A Series of Inverse of Ideals in Discrete Valuation Domains (DVR)

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Abstract. This article discusses some properties related to fractional ideals of valuation domains and their duals. These properties are used to prove the conjecture that there is a series of inverses of ideals in a DVR which factors are simple modules over the DVR. The conjecture arises because there is a property which states that every two ideals of valuation domains can be comparable and in addition, these ideals form a series. While there is also another property in PID, that is every ideal of PID has an inverse. By considering more properties of valuation domains and PID it can be proven that the inverses of ideals of DVR can be comparable and form a series which factors are simple modules.

Keywords: dual of ideals, discrete valuation domains, fractional ideals, PID, simple modules, valuation domains.

1 Introduction

In [1] and [2] valuation domains are domains which have a property that every two its ideals are comparable. That is to say, if I, J are two ideals of a valuation domain R, then $I \subseteq J$ or $J \subseteq I$. In general, valuation domains are not PID and vice versa. PID is a domain in which every its ideal can be generated by an element [3]. There is an example of a domain which is a valuation domain but not a PID. Note the domain below.

$$O = \cup_k O_k, k = 0,1,2,\cdots \tag{1}$$

where

$$\begin{aligned} O_k &= K_{k_{P_k}} \\ &= \left\{ \frac{f\left(x^{\frac{1}{2^k}}\right)}{g\left(x^{\frac{1}{2^k}}\right)} \middle| f\left(x^{\frac{1}{2^k}}\right) \in K_k,\ g\left(x^{\frac{1}{2^k}}\right) \in K_k - P_k \right\} \end{aligned} \tag{2}$$

K is a field with transcendental $x, K_k = K\left[x^{\frac{1}{2^k}}\right]$ and P_k is a maximal ideal in K_k with $x^{\frac{1}{2^k}}$ as its generator. Based on [4] the domain O is a valuation domain but not a PID because it is non-Noetherian domain (a domain R is called Noetherian domain if every ascending chain of its ideals terminates at an ideal of the chain [5]. It means, non-Noetherian domains do not satisfy

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that condition). This is because base on [6] a PID must be a Noetherian domain. The proof detail of the example can be seen in [7].There is also another example for the latter statement. Domain $\mathbb{Z}$ is a PID but not a valuation domain because there are two ideals $2\mathbb{Z}$ and $3\mathbb{Z}$ of $\mathbb{Z}$ which are not comparable. In fact, based on [8] and [9] there is a domain which is a valuation domain as well as a PID at once. This domain is called a DVR. There are some examples of DVR in [10]. One of the example is domain $\mathbb{Z}_P = \left\{\frac{a}{b}\middle|a, b \in \mathbb{Z}, b \notin P\right\}$ in which P is any prime ideal of $\mathbb{Z}$ (an ideal P of a domain R is a prime ideal if $ab \in P$ then $a \in P$ or $b \in P$ [11]).Because a DVR is a PID and a valuation domain, then every ideal of DVR is generated by an element and moreover, every two ideals of DVR are comparable so that there is a series of the ideals.

In [14], it is stated that every ideal of PID has an inverse. Therefore, it will be discussed, whether those properties of idealsof DVR, which have been mentioned above, can be extended to their inverses.

2 Methods

It is known that ideals of DVR form a series and moreover every ideal also has an inverse. Therefore, there arises a conjecture that there is also a series of inverses of ideals. This statement is one of the main results of this article. To prove the conjecture, there are some results which are developed from previous properties and these results are used in the process of main result's proof. Another main result is connected to factors of the series of inverses of ideals. These factors are simple modules in which this property is inherited from factors of series of ideals. It is because there is no module between inverses of two ideals.

3 Result and Discussion

Definition 1
Let R be a domain and K is its quotient field. A fractional ideal of R is an $R-$submodule I of K such that $aI \subseteq R$ for some $a \in R$ [12].

A trivial example of a fractional ideal is an ideal of R. Note that every fractional ideal is an $R-$submodule of K. However, not all $R-$submodule of K is a fractional ideal. For example, consider $\mathbb{Q}$ and $\mathbb{R}$ be $\mathbb{Z}-$modules. We can see that $\mathbb{Q}$ is an $\mathbb{Z}-$submodule of $\mathbb{R}$, but it is not a fractional ideal of $\mathbb{Z}$ because there is no $a \in \mathbb{Z}$ such that $a\mathbb{Q} \subseteq \mathbb{Z}$.

Definition 2
Let R be a domain, K is a quotient field of R and I is a fractional ideal of this domain. The dual of this fractional ideal is a set $I^{-1} = \left\{\frac{a}{b} \in K\middle|\frac{a}{b}I \subseteq R\right\}$ [13].

This dual set is not an empty set because at least $R \subseteq I^{-1}$. Moreover, I^{-1} is an $R-$submodule of K. Base on [14] if this fractional ideal has the property that $II^{-1} = R$, then this fractional ideal is invertible and I^{-1} becomes its inverse. The example of a dual of an ideal $\mathbb{Z}4$ is the set $I^{-1} = \left\{\frac{a}{4}\middle|a \in \mathbb{Z}\right\}$. These two sets can be seen as a $\mathbb{Z}-$submodule of $\mathbb{Q}$.

Lemma 3
Let I, Jbe ideals of a domain R. If $I \subseteq J$, then $J^{-1} \subseteq I^{-1}$.
Proof.
Let $\frac{a}{b}$belongs toJ^{-1}. Since$I \subseteq J$, $\frac{a}{b}I \subseteq \frac{a}{b}J \subseteq R$. By the definition of dual, $\frac{a}{b} \in I^{-1}$. Hence we have $J^{-1} \subseteq I^{-1}$.

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In valuation domains, the set of ideals, as well as, the set of duals are totally ordered as follows.

Corollary 4
Let R be a valuation domain and$X = \{I^{-1} | I \text{ is an ideal of } R\}$. Then X is a totally ordered set by inclusion.
Proof.
By Lemma 3.

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Corollary 5
Let M be the maximal ideal of a valuation domain R. Then M^{-1} is the minimal R −submodule of K in X. Furthermore, this minimal submodule is unique.
Proof.
Let I is any ideal in R. Then $I \subseteq M$ and based on lemma 3, $M^{-1} \subseteq I^{-1}$. Hence M^{-1} is a minimal R −submodule in X.
Now let M' becomes another minimal R −submodule of K in X. We know that X is totally ordered set. Then $M' \subseteq M^{-1}$ and at the same time, by the minimality of M^{-1}, we have $M^{-1} \subseteq M'$. Therefore, $M' = M^{-1}$.

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The converse of Lemma 3 satisfies if the ideals are invertible. Because of the invertibility of ideals, 1 can be expressed as a linear combination of elements of I^{-1} with coefficients in I.

Lemma 6
Let I, J be invertible ideals of a domain R. If $I^{-1} \subseteq J^{-1}$ then $J \subseteq I$.
Proof.
Let $A = \{a \in R | aI^{-1} \subseteq R\}$. It can be shown that A is an ideal of R. Further,$I \subseteq A$. Now let $a \in A$. Note that

$$\begin{aligned} a &= a.1 \\ &= a\left(\sum_{i=1}^{n} \alpha_i b_i\right) \end{aligned} \tag{3}$$

where $\alpha_i \in I^{-1}, b_i \in I$. Hence we have

$$a\left(\sum_{i=1}^{n} \alpha_i b_i\right) = \sum_{i=1}^{n} \underbrace{\underbrace{(a\alpha_i)}_{\in R} b_i}_{\in I}. \tag{4}$$

Accordingly, $a \in I$. It shows that$A = I$. We can now proceed analogously to the proof of $B = J$ where $B = \{b \in R | bJ^{-1} \subseteq R\}$.
Let $b \in J$. Since$I^{-1} \subseteq J^{-1}$,$bI^{-1} \subseteq bJ^{-1} \subseteq R$. It shows$b \in A = I$. Therefore we have $J \subseteq I$.

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Assume I be an ideal of a DVR R. Then I must be a principal ideal, that is $I = Ra$ for some $a \in R$. We follow [15] in assuming that a can be written uniquely as $a = up^k$ for some $k \in \mathbb{Z}_{\geq 0}, u$ is a unit and p is a prime element of R. Then $I = Rp^k$. We have thus proved that there is no proper ideal between Rp^{n+1} and Rp^n for any $n \in \mathbb{Z}_{\geq 0}$. As the consequence of this result, we have this series which has factors as simple R −modules below

$$0 \subseteq \cdots \subseteq Rp^2 \subseteq Rp \subseteq R. \tag{5}$$

Take for an example there is a discrete valuation domain $F[\![x]\!]$ in [16], which is a set of power series with the coefficients in F. In this domain, there is a series of ideals of $F[\![x]\!]$

$$\cdots \subseteq (x^2) \subseteq (x) \subseteq (1) = F[\![x]\!] \tag{6}$$

in which the factors are simple $F[\![x]\!]$ −modules.

Note that for ideal Rp^n of$R$ with $n \in \mathbb{Z}_{\geq 0}$, the inverse of this ideal is $R\frac{1}{p^n}$. Hence, if we have $Rp^{n+1} \subseteq Rp^n$, then based on lemma 3 we have $R\left(\frac{1}{p^n}\right) \subseteq R\left(\frac{1}{p^{n+1}}\right)$. Now we consider some properties of R −submodule of K between $R\left(\frac{1}{p^n}\right)$ and $R\left(\frac{1}{p^{n+1}}\right)$.

Lemma 7
Let R be a DVR and K is its quotient field. If there is an R −submodule of K, namelyN', which satisfies

$$R\left(\frac{1}{p^n}\right) \subset N' \subset R\left(\frac{1}{p^{n+1}}\right) \tag{7}$$

for $n \in \mathbb{Z}_{\geq 0}$, then N' is a fractional ideal. Moreover, the dual of N' is $N = \{r \in R | rN' \subseteq R\}$.
Proof.
Note that$p^{n+1}N' \subseteq p^{n+1}R\left(\frac{1}{p^{n+1}}\right) = R$. Hence N' is a fractional ideal of R. Because a DVR is a PID, then every nonzero fractional ideal of R is invertible. ThusN' has a dual.Let N be a dual of N'.Our next claim is that $N = \bar{N}$ where $\bar{N} = \{r \in R | rN' \subseteq R\}$.
Note that for any element r of $\bar{N}$, we have

$$r = 1.r$$

$$= \left(\sum_{i=1}^{k} \alpha_i \beta_i\right) r, \text{ for some } \alpha_i \in N, \beta_i \in N'$$
$$= \sum_{i=1}^{k} \alpha_i \underbrace{(\beta_i r)}_{\in R}. \qquad (8)$$

Since$N$ is an R −submodule of K, we have $r = \sum_{i=1}^{k} \alpha_i \underbrace{(\beta_i r)}_{\in R} \in N$. Therefore $\overline{N} \subseteq N$.

Let $y \in N'$ where $y \in R\left(\frac{1}{p^{n+1}}\right) - R\left(\frac{1}{p^n}\right)$. This element must be exist since $R\left(\frac{1}{p^n}\right) \subset N' \subset R\left(\frac{1}{p^{n+1}}\right)$. We note that $y = \frac{u}{p^{n+1}}$ with u is a unit of$R$. If u is a non unit element of R, we have $u = vp^k$ with v is a unit of R and $k \in \mathbb{Z}_{>0}$. Thus $y = \frac{v}{p^{n+1-k}} \in R\left(\frac{1}{p^n}\right)$. This contradicts our assumption that $y \notin R\left(\frac{1}{p^n}\right)$. Now let $x$be an arbitrary element of N. Then $xy \in R$. Note that

$$x\frac{u}{p^{n+1}} = xy \in R$$
$$x\frac{u}{p^{n+1}} = r, \text{ for some } r \in R \qquad (9)$$
$$x = u^{-1} r p^{n+1} \in R.$$

Hence $N \subseteq \overline{N}$. Therefore we have $N = \overline{N}$.

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Definition 8
Let M be an R −module. Module M is said to be simple if it has no non-trivial submodules[17].

Some examples of simple modules are a field (which is a module over itself) and$\mathbb{Z}_p$ as a $\mathbb{Z}$ −module in which p is a prime number as can be seen in [18] and [19]. Another example is one-dimensional vector spaces as explained in [20].

Theorem 9
If R is a DVR, then $R\left(\frac{1}{p^{n+1}}\right)/R\left(\frac{1}{p^n}\right)$ is an R −module simple.

Proof.
Suppose that $R\left(\frac{1}{p^{n+1}}\right)/R\left(\frac{1}{p^n}\right)$ is not a simple R −module. Then there exist a porper R −submodule N' such that $R\left(\frac{1}{p^n}\right) \subset N' \subset R\left(\frac{1}{p^{n+1}}\right)$. Bylemma 11, the inverse of N' is $N = \{r \in R | rN' \subseteq R\}$. This set is an ideal of R and satisfies $Rp^{n+1} \subseteq N \subseteq Rp^n$. Consequently, either$N = Rp^{n+1}$ or $N = Rp^n$. If $N = Rp^{n+1}$, then $N' = R\left(\frac{1}{p^{n+1}}\right)$. It contradicts the fact that $R\left(\frac{1}{p^n}\right) \subset N' \subset R\left(\frac{1}{p^{n+1}}\right)$. Therefore,$R\left(\frac{1}{p^{n+1}}\right)/R\left(\frac{1}{p^n}\right)$ is an R −module simple.

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Corollary 9

In DVR, the series

$$R \subseteq R\frac{1}{p} \subseteq R\frac{1}{p^2} \subseteq \cdots \subseteq K. \tag{10}$$

has factors which are simple R −modules

Proof.
Let R be a DVR. Base on theorem 9, $R\left(\frac{1}{p^{n+1}}\right)/R\left(\frac{1}{p^n}\right)$ is an $R-$ module simple for every n. Therefore, the factors of series (10) are $R-$ module simple.

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4 Conclusion

We know that all ideals in a valuation domain can be ordered so that they makes a totally ordered set. This property is inherited to the set of its dual. Thus, the duals set of idealsof a valuation domain forms a totally ordered set. Moreover, in a special case of a valuation domain, which is in a DVR, we have an inverse series of ideals of this domain. This series is

$$R \subseteq R\frac{1}{p} \subseteq R\frac{1}{p^2} \subseteq \cdots \subseteq K$$

where $R\left(\frac{1}{p^{n+1}}\right)/R\left(\frac{1}{p^n}\right)$ are simple R −modules.

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