will not be used in this article because those still have deficiency such as the multiplication of an interval number and its invers does not generate an identity of interval number and division is undefined if several conditions are not fulfilled. In [14-16], suppose there is an interval number $\tilde{a} = [\underline{a}, \overline{a}]$ then the inverse of $\tilde{a}$ cannot be determined if $\underline{a} < 0$ and $\overline{a} = 0$. In [10-11],[14] and [17], the invers of an interval number $\tilde{a} = [\underline{a}, \overline{a}]$ is undefined if $\underline{a} = 0$ and $\overline{a} = 0$. Whereas in [1-4],[7-9],[14-15] and [17-19], the invers of an interval number $\tilde{a}$ is undefined if

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$0 \in \tilde{a}$. Other opinions regarding the formula for invers of interval numbers are given in [1-4],[14-15],[90-11],[13] and [20], but the results of multiplication of any interval numbers and its invers does not generate identity $\tilde{I} = [1,1]$. In this case, new formula given in [21] will be used based on the definition of positive fuzzy triangular numbers shown in [22-27]. In [21], the general formula of the inverse of an interval number is also given so that the multiplication of the interval and its invers is an identity of interval numbers.

In [1] interval numbers with powers of positive integers are introduced. The general formula for interval number $\tilde{a}$ to the power of k where k is a positive integer are $\tilde{a}^k = \left[\underline{a}^k, \overline{a}^k\right]$ if $\underline{a}^k < \overline{a}^k$ and $\tilde{a}^k = \left[\overline{a}^k, \underline{a}^k\right]$ if $\overline{a}^k < \underline{a}^k$. But, the exponential property does not apply if you use this formula, so that $\tilde{a}^m \otimes \tilde{a}^n \neq \tilde{a}^{m+n}$. Using the multiplication formula given in [21], the exponential property does not apply. In [6], the multiplication formula given in [6-13] is used such as using the maximum and minimum, but even if we used this multiplication formula, the exponential property still does not apply if $\underline{a} \leq 0$. In order for the exponential property applied to interval numbers, we need to determine a new formula for interval numbers with positive integer powers using multiplication operation of interval numbers given in [21]. Based on this new formula for interval numbers with powers of positive integers, we can construct the formula for interval numbers with powers of fractional numbers. Using these new formulas, we can show that the exponential property applies to interval numbers so that $\tilde{a}^m \otimes \tilde{a}^n = \tilde{a}^{m+n}$ for any interval number $\tilde{a}$ and m, n are positive rational numbers.

2 Literature Study

2.1 Algebra of Interval Number

General form of any interval number $\tilde{a}$ is $\tilde{a} = \left[\underline{a}, \overline{a}\right] \in IR$ where $IR = \left\{\tilde{a} = \left[\underline{a}, \overline{a}\right] \mid \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \leq \overline{a}\right\}$. Let $\tilde{a} = \left[\underline{a}, \overline{a}\right]$ dan $\tilde{b} = \left[\underline{b}, \overline{b}\right]$ be some interval numbers where $\tilde{a}, \tilde{b} \in IR^*$ and IR^* defined as $IR^* = \left\{\tilde{x} = \left[\underline{x}, \overline{x}\right] \mid \underline{x}, \overline{x} \in \mathbb{R}\right\}$. In [29], the algebra of interval numbers is shown as follows:

i. $\tilde{a} \oplus \tilde{b} = \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right]$

ii. $\tilde{a} \ominus \tilde{b} = \left[\underline{a} - \overline{b}, \overline{a} - \underline{b}\right]$

iii. $k\tilde{a} = \begin{cases} \left[k\underline{a}, k\overline{a}\right], & k \geq 0 \\ \left[k\overline{a}, k\underline{a}\right], & k < 0 \end{cases}$

iv. $\tilde{a} \otimes \tilde{b} = \left[\underline{a} \cdot m(\tilde{b}) + \underline{b} \cdot m(\tilde{a}) - m(\tilde{a})m(\tilde{b}), \overline{a} \cdot m(\tilde{b}) + \overline{b} \cdot m(\tilde{a}) - m(\tilde{a})m(\tilde{b})\right]$

v. $\frac{\tilde{a}}{\tilde{b}} = \tilde{a} \otimes \frac{1}{\tilde{b}}$

where $\frac{1}{\tilde{b}} = \left[\frac{2 \cdot m(\tilde{b}) - \underline{b}}{m(\tilde{b})^2}, \frac{2 \cdot m(\tilde{b}) - \overline{b}}{m(\tilde{b})^2}\right]$

2.1 Midpoint Theorems

Definition 2.1.1 In [1-8], [11] and [17], midpoint of an interval number $\tilde{a} = \left[\underline{a}, \overline{a}\right]$ is defined as follows

$$m(\tilde{a}) = \frac{\underline{a} + \overline{a}}{a}$$

Theorem 2.1.2 Let $\tilde{a} = [\underline{a}, \overline{a}]$ and $\tilde{b} = [\underline{b}, \overline{b}]$ be any interval numbers, then

1. $m(\tilde{a} \otimes \tilde{b}) = m(\tilde{a}) \cdot m(\tilde{b})$
2. $m(\tilde{a}^n) = m(\tilde{a})^n$
3. $m\left(\frac{1}{\tilde{a}}\right) = \frac{1}{m(\tilde{a})}$

Proof :

1. $m(\tilde{a} \otimes \tilde{b}) = \frac{\underline{a} \cdot m(\tilde{b}) + \underline{b} \cdot m(\tilde{a}) - m(\tilde{a})m(\tilde{b}) + \overline{a} \cdot m(b) + \overline{b} \cdot m(\tilde{a}) - m(\tilde{a})m(\tilde{b})}{2}$

$$= \frac{m(\tilde{b})(\underline{a} + \overline{a}) + m(\tilde{a})(\underline{b} + \overline{b}) - 2m(\tilde{a})m(\tilde{b})}{2}$$
$$= m(\tilde{b})m(\tilde{a}) + m(\tilde{a})m(\tilde{b}) - m(\tilde{a})m(\tilde{b})$$
$$= m(\tilde{a})m(\tilde{b})$$

2. $m\left(\frac{1}{\tilde{a}}\right) = \frac{1}{2}\left(\frac{2 \cdot m(\tilde{a}) - \underline{a}}{m(\tilde{a})^2} + \frac{2 \cdot m(\tilde{a}) - \overline{a}}{m(\tilde{a})^2}\right)$

$$= \frac{2m(\tilde{a})}{2m(\tilde{a})^2} - \frac{\underline{a}}{2m(\tilde{a})^2} + \frac{2m(\tilde{a})}{2m(\tilde{a})^2} - \frac{\overline{a}}{2m(\tilde{a})^2}$$
$$= \frac{2}{m(\tilde{a})} - \frac{m(\tilde{a})}{m(\tilde{a})^2}$$
$$= \frac{1}{m(\tilde{a})}$$

3. $m(\tilde{a}^n) = m(\tilde{a}_1 \otimes \tilde{a}_2 \otimes \cdots \otimes \tilde{a}_n)$

$$= m((\tilde{a}_1 \otimes \tilde{a}_2 \otimes \cdots \otimes \tilde{a}_{n-1}) \otimes \tilde{a}_n)$$
$$= m(\tilde{a}_1 \otimes \tilde{a}_2 \otimes \cdots \otimes \tilde{a}_{n-1}) \cdot m(\tilde{a}_n)$$
$$= m(\tilde{a}_1) \cdot m(\tilde{a}_2) \cdots m(\tilde{a}_{n-1}) \cdot m(\tilde{a}_n)$$
$$= m(\tilde{a})^n$$

3 Result and Discussion

3.1 Intervals Numbers with Positive Integers Powers

In this section, we will show the general form of interval numbers with powers of positive integers and fractions. Then, we will also show several properties that apply to these interval numbers

Theorem 3.1.1 Let $\tilde{a}$ be any interval number and n is positive integer, then

$$\tilde{a}^n = [(n\underline{a}) \cdot m(\tilde{a}^{n-1}) - (n-1) \cdot m(\tilde{a}^n), (n\overline{a}) \cdot m(\tilde{a}^{n-1}) - (n-1) \cdot m(\tilde{a}^n)]$$

Proof :
We will use induction method. Let S be a set where $P(n)$ is true for $n \in \mathbb{N}$. If $n = 1$, then

$$\tilde{a} = [(1 \cdot \underline{a}) \cdot m(\tilde{a})^{1-1} - (1-1) \cdot m(\tilde{a})^{1-1}, (1 \cdot \overline{a}) \cdot m(\tilde{a})^{1-1} - (1-1) \cdot m(\tilde{a})^{1-1}]$$

$\tilde{a} = [\underline{a}, \overline{a}]$

Therefore, $P(1)$ is true dan $1 \in S$. Next, we assume that $P(k)$ is true and wish to infer from this assumption that $P(k+1)$ is also true. If $k \in S$,then

$$\tilde{a}^k = \left[(k\underline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k), (k\overline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\right]$$

If we multiply $\tilde{a}$ to both sides, then

$$\begin{aligned}
\tilde{a}^k \otimes \tilde{a} &= \left[(k\underline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k), (k\overline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\right] \otimes \tilde{a} \\
&= \Big[\Big((k\underline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\Big) m(\tilde{a}) + \underline{a} \cdot m(\tilde{a}^k) - m(\tilde{a}^k) \\
&\qquad \cdot m(\tilde{a}), \big((k\overline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\big) m(\tilde{a}) + \overline{a} \cdot m(\tilde{a}^k) \\
&\qquad - m(\tilde{a}^k) \cdot m(\tilde{a})\Big] \\
&= \big[(k\underline{a}) \cdot m(\tilde{a})^k - (k-1) \cdot m(\tilde{a})^{k+1} + \underline{a} \cdot m(\tilde{a})^k - m(\tilde{a})^{k+1}, (k\overline{a}) \cdot m(\tilde{a})^k \\
&\qquad - (k-1) \cdot m(\tilde{a})^{k+1} + \overline{a} \cdot m(\tilde{a})^k - m(\tilde{a})^{k+1}\big] \\
&= \big[\big(\underline{a} \cdot m(\tilde{a})^k\big)(k+1) - m(\tilde{a})^{k+1}(k-1+1), \big(\overline{a} \cdot m(\tilde{a})^k\big)(k+1) \\
&\qquad - m(\tilde{a})^{k+1}(k-1+1)\big] \\
&= \big[(k+1)\big(\underline{a} \cdot m(\tilde{a})^k\big) - ((k+1)-1)m(\tilde{a})^{k+1}, (k+1)\big(\overline{a} \cdot m(\tilde{a})^k\big) \\
&\qquad - ((k+1)-1)m(\tilde{a})^{k+1}\big] \\
&= \tilde{a}^{k+1}
\end{aligned}$$

Therefore, $P(k+1)$ is true and $(k+1) \in S$. Based on principle of mathematical induction, this formula holds for all $n \in \mathbb{N}$.

Theorem 3.1.2 Let $\tilde{a}$ be any interval number and p, q are positive integers, then

$$\tilde{a}^p \otimes \tilde{a}^q = \tilde{a}^{p+q}$$

Proof :

$$\begin{aligned}
\tilde{a}^p \otimes \tilde{a}^q &= \left[(p\underline{a}) \cdot m(\tilde{a}^{p-1}) - (p-1) \cdot m(\tilde{a}^p), (p\overline{a}) \cdot m(\tilde{a}^{p-1}) - (p-1) \cdot m(\tilde{a}^p)\right] \\
&\qquad \otimes \big[(q\underline{a}) \cdot m(\tilde{a}^{q-1}) - (q-1) \cdot m(\tilde{a}^q), (q\overline{a}) \cdot m(\tilde{a}^{q-1}) - (q-1) \\
&\qquad \cdot m(\tilde{a}^q)\big] \\
&= \Big[\Big((p\underline{a}) \cdot m(\tilde{a}^{p-1}) - (p-1) \cdot m(\tilde{a}^p)\Big) m(\tilde{a}^q) \\
&\qquad + \Big((q\underline{a}) \cdot m(\tilde{a}^{q-1}) - (q-1) \cdot m(\tilde{a}^q)\Big) m(\tilde{a}^p) \\
&\qquad - m(\tilde{a}^p) m(\tilde{a}^q), \big((p\overline{a}) \cdot m(\tilde{a}^{p-1}) - (p-1) \cdot m(\tilde{a}^p)\big) m(\tilde{a}^q) \\
&\qquad + \Big((q\underline{a}) \cdot m(\tilde{a}^{q-1}) - (q-1) \cdot m(\tilde{a}^q)\Big) m(\tilde{a}^p) - m(\tilde{a}^p) m(\tilde{a}^q)\Big]
\end{aligned}$$

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vi. $\tilde{a} \oplus \tilde{b} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$

vii. $\tilde{a} \ominus \tilde{b} = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$

viii. $k\tilde{a} = \begin{cases} [k\underline{a}, k\overline{a}], & k \geq 0 \\ [k\overline{a}, k\underline{a}], & k < 0 \end{cases}$

ix. $\tilde{a} \otimes \tilde{b} = [\underline{a} \cdot m(\tilde{b}) + \underline{b} \cdot m(\tilde{a}) - m(\tilde{a})m(\tilde{b}), \overline{a} \cdot m(\tilde{b}) + \overline{b} \cdot m(\tilde{a}) - m(\tilde{a})m(\tilde{b})]$

x. $\frac{\tilde{a}}{\tilde{b}} = \tilde{a} \otimes \frac{1}{\tilde{b}}$

where $\frac{1}{\tilde{b}} = \left[\frac{2 \cdot m(\tilde{b}) - \underline{b}}{m(\tilde{b})^2}, \frac{2 \cdot m(\tilde{b}) - \overline{b}}{m(\tilde{b})^2}\right]$

2.1 Midpoint Theorems

Definition 2.1.1 In [1-8],[11] and [17], midpoint of an interval number $\tilde{a} = [\underline{a}, \overline{a}]$ is defined as follows

$$m(\tilde{a}) = \frac{\underline{a} + \overline{a}}{a}$$

Theorem 2.1.2 Let $\tilde{a} = [\underline{a}, \overline{a}]$ and $\tilde{b} = [\underline{b}, \overline{b}]$ be any interval numbers, then

4. $m(\tilde{a} \otimes \tilde{b}) = m(\tilde{a}) \cdot m(\tilde{b})$
5. $m(\tilde{a}^n) = m(\tilde{a})^n$
6. $m\left(\frac{1}{\tilde{a}}\right) = \frac{1}{m(\tilde{a})}$

Proof :

4. $m(\tilde{a} \otimes \tilde{b}) = \frac{\underline{a} \cdot m(\tilde{b}) + \underline{b} \cdot m(\tilde{a}) - m(\tilde{a})m(\tilde{b}) + \overline{a} \cdot m(b) + \overline{b} \cdot m(\tilde{a}) - m(\tilde{a})m(\tilde{b})}{2}$

$$= \frac{m(\tilde{b})(\underline{a} + \overline{a}) + m(\tilde{a})(\underline{b} + \overline{b}) - 2m(\tilde{a})m(\tilde{b})}{2}$$
$$= m(\tilde{b})m(\tilde{a}) + m(\tilde{a})m(\tilde{b}) - m(\tilde{a})m(\tilde{b})$$
$$= m(\tilde{a})m(\tilde{b})$$

5. $m\left(\frac{1}{\tilde{a}}\right) = \frac{1}{2}\left(\frac{2 \cdot m(\tilde{a}) - \underline{a}}{m(\tilde{a})^2} + \frac{2 \cdot m(\tilde{a}) - \overline{a}}{m(\tilde{a})^2}\right)$

$$= \frac{2m(\tilde{a})}{2m(\tilde{a})^2} - \frac{\underline{a}}{2m(\tilde{a})^2} + \frac{2m(\tilde{a})}{2m(\tilde{a})^2} - \frac{\overline{a}}{2m(\tilde{a})^2}$$
$$= \frac{2}{m(\tilde{a})} - \frac{m(\tilde{a})}{m(\tilde{a})^2}$$
$$= \frac{1}{m(\tilde{a})}$$

6. $m(\tilde{a}^n) = m(\tilde{a}_1 \otimes \tilde{a}_2 \otimes \cdots \otimes \tilde{a}_n)$

$$= m((\tilde{a}_1 \otimes \tilde{a}_2 \otimes \cdots \otimes \tilde{a}_{n-1}) \otimes \tilde{a}_n)$$
$$= m(\tilde{a}_1 \otimes \tilde{a}_2 \otimes \cdots \otimes \tilde{a}_{n-1}) \cdot m(\tilde{a}_n)$$
$$= m(\tilde{a}_1) \cdot m(\tilde{a}_2) \cdots m(\tilde{a}_{n-1}) \cdot m(\tilde{a}_n)$$
$$= m(\tilde{a})^n$$

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Proof :

We will use induction method. Let S be a set where $P(n)$ is true for $n \in \mathbb{N}$. If $n = 1$, then

$$\tilde{a} = \left[\left(1 \cdot \underline{a}\right) \cdot m(\tilde{a})^{1-1} - (1-1) \cdot m(\tilde{a})^{1-1}, (1 \cdot \overline{a}) \cdot m(\tilde{a})^{1-1} - (1-1) \cdot m(\tilde{a})^{1-1}\right]$$
$$\tilde{a} = \left[\underline{a}, \overline{a}\right]$$

Therefore, $P(1)$ is true dan $1 \in S$. Next, we assume that $P(k)$ is true and wish to infer from this assumption that $P(k+1)$ is also true. If $k \in S$,then

$$\tilde{a}^k = \left[\left(k\underline{a}\right) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k), (k\overline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\right]$$

If we multiply $\tilde{a}$ to both sides, then

$$\begin{aligned}
\tilde{a}^k \otimes \tilde{a} &= \left[\left(k\underline{a}\right) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k), (k\overline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\right] \otimes \tilde{a} \\
&= \Big[\Big(\left(k\underline{a}\right) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\Big) m(\tilde{a}) + \underline{a} \cdot m(\tilde{a}^k) - m(\tilde{a}^k) \\
&\qquad \cdot m(\tilde{a}), \left((k\overline{a}) \cdot m(\tilde{a}^{k-1}) - (k-1) \cdot m(\tilde{a}^k)\right) m(\tilde{a}) + \overline{a} \cdot m(\tilde{a}^k) \\
&\qquad - m(\tilde{a}^k) \cdot m(\tilde{a})\Big] \\
&= \left[\left(k\underline{a}\right) \cdot m(\tilde{a})^k - (k-1) \cdot m(\tilde{a})^{k+1} + \underline{a} \cdot m(\tilde{a})^k - m(\tilde{a})^{k+1}, (k\overline{a}) \cdot m(\tilde{a})^k \right. \\
&\qquad \left. - (k-1) \cdot m(\tilde{a})^{k+1} + \overline{a} \cdot m(\tilde{a})^k - m(\tilde{a})^{k+1}\right] \\
&= \left[\left(\underline{a} \cdot m(\tilde{a})^k\right)(k+1) - m(\tilde{a})^{k+1}(k-1+1), (\overline{a} \cdot m(\tilde{a})^k)(k+1) \right. \\
&\qquad \left. - m(\tilde{a})^{k+1}(k-1+1)\right] \\
&= \left[(k+1)\left(\underline{a} \cdot m(\tilde{a})^k\right) - ((k+1)-1) m(\tilde{a})^{k+1}, (k+1)(\overline{a} \cdot m(\tilde{a})^k) \right. \\
&\qquad \left. - ((k+1)-1) m(\tilde{a})^{k+1}\right] \\
&= \tilde{a}^{k+1}
\end{aligned}$$

Therefore, $P(k+1)$ is true and $(k+1) \in S$. Based on principle of mathematical induction, this formula holds for all $n \in \mathbb{N}$.

Theorem 3.1.2 Let $\tilde{a}$ be any interval number and p, q are positive integers, then

$$\tilde{a}^p \otimes \tilde{a}^q = \tilde{a}^{p+q}$$

Proof :

$$\begin{aligned}\tilde{a}^p \otimes \tilde{a}^q &= \left[\left(p\underline{a}\right)\cdot m(\tilde{a}^{p-1}) - (p-1)\cdot m(\tilde{a}^p), (p\overline{a})\cdot m(\tilde{a}^{p-1}) - (p-1)\cdot m(\tilde{a}^p)\right] \\ &\quad \otimes \left[\left(q\underline{a}\right)\cdot m(\tilde{a}^{q-1}) - (q-1)\cdot m(\tilde{a}^q), (q\overline{a})\cdot m(\tilde{a}^{q-1}) - (q-1) \cdot m(\tilde{a}^q)\right] \\ &= \Big[\Big(\left(p\underline{a}\right)\cdot m(\tilde{a}^{p-1}) - (p-1)\cdot m(\tilde{a}^p)\Big) m(\tilde{a}^q) \\ &\quad + \Big(\left(q\underline{a}\right)\cdot m(\tilde{a}^{q-1}) - (q-1)\cdot m(\tilde{a}^q)\Big) m(\tilde{a}^p) \\ &\quad - m(\tilde{a}^p) m(\tilde{a}^q), \left((p\overline{a})\cdot m(\tilde{a}^{p-1}) - (p-1)\cdot m(\tilde{a}^p)\right) m(\tilde{a}^q) \\ &\quad + \Big(\left(q\underline{a}\right)\cdot m(\tilde{a}^{q-1}) - (q-1)\cdot m(\tilde{a}^q)\Big) m(\tilde{a}^p) - m(\tilde{a}^p) m(\tilde{a}^q)\Big] \\ &= \left[\left(p\underline{a}\right)\cdot m(\tilde{a}^{p-1})\cdot m(\tilde{a}^q) - (p-1)\cdot m(\tilde{a}^p)\cdot m(\tilde{a}^q) + \left(q\underline{a}\right)\cdot m(\tilde{a}^{q-1})\cdot m(\tilde{a}^p)\right. \\ &\quad - (q-1)\cdot m(\tilde{a}^q)\cdot m(\tilde{a}^p) - m(\tilde{a}^p) m(\tilde{a}^q), (p\overline{a})\cdot m(\tilde{a}^{p-1}) \\ &\quad \cdot m(\tilde{a}^q) - (p-1)\cdot m(\tilde{a}^p)\cdot m(\tilde{a}^q) + (q\overline{a})\cdot m(\tilde{a}^{q-1})\cdot m(\tilde{a}^p) \\ &\quad \left. - (q-1)\cdot m(\tilde{a}^q)\cdot m(\tilde{a}^p) - m(\tilde{a}^p) m(\tilde{a}^q)\right] \\ &= \left[\left(p\underline{a}\right)\cdot m(\tilde{a})^{p-1+q} - (p-1)\cdot m(\tilde{a})^{p+q} + \left(q\underline{a}\right)\cdot m(\tilde{a})^{p-1+q} - (q-1)\right. \\ &\quad \cdot m(\tilde{a})^{p+q} - m(\tilde{a})^{p+q}, (p\overline{a})\cdot m(\tilde{a})^{p-1+q} - (p-1)\cdot m(\tilde{a})^{p+q} \\ &\quad \left. + (q\overline{a})\cdot m(\tilde{a})^{p-1+q} - (q-1)\cdot m(\tilde{a})^{p+q} - m(\tilde{a})^{p+q}\right] \\ &= \left[(p+q)\cdot \underline{a} m(\tilde{a})^{p-1+q} - (p+q-1)\cdot m(\tilde{a})^{p+q}, (p+q)\cdot \overline{a} m(\tilde{a})^{p-1+q}\right. \\ &\quad \left. - (p+q-1)\cdot m(\tilde{a})^{p+q}\right] \\ &= \tilde{a}^{p+q}\end{aligned}$$

3.2 Interval Numbers with Positive Fractional Powers

Theorem 3.2.1 Let $\tilde{x}$ be any interval number and a, b are positive integers, then

$$\tilde{x}^{\frac{a}{b}} = \left[\frac{a\underline{x} + (b-a)m(\tilde{x})}{b\cdot m(\tilde{x})^{1-\frac{a}{b}}}, \frac{a\overline{x} + (b-a)m(\tilde{x})}{b\cdot m(\tilde{x})^{1-\frac{a}{b}}}\right]$$

Proof :

We will use induction method. Let S be a set where $P(n)$ is true for $n \in \mathbb{Q}^+$. If $n = 1$, then

$$\begin{aligned}\tilde{x} &= \left[\frac{1\cdot \underline{x} + (1-1)m(\tilde{x})}{1\cdot m(\tilde{x})^{1-1}}, \frac{1\cdot \overline{x} + (1-1)m(\tilde{x})}{1\cdot m(\tilde{x})^{1-1}}\right] \\ \tilde{x} &= \left[\underline{x}, \overline{x}\right]\end{aligned}$$

Therefore, $P(1)$ is true dan $1 \in S$. Next, we assume that $P(k)$ is true and wish to infer from this assumption that $P(k+1)$ is also true. If $k \in S$ where $k = \frac{c}{d}$, then

$$\tilde{x}^{\frac{c}{d}} = \left[\frac{c\underline{x} + (d-c)m(\tilde{x})}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}}, \frac{c\overline{x} + (d-c)m(\tilde{x})}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}}\right]$$

If we multiply $\tilde{x}$ to both sides, then

$$\tilde{x}^{\frac{c}{d}} \otimes \tilde{x} = \left[\frac{c\underline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}, \frac{c\overline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right] \otimes \tilde{x}$$

$$= \left[\left(\frac{c\underline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right) m(\tilde{x}) + \underline{x} \cdot m\left(\tilde{x}^{\frac{c}{d}}\right) - m\left(\tilde{x}^{\frac{c}{d}}\right) \cdot m(\tilde{x}), \left(\frac{c\overline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right) m(\tilde{x}) + \overline{x} \cdot m\left(\tilde{x}^{\frac{c}{d}}\right) - m\left(\tilde{x}^{\frac{c}{d}}\right) \cdot m(\tilde{x})\right]$$

$$= \left[\left(\frac{1}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right) c\underline{x}m(\tilde{x}) + (d-c)m(\tilde{x})^2 + \underline{x}d \cdot m(\tilde{x})^{\frac{c}{d}} m(\tilde{x})^{1-\frac{c}{d}} - d \cdot m(\tilde{x})^{\frac{c}{d}+1} m(\tilde{x})^{1-\frac{c}{d}}, \left(\frac{1}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right) c\overline{x}m(\tilde{x}) + (d-c)m(\tilde{x})^2 + \overline{x}d \cdot m(\tilde{x})^{\frac{c}{d}} m(\tilde{x})^{1-\frac{c}{d}} - d \cdot m(\tilde{x})^{\frac{c}{d}+1} m(\tilde{x})^{1-\frac{c}{d}}\right]$$

$$= \left[\left(\frac{1}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right) c\underline{x}m(\tilde{x}) + (d-c)m(\tilde{x})^2 + \underline{x}d \cdot m(\tilde{x}) - d \cdot m(\tilde{x})^2, \left(\frac{1}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right) c\overline{x}m(\tilde{x}) + (d-c)m(\tilde{x})^2 + \overline{x}d \cdot m(\tilde{x}) - d \cdot m(\tilde{x})^2\right]$$

$$= \left[\left(\frac{(c+d) \cdot \underline{x} + (d-(c+d))m(\tilde{x})}{d \cdot m(\tilde{x})^{-\frac{c}{d}}}\right), \left(\frac{(c+d) \cdot \overline{x} + (d-(c+d))m(\tilde{x})}{d \cdot m(\tilde{x})^{-\frac{c}{d}}}\right)\right]$$

$$= \left[\left(\frac{(c+d) \cdot \underline{x} + (d-(c+d))m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\left(\frac{c+d}{d}\right)}}\right), \left(\frac{(c+d) \cdot \overline{x} + (d-(c+d))m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\left(\frac{c+d}{d}\right)}}\right)\right]$$

$$= \tilde{x}^{\frac{c+d}{d}}$$

$$= \tilde{x}^{\frac{c}{d}+1}$$

Therefore, $P(k+1)$ is true and $(k+1) \in S$. Next, we assume that $P(k-1)$ is also true where $k = \frac{c}{d}$. To prove this, we will use invers formula of interval numbers.

$$\tilde{x}^{\frac{c}{d}} \otimes \tilde{x}^{-1} = \left[\frac{c\underline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}, \frac{c\overline{x} + (d-c)m(\tilde{x})}{d \cdot m(\tilde{x})^{1-\frac{c}{d}}}\right] \otimes \left[\frac{2 \cdot m(\tilde{x}) - \underline{x}}{m(\tilde{x})^2}, \frac{2 \cdot m(x) - \overline{x}}{m(\tilde{x})^2}\right]$$

$$
\begin{aligned}
&= \Bigg[\left(\frac{c\underline{x} + (d-c)m(\tilde{x})}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}}\right)m(\tilde{x}^{-1}) + \left(\frac{2\cdot m(\tilde{x}) - \underline{x}}{m(\tilde{x})^2}\right)m\left(\tilde{x}^{\frac{c}{d}}\right) - m\left(\tilde{x}^{\frac{c}{d}}\right) \\
&\qquad \cdot m(\tilde{x}^{-1}), \left(\frac{c\overline{x} + (d-c)m(\tilde{x})}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}}\right)m(\tilde{x}^{-1}) + \left(\frac{2\cdot m(x) - \overline{x}}{m(\tilde{x})^2}\right)m\left(\tilde{x}^{\frac{c}{d}}\right) \\
&\qquad - m\left(\tilde{x}^{\frac{c}{d}}\right)\cdot m(\tilde{x}^{-1})\Bigg] \\
&= \Bigg[\frac{c\underline{x}\cdot m(\tilde{x}^{-1}) + (d-c)m(\tilde{x})\cdot m(\tilde{x}^{-1})}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}} + \frac{2\cdot m(\tilde{x})m\left(\tilde{x}^{\frac{c}{d}}\right) - \underline{x}\cdot m\left(\tilde{x}^{\frac{c}{d}}\right)}{m(\tilde{x})^2} \\
&\qquad - m\left(\tilde{x}^{\frac{c}{d}}\right)\cdot m(\tilde{x}^{-1}), \frac{c\overline{x}\cdot m(\tilde{x}^{-1}) + (d-c)m(\tilde{x})\cdot m(\tilde{x}^{-1})}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}} \\
&\qquad + \frac{2\cdot m(\tilde{x})m\left(\tilde{x}^{\frac{c}{d}}\right) - \overline{x}\cdot m\left(\tilde{x}^{\frac{c}{d}}\right)}{m(\tilde{x})^2} - m\left(\tilde{x}^{\frac{c}{d}}\right)\cdot m(\tilde{x}^{-1})\Bigg] \\
&= \Bigg[\frac{c\underline{x}\cdot m(\tilde{x}^{-1})m(\tilde{x})^2 + (d-c)m(\tilde{x})m(\tilde{x}^{-1})m(\tilde{x})^2}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}\cdot m(\tilde{x})^2} \\
&\qquad + \frac{2d\cdot m(\tilde{x})m\left(\tilde{x}^{\frac{c}{d}}\right)m(\tilde{x})^{1-\frac{c}{d}} - d\underline{x}\cdot m\left(\tilde{x}^{\frac{c}{d}}\right)m(\tilde{x})^{1-\frac{c}{d}}}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}\cdot m(\tilde{x})^2} \\
&\qquad - \frac{m\left(\tilde{x}^{\frac{c}{d}-1}\right)\cdot d\cdot m(\tilde{x})^{1-\frac{c}{d}}\cdot m(\tilde{x})^2}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}\cdot m(\tilde{x})^2}, \quad \frac{c\overline{x}\cdot m(\tilde{x}^{-1})m(\tilde{x})^2}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}\cdot m(\tilde{x})^2} \\
&\qquad + \frac{(d-c)m(\tilde{x})m(\tilde{x}^{-1})m(\tilde{x})^2}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}\cdot m(\tilde{x})^2} \\
&\qquad + \frac{2d\cdot m(\tilde{x})m\left(\tilde{x}^{\frac{c}{d}}\right)m(\tilde{x})^{1-\frac{c}{d}} - d\overline{x}\cdot m\left(\tilde{x}^{\frac{c}{d}}\right)m(\tilde{x})^{1-\frac{c}{d}}}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}\cdot m(\tilde{x})^2} \\
&\qquad - \frac{m\left(\tilde{x}^{\frac{c}{d}-1}\right)\cdot d\cdot m(\tilde{x})^{1-\frac{c}{d}}\cdot m(\tilde{x})^2}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}\cdot m(\tilde{x})^2}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \Bigg[\frac{c\underline{x}\cdot m(\tilde{x}) + (d-c)m(\tilde{x})^2 + 2d\cdot m(\tilde{x})^2 - d\underline{x}\cdot m(\tilde{x}) - d\cdot m(\tilde{x})^2}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}\cdot m(\tilde{x})^2}, \\
&\frac{c\overline{x}\cdot m(\tilde{x}) + (d-c)m(\tilde{x})^2 + 2d\cdot m(\tilde{x})^2 - d\overline{x}\cdot m(\tilde{x}) - d\cdot m(\tilde{x})^2}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}\cdot m(\tilde{x})^2}\Bigg]
\end{aligned}
$$

$$= \left[\frac{\underline{x} \cdot m(\tilde{x})(c-d) + m(\tilde{x})^2(d-c+2d-d)}{d \cdot m(\tilde{x})^{1-\frac{c}{d}} \cdot m(\tilde{x})^2}, \frac{\overline{x} \cdot m(\tilde{x})(c-d) + m(\tilde{x})^2(d-c+2d-d)}{d \cdot m(\tilde{x})^{1-\frac{c}{d}} \cdot m(\tilde{x})^2}\right]$$

$$= \left[\frac{\underline{x} \cdot (c-d) + m(\tilde{x})\big(d-(c-d)\big)}{d \cdot m(\tilde{x})^{1-\frac{c}{d}} \cdot m(\tilde{x})}, \quad \frac{\overline{x} \cdot m(\tilde{x})(c-d) + m(\tilde{x})^2\big(d-(c-d)\big)}{d \cdot m(\tilde{x})^{1-\frac{c}{d}} \cdot m(\tilde{x})}\right]$$

$$= \left[\frac{\underline{x} \cdot (c-d) + m(\tilde{x})\big(d-(c-d)\big)}{d \cdot m(\tilde{x})^{1-\left(\frac{c}{d}-1\right)}}, \quad \frac{\overline{x} \cdot m(\tilde{x})(c-d) + m(\tilde{x})^2\big(d-(c-d)\big)}{d \cdot m(\tilde{x})^{1-\left(\frac{c}{d}-1\right)}}\right]$$

$$= (\tilde{x})^{\frac{c-d}{d}}$$

$$= (\tilde{x})^{\frac{c}{d}-1}$$

Therefore, $P(k-1)$ is true and $(k-1) \in S$. Based on principle of mathematical induction, this formula holds for all $n \in \mathbb{Q}^{+}$.

Theorem 3.2.2 Let $\tilde{x}$ be any interval number and a, b are positive integers, then

$$m\left(\tilde{x}^{\frac{a}{b}}\right) = m(\tilde{x})^{\frac{a}{b}}$$

Proof :

$$m\left(\tilde{x}^{\frac{a}{b}}\right) = \frac{a\underline{x} + (b-a)m(\tilde{x}) + (b-a)m(\tilde{x})}{2b \cdot m(\tilde{x})^{1-\frac{a}{b}}}$$

$$= \frac{a\big(\underline{x} + \overline{x}\big) + 2(b-a)m(\tilde{x})}{2b \cdot m(\tilde{x})^{1-\frac{a}{b}}}$$

$$= \frac{am(\tilde{x})}{b \cdot m(\tilde{x})^{1-\frac{a}{b}}} + \frac{2bm(\tilde{x})}{2b \cdot m(\tilde{x})^{1-\frac{a}{b}}} - \frac{2am(\tilde{x})}{2b \cdot m(\tilde{x})^{1-\frac{a}{b}}}$$

$$= \frac{a}{b} m(\tilde{x})^{\frac{a}{b}} + m(\tilde{x})^{\frac{a}{b}} - \frac{a}{b} m(\tilde{x})^{\frac{a}{b}}$$

$$= m(\tilde{x})^{\frac{a}{b}}$$

Theorem 3.2.3 Let $\tilde{x}$ be any interval number and a, b, c, d are positive integers, then

$$\tilde{x}^{\frac{a}{b}} \otimes \tilde{x}^{\frac{c}{d}} = \tilde{x}^{\frac{a}{b}+\frac{c}{d}}$$

Proof :

$$
\begin{aligned}
\tilde{x}^{\frac{a}{b}} \otimes \tilde{x}^{\frac{c}{d}} = & \left[\frac{a\underline{x}+(b-a)m(\tilde{x})}{b\cdot m(\tilde{x})^{1-\frac{a}{b}}}, \frac{a\overline{x}+(b-a)m(\tilde{x})}{b\cdot m(\tilde{x})^{1-\frac{a}{b}}}\right] \\
& \otimes \left[\frac{c\underline{x}+(d-c)m(\tilde{x})}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}}, \frac{c\overline{x}+(d-c)m(\tilde{x})}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}}\right] \\
= & \left[\left(\frac{a\underline{x}+(b-a)m(\tilde{x})}{b\cdot m(\tilde{x})^{1-\frac{a}{b}}}\right) m\left(\tilde{x}^{\frac{c}{d}}\right) + \left(\frac{c\underline{x}+(d-c)m(\tilde{x})}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}}\right) m\left(\tilde{x}^{\frac{a}{b}}\right)\right. \\
& - m\left(\tilde{x}^{\frac{a}{b}}\right) m\left(\tilde{x}^{\frac{c}{d}}\right), \left(\frac{a\overline{x}+(b-a)m(\tilde{x})}{b\cdot m(\tilde{x})^{1-\frac{a}{b}}}\right) m\left(\tilde{x}^{\frac{c}{d}}\right) \\
& \left. + \left(\frac{c\overline{x}+(d-c)m(\tilde{x})}{d\cdot m(\tilde{x})^{1-\frac{c}{d}}}\right) m\left(\tilde{x}^{\frac{a}{b}}\right) - m\left(\tilde{x}^{\frac{a}{b}}\right) m\left(\tilde{x}^{\frac{c}{d}}\right)\right] \\
= & \left[\frac{a\underline{x}\cdot m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}}{b\cdot m(\tilde{x})} + \frac{(b-a)m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}}{b} + \frac{c\underline{x}\cdot m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}}{d\cdot m(\tilde{x})} + \frac{(d-c)m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}}{d}\right. \\
& - m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}, \frac{a\overline{x}\cdot m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}}{b\cdot m(\tilde{x})} + \frac{(b-a)m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}}{b} + \frac{c\overline{x}\cdot m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}}{d\cdot m(\tilde{x})} \\
& \left. + \frac{(d-c)m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}}{d} - m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}\right] \\
= & \left[m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}\left(\frac{a\underline{x}}{b\cdot m(\tilde{x})} + \frac{(b-a)}{b} + \frac{c\underline{x}}{d\cdot m(\tilde{x})} + \frac{(d-c)}{d}\right.\right. \\
& \left. - 1\right), m(\tilde{x})^{\frac{a}{b}+\frac{c}{d}}\left(\frac{a\overline{x}}{b\cdot m(\tilde{x})} + \frac{(b-a)}{b} + \frac{c\overline{x}}{d\cdot m(\tilde{x})} + \frac{(d-c)}{d}\right. \\
& \left.\left. - 1\right)\right] \\
= & \left[m(\tilde{x})^{\frac{ad+bc}{bd}}\left(\frac{\underline{x}}{m(\tilde{x})}\left(\frac{a}{b}+\frac{c}{d}\right) + \frac{bd-ad-bc}{bd}\right), m(\tilde{x})^{\frac{ad+bc}{bd}}\left(\frac{\overline{x}}{m(\tilde{x})}\left(\frac{a}{b}+\frac{c}{d}\right)\right.\right. \\
& \left.\left. + \frac{bd-ad-bc}{bd}\right)\right] \\
= & \left[m(\tilde{x})^{\frac{ad+bc}{bd}}\left(\frac{\underline{x}}{m(\tilde{x})}\left(\frac{ad+bc}{bd}\right)\right.\right. \\
& \left. + \frac{(bd-ad-bc)m(\tilde{x})}{bd\cdot m(\tilde{x})}\right), m(\tilde{x})^{\frac{ad+bc}{bd}}\left(\frac{\overline{x}}{m(\tilde{x})}\left(\frac{ad+bc}{bd}\right)\right. \\
& \left.\left. + \frac{(bd-ad-bc)m(\tilde{x})}{bd\cdot m(\tilde{x})}\right)\right] \\
= & \left[\frac{\underline{x}(ad+bc)+(bd-ad-bc)m(\tilde{x})}{bd\cdot m(\tilde{x})^{1-\frac{ad+bc}{bd}}}, \frac{\overline{x}(ad+bc)+(bd-ad-bc)m(\tilde{x})}{bd\cdot m(\tilde{x})^{1-\frac{ad+bc}{bd}}}\right] \\
= & \tilde{x}^{\frac{ad+bc}{bd}} \\
= & \tilde{x}^{\frac{a}{b}+\frac{c}{d}}
\end{aligned}
$$

Corollary 3.2.4 Let $\tilde{a}$ be any interval number and n is positive integer, then

$$(\tilde{a}^n)^{\frac{1}{n}} = \tilde{a}$$

Proof :

$$(\tilde{a}^n)^{\frac{1}{n}} = \left(\left(n\underline{a}\right) \cdot m(\tilde{a}^{n-1}) - (n-1) \cdot m(\tilde{a}^n), (n\overline{a}) \cdot m(\tilde{a}^{n-1}) - (n-1) \cdot m(\tilde{a}^n)\right)^{\frac{1}{n}}$$

$$= \left[\frac{1}{n \cdot m(\tilde{a}^n)^{1-\frac{1}{n}}}\left(\left(n\underline{a}\right) \cdot m(\tilde{a}^{n-1}) - (n-1) \cdot m(\tilde{a}^n) + (n-1) \cdot m(\tilde{a}^n)\right), \frac{1}{n \cdot m(\tilde{a}^n)^{1-\frac{1}{n}}}\left((n\overline{a}) \cdot m(\tilde{a}^{n-1}) - (n-1) \cdot m(\tilde{a}^n) + (n-1) \cdot m(\tilde{a}^n)\right)\right]$$

$$= \left[\frac{n\underline{a} \cdot m(\tilde{a})^{n-1}}{n \cdot m(\tilde{a})^{n\left(1-\frac{1}{n}\right)}}, \frac{n\overline{a} \cdot m(\tilde{a})^{n-1}}{n \cdot m(\tilde{a})^{n\left(1-\frac{1}{n}\right)}}\right]$$

$$= \left[\underline{a}, \overline{a}\right]$$

Corollary 3.2.5 Let $\tilde{a}$ be any interval number and b, c are positive integers, then

$$\left(\tilde{a}^{\frac{b}{c}}\right)^{\frac{c}{b}} = \tilde{a}$$

Proof :

$$\left(\tilde{a}^{\frac{b}{c}}\right)^{\frac{c}{b}} = \left[\frac{b\underline{a} + (c-b)m(\tilde{a})}{c \cdot m(\tilde{a})^{1-\frac{b}{c}}}, \frac{b\overline{a} + (c-b)m(\tilde{a})}{c \cdot m(\tilde{a})^{1-\frac{b}{c}}}\right]^{\frac{c}{b}}$$

$$= \left[\frac{1}{b \cdot m(\tilde{a})^{\frac{b}{c}\left(1-\frac{c}{b}\right)}}\left(\frac{c\left(b\underline{a} + (c-b)m(\tilde{a})\right)}{c \cdot m(\tilde{a})^{1-\frac{b}{c}}} + (b-c)m(\tilde{a})^{\frac{b}{c}}\right), \frac{1}{b \cdot m(\tilde{a})^{\frac{b}{c}\left(1-\frac{c}{b}\right)}}\left(\frac{c(b\overline{a} + (c-b)m(\tilde{a}))}{c \cdot m(\tilde{a})^{1-\frac{b}{c}}} + (b-c)m(\tilde{a})^{\frac{b}{c}}\right)\right]$$

$$
\begin{aligned}
&= \Bigg[\frac{1}{b\cdot m(\tilde{a})^{\frac{b}{c}-1}}\Bigg(\frac{b\underline{a}+(c-b)m(\tilde{a})}{m(\tilde{a})^{1-\frac{b}{c}}} \\
&\qquad - \frac{(c-b)m(\tilde{a})^{\frac{b}{c}}\cdot m(\tilde{a})^{1-\frac{b}{c}}}{m(\tilde{a})^{1-\frac{b}{c}}}\Bigg), \frac{1}{b\cdot m(\tilde{a})^{\frac{b}{c}-1}}\Bigg(\frac{b\overline{a}+(c-b)m(\tilde{a})}{m(\tilde{a})^{1-\frac{b}{c}}} \\
&\qquad - \frac{(c-b)m(\tilde{a})^{\frac{b}{c}}\cdot m(\tilde{a})^{1-\frac{b}{c}}}{m(\tilde{a})^{1-\frac{b}{c}}}\Bigg)\Bigg] \\
&= \Bigg[\Bigg(\frac{b\underline{a}+(c-b)m(\tilde{a})}{b\cdot m(\tilde{a})^{\frac{b}{c}-1+1-\frac{b}{c}}} - \frac{(c-b)m(\tilde{a})^{\frac{b}{c}+1-\frac{b}{c}}}{b\cdot m(\tilde{a})^{\frac{b}{c}-1+1-\frac{b}{c}}}\Bigg), \Bigg(\frac{b\overline{a}+(c-b)m(\tilde{a})}{b\cdot m(\tilde{a})^{\frac{b}{c}-1+1-\frac{b}{c}}} \\
&\qquad - \frac{(c-b)m(\tilde{a})^{\frac{b}{c}+1-\frac{b}{c}}}{b\cdot m(\tilde{a})^{\frac{b}{c}-1+1-\frac{b}{c}}}\Bigg)\Bigg] \\
&= [\underline{a}, \overline{a}]
\end{aligned}
$$

Corollary 3.2.6 Let $\tilde{a}$ is any interval number and b, c are positive integers, then

$$\left(\tilde{a}^{\frac{r}{s}}\right)^{\frac{t}{u}} = \left(\tilde{a}^{\frac{t}{u}}\right)^{\frac{r}{s}}$$

Proof :

$$
\begin{aligned}
\left(\tilde{a}^{\frac{r}{s}}\right)^{\frac{t}{u}} &= \left[\frac{r\underline{a}+(s-r)m(\tilde{a})}{s\cdot m(\tilde{a})^{1-\frac{r}{s}}}, \frac{r\overline{a}+(s-r)m(\tilde{a})}{s\cdot m(\tilde{a})^{1-\frac{r}{s}}}\right]^{\frac{t}{u}} \\
&= \Bigg[\frac{1}{u\cdot m(\tilde{a})^{\frac{r}{s}\left(1-\frac{t}{u}\right)}}\Bigg(\frac{tr\underline{a}+t(s-r)m(\tilde{a})}{s\cdot m(\tilde{a})^{1-\frac{r}{s}}} \\
&\quad + \frac{(u-t)m(\tilde{a})^{\frac{r}{s}}\left(s\cdot m(\tilde{a})^{1-\frac{r}{s}}\right)}{s\cdot m(\tilde{a})^{1-\frac{r}{s}}}\Bigg), \frac{1}{u\cdot m(\tilde{a})^{\frac{r}{s}\left(1-\frac{t}{u}\right)}}\Bigg(\frac{tr\overline{a}+t(s-r)m(\tilde{a})}{s\cdot m(\tilde{a})^{1-\frac{r}{s}}} \\
&\quad + \frac{(u-t)m(\tilde{a})^{\frac{r}{s}}\left(s\cdot m(\tilde{a})^{1-\frac{r}{s}}\right)}{s\cdot m(\tilde{a})^{1-\frac{r}{s}}}\Bigg)\Bigg] \\
&= \Bigg[\Bigg(\frac{tr\underline{a}+t(s-r)m(\tilde{a})}{u\cdot s\cdot m(\tilde{a})^{1-\frac{tr}{us}}} + \frac{s(u-t)m(\tilde{a})}{u\cdot s\cdot m(\tilde{a})^{1-\frac{tr}{us}}}\Bigg), \Bigg(\frac{tr\overline{a}+t(s-r)m(\tilde{a})}{u\cdot s\cdot m(\tilde{a})^{1-\frac{tr}{us}}} \\
&\qquad + \frac{s(u-t)m(\tilde{a})}{u\cdot s\cdot m(\tilde{a})^{1-\frac{tr}{us}}}\Bigg)\Bigg]
\end{aligned}
$$

$$= \left[\left(\frac{tr\underline{a} + m(\tilde{a})(ts - tr + us - ts)}{u \cdot s \cdot m(\tilde{a})^{1-\frac{tr}{us}}}\right), \left(\frac{tr\overline{a} + m(\tilde{a})(ts - tr + us - ts)}{u \cdot s \cdot m(\tilde{a})^{1-\frac{tr}{us}}}\right)\right]$$

$$= \left[\left(\frac{tr\underline{a} + m(\tilde{a})(ru - tr + su - ru)}{u \cdot s \cdot m(\tilde{a})^{1-\frac{tr}{us}}}\right), \left(\frac{tr\overline{a} + m(\tilde{a})(ru - tr + su - ru)}{u \cdot s \cdot m(\tilde{a})^{1-\frac{tr}{us}}}\right)\right]$$

$$= \left[\left(\frac{tr\underline{a} + r(u - t)m(\tilde{a})}{u \cdot s \cdot m(\tilde{a})^{1-\frac{tr}{us}}} + \frac{u(s - r)m(\tilde{a})}{u \cdot s \cdot m(\tilde{a})^{1-\frac{tr}{us}}}\right), \left(\frac{tr\overline{a} + r(u - t)m(\tilde{a})}{u \cdot s \cdot m(\tilde{a})^{1-\frac{tr}{us}}} + \frac{u(s - r)m(\tilde{a})}{u \cdot s \cdot m(\tilde{a})^{1-\frac{tr}{us}}}\right)\right]$$

$$= \left[\frac{1}{s \cdot m(\tilde{a})^{\frac{t}{u}\left(1-\frac{r}{s}\right)}}\left(\frac{tr\underline{a} + r(u - t)m(\tilde{a})}{u \cdot m(\tilde{a})^{1-\frac{t}{u}}} + \frac{(s - r)m(\tilde{a})^{\frac{t}{u}}\left(u \cdot m(\tilde{a})^{1-\frac{t}{u}}\right)}{u \cdot m(\tilde{a})^{1-\frac{t}{u}}}\right), \frac{1}{s \cdot m(\tilde{a})^{\frac{t}{u}\left(1-\frac{r}{s}\right)}}\left(\frac{tr\overline{a} + r(u - t)m(\tilde{a})}{u \cdot m(\tilde{a})^{1-\frac{t}{u}}} + \frac{(s - r)m(\tilde{a})^{\frac{t}{u}}\left(u \cdot m(\tilde{a})^{1-\frac{t}{u}}\right)}{u \cdot m(\tilde{a})^{1-\frac{t}{u}}}\right)\right]$$

$$= \left[\frac{t\underline{a} + (u - t)m(\tilde{a})}{u \cdot m(\tilde{a})^{1-\frac{t}{u}}}, \frac{t\overline{a} + (u - t)m(\tilde{a})}{u \cdot m(\tilde{a})^{1-\frac{t}{u}}}\right]^{\frac{r}{s}}$$

$$= \left(\tilde{a}^{\frac{t}{u}}\right)^{\frac{r}{s}}$$

Using the multiplication formula given in [21], interval numbers can be raised to different positive integer powers to obtain the pattern of $\tilde{a}^n$ where n is any positive integers. Based on this pattern, a new formula can be formed for interval numbers with positive integer powers. If we using the previous formula of interval numbers with powers of positive integers which was given in [6], one of the basic properties in exponents does not apply such as $\tilde{a}^m \otimes \tilde{a}^n \neq \tilde{a}^{m+n}$. But, by using the new formula for interval numbers with positive integer powers, it has been shown above that $\tilde{a}^m \otimes \tilde{a}^n = \tilde{a}^{m+n}$ where m, n are positive integers so that the exponential property apply to interval numbers with powers of positive integers. Furthermore, using the new formula of interval numbers with powers of positive integers, we can construct the formula for interval numbers with powers of fractional numbers. Let say $\tilde{a}^{\frac{1}{n}} = \tilde{b}$, then $\left(\tilde{a}^{\frac{1}{n}}\right)^n = \tilde{b}^n$. Use the new formula for $\tilde{b}^n$ and calculate to find $\underline{b}$ and $\overline{b}$. Therefore, we got the formula for interval numbers with powers of fractional numbers $1/n$ or the formula for $\tilde{a}^{\frac{1}{n}}$ is obtained. Then, we can determine the formula for $\tilde{a}^{\frac{m}{n}}$ by calculate $\left(\tilde{a}^{\frac{1}{n}}\right)^m$. It can be shown that the exponential property and several other properties also apply to this formula of interval numbers with positive rational powers.

4 Conclusion

Based on the multiplication formula we used, a pattern is obtained to determined the general form of interval numbers with positive integer powers and it has been shown that the exponential property applies to this formula. Using this formula for interval numbers with positive integer powers, a general formula for interval numbers with fractional powers is formed. To show that the formula is valid, it is also shown that the exponential properties apply to the formula. By using the formulas we provided above, future research endeavors can explore about the convergence of sequences of interval numbers with positive integers and fractional powers and the properties that apply to those sequences.

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