Study of the Structural Relationship Between the Addresses of Nodes of an Interconnection Network

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Abstract. In an interconnection network, the structural relationship between addresses of nodes often resonates with the network's topology, directing how data is transferred between nodes. This type of structural relationship is essential for efficient routing and communication. In this paper, we studied the structural relationship between the addresses of nodes and vectors of a hypercube interconnection network. We found that the relationship between the addresses works beautifully as it works in the vector analysis of the nodes of the hypercube; this is an isomorphic behavior of the interconnection network.

Keywords: Hypercube (HC), Structural Relationship, Hamming Distance, Binary Relation, Tripod, Address of Nodes

1 Introduction

In Parallel and Distributed Systems, an interconnection network is a topology of interconnected processing elements called nodes/processors. It is sketched by a structure, where each node is connected to exactly *n* other nodes based on the unique addressing of the nodes. The structure is usually used to expedite communication and data exchange between nodes/processors [1] [2]. The addresses of nodes can affect the performance of the interconnection network. The structural relationships of nodes play an important role in simplifying routing, improving efficiency, and reducing latency [3]. In this paper, we explored the structural relationship between the nodes of a Hypercube (HC) interconnection network [4]. The addressing of the nodes of a hypercube is a power set of 2^n, which can be represented by binary numbers; therefore, it is called the binary complete set of a hypercube. Assuming that the nodes are in Hamming distance, and finding the characteristics of addressing with Hamming distance between the nodes of a hypercube. Each node has tripod connectivity with degree 3, having a Hamming distance [19]. We will find the structural relation between the adjacency nodes of the hypercube so that binary relations between the nodes can be derived. A binary relation determines the connection or relationship between the nodes/processors inside the interconnection network. The relationship can be represented as an ordered pair *(a, b)*, where *a* and *b* represent nodes. We assumed that a node is connected to itself, but the connection of a node to itself does not make any difference in the binary relations of tripod connectivity. A tripod connectivity has three connections between four nodes, with the Hamming distance, or

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we can say that the tripod connectivity between nodes is the Hamming distance in binary relations. The Hamming distance of the addressing of connected nodes behaves as it holds binary relations between nodes on the counterpart of the binary relation of addressing nodes.

We have used binary relation properties (reflexive, symmetric, antisymmetric, and transitive relations) to study the relations between the nodes of the hypercube and found that the binary relations of the addressing nodes and the binary relations of vectors of the connectivity matrix remain the same [15]. Finally, to verify and validate the study, the tautological statements have been used, which play an important role in the study of the binary relation of an interconnection network. The tautology is a well-defined formula and rules of inference, where the logical statement may be either true or false [21]. The tautology is based on mathematical logical operations.

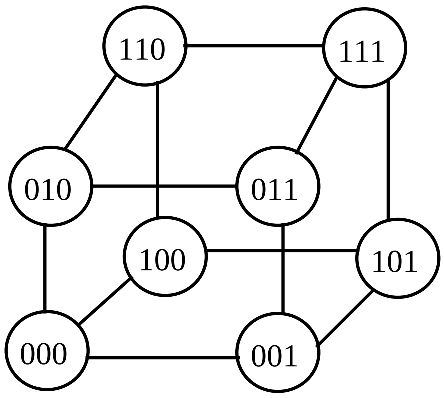

Fig. 1. Hypercube with degree 3.

The binary addresses in the above hypercube in Fig 1 are defined as $000, 001, 010, 011, 100, 101, 110, 111$.

2 Connectivity Matrix of the Hypercube

The connectivity matrix is used to show the interconnections between the nodes of an interconnection network. Each row and column (basically, cell $[i, j]$) represents a potential interconnection between two or more nodes. The hypercube connectivity matrix is defined as:

$$CM_{ij} = \begin{cases} 1, \text{if connection exists between those two nodes} \\ 0, \text{if no connection} \end{cases} \quad (1)$$

The following is the hypercube connectivity matrix based on Fig 1:

Table 1. The Hypercube Connectivity Matrix.

	000	001	010	011	100	101	110	111
000	0	1	1	0	1	0	0	0
001	1	0	0	1	0	1	0	0
010	1	0	0	1	0	0	1	0
011	0	1	1	0	0	0	0	1
100	1	0	0	0	0	1	1	0

101	0	1	0	0	1	0	0	1
110	0	0	1	0	1	1	0	0
111	0	0	0	1	0	1	1	0

Let n be 3 in the connectivity matrix of a Hypercube interconnection network. In Fig 1, there are 8 tripods and modules; all modules in the hypercube have the modular properties called modularity. Below lemma that proves the modular properties of the tripod of hypercube interconnection networks.

Lemma 1: A tripod is a modular lattice in the addressing of the nodes in the Hypercube for binary relations.

Proof: Let's define a tripod as a modular lattice in the following manner:

$$c \leq a) \Rightarrow (a \wedge (b \vee c)) \Leftrightarrow ((a \wedge b)\ \vee c) \tag{2}$$

Considering the addressing value of the hypercube from Fig 1, for the tripod node address 000.

$$(100 \Rightarrow 001) \Rightarrow (001 \wedge (010 \vee 100)) \Leftrightarrow ((001 \wedge 010) \vee 100) \tag{3}$$

Hence, above (2) shows the tautology that holds the modularity in the tripod of the addressing of the hypercube for binary relations. In the same fashion, we can find the tautology for other tripods, and they also hold modularity in the tripods of the hypercube.

Lemma 2: The degree of each node in an *n*-dimensional hypercube is *n* neighbours, where *n* is 3.

Proof: Let us consider the addresses of nodes in Fig 1, where we flip every *n* bits in the binary address. After each flip, a unique neighbor is found as a result, which differs in exactly one bit. In the hypercube, node 000 has exactly three other neighbors, namely 001, 010, and 100, which are tripod members of the current node. Hence, each node has a degree *n* corresponding to 3 neighbors in 3 dimensions and one neighbor per dimension.

Lemma 3: In an *n*-dimensional hypercube interconnection network, the total number of edges is $n * 2^{n-1}$

Proof: The addressing of the nodes of the hypercube is the power set of 2^n**,** each node has *n* edges, as elaborated in Lemma 2. Now the total sum of edge degree is $2*2^n$, since edges are shared between 2 nodes, so that the edges of the Hypercube (HC) will be:

$$The\ total\ number\ of\ edges = \frac{n*2^n}{2} = n * 2^{n-1} \tag{4}$$

From Lemma 2, considering $n = 3$ and the total number of nodes of the hypercube is 8 nodes, then in this case:

$$The\ total\ number\ of\ edges = n * 2^{n-1} \tag{5}$$

$$3 * 2^{3-1} = 12$$

The total number of edges in Fig 1 is 12, which can also be validated through Table 1. Hence, the lemma holds.

3 Structural Relationship of the vectors and addresses of nodes

Table 2 shows the structural relationships and transition matrix between the connectivity of addresses of nodes of an Interconnection Network, where we represent relationships with the help of properties of binary relations.

Table 2. Representation of the transition matrix for a binary relation.

	000	001	010	011	100	101	110	111
000	<000, 000>	<000, 001> **1**	<000, 010> **1**	<000, 011>	<000, 100> **1**	<000, 101>	<000, 110>	<000, 111>
001	<001, 000> **1**	<001, 001>	<001, 010>	<001, 011> **1**	<001, 100>	<001, 101> **1**	<001, 110>	<001, 111>
010	<010, 000> **1**	<010, 001>	<010, 010>	<010, 011> **1**	<010, 100>	<010, 101>	<010, 110> **1**	<010, 111>
011	<011, 000>	<011, 001> **1**	<011, 010> **1**	<011, 011>	<011, 100>	<011, 101>	110	<011, 111> **1**
100	<100, 000> **1**	<100, 001>	<100, 010>	<100, 011>	<100, 100>	<100, 101> **1**	<100, 110> **1**	<100, 111>
101	<101, 000>	<101, 001> **1**	<101, 010>	<101, 011>	<101, 100> **1**	<101, 101>	<101, 110>	<101, 111> **1**
110	<110, 000>	<110, 001>	<110, 010> **1**	<110, 011>	<110, 100> **1**	<110, 101>	<110, 110>	<110, 111> **1**
111	<111, 000>	<111, 001>	<111, 010>	<111, 011> **1**	<111, 100>	<111, 101> **1**	<111, 110> **1**	<111, 111>

Lemma 4: The shortest path between two nodes (e.g. 000,001) in a hypercube interconnection network is equal to the Hamming distance between the addresses of the nodes, and it follows the symmetric property.

Proof: By considering the flipping of binary bits from Lemma 2, let's make the address of node a into the address of node b and vice versa, at the hamming distance we flip all bits, here every flip is one link. This conversion is nothing but a symmetrical relation that follows the rules of equality. That is if $a = b \text{ } then \text{ } b = a$.

Example 1

$(000 \wedge 001) \rightarrow (001 \wedge 000) = 111$ (Tautology)

Here we get a tautology that verifies the symmetrical property. Hence, Lemma holds. The binary relation properties are elaborated in the sections below.

3.1 Reflexive relationship

A reflexive relation is a type of binary relation on a set where every element in the set is related to itself [10]. This means that if a hypercube node is connected to itself or a self-loop connection is called a reflexive node, it is a fundamental property of graph theory [11]. A binary relation R in a set is reflexive if, for every $a \in X,\ a\ R\ a, < a, a >$ that is:

$$X \longrightarrow (a\ R\ a) \text{ or } X \longrightarrow (a\ \wedge\ a) \quad (6)$$

Example 2

$$001 \Rightarrow (001\ \wedge\ 001) = 111 \quad (7)$$

Employing implication get 111, which is a tautology; hence, it holds that the reflexive property.

3.2 Symmetrical relationship

In a hypercube interconnection network "Symmetric" refers to those nodes that have the same connectivity ornament, as all other nodes in the network, meaning it is connected to the same number of neighbouring nodes with identical links characteristics, essentially displaying a

stabilized and consistent structure throughout the network, essentially, no single node has a different connection ornament compared to others [5] [8] [9]. A relation R in a set X is symmetric if, for every a and b in X, whenever $a\ R\ b$, then $b\ R\ a$. That is:

$$(a\ R\ b) \rightarrow b\ R\ a)\ or\ (a \wedge b) \rightarrow (b \wedge a) \quad (8)$$

Example 3

$$(10010100 \wedge 10010010) \rightarrow (10010010 \wedge 10010100) = 11111111 \text{ (Tautology)} \quad (9)$$

The symmetric property also holds for the addresses of nodes in the Interconnection Network.

$$(001 \wedge 010) \rightarrow (010 \wedge 001) = 111 \text{ (Tautology)} \quad (10)$$

Lemma 5: The binary relation of addressing of the tripods in Hamming distance and the binary relation of vectors of the connectivity matrix remain the same.

Proof: The addressing of nodes is the power set of 2^n of n nodes. Here n it is 3, so each node has three neighbours, and the neighbours of each node are in Hamming distance; therefore, we call each node a tripod, such as node 000 has three neighbours, namely $001, 010, 100$, node 001 has three neighbours $000, 011, 101$, node 010 has three neighbours $000, 011, 110$, node 011 has three neighbours $001, 010, 111$, node 100 has three neighbours $000, 101, 110$, node 101 has three neighbours $001, 100, 111$, node 110 has three neighbours $010, 100$,111 and node 111 has three neighbours $011, 101, 110$.

We can say that each node is connected to n neighbours, and the address of the node is a binary complete set. We successively derived the Hamming distance of three bits of n numbers, and we found that the first set of addresses has a Hamming distance of four elements in the set $e.g.\{000, 001, 010, 100\}$. The second step is to find out the Hamming distance of the second element of the set 001 as $000, 011, 101$, etc., and so on. We have the following set without repeating the previous nodes, and we can write the nodes with the Hamming distance.

Table 3: Successive Hamming distance.

1	2	3	4	5	6	7
000	001	010	100	011	101	110
001	011	011	101	111	111	111
010	101	110	110			
100						

From Table 3, We can derive the Hamming distance by excluding 111 from the sets 5, 6, and 7 as 011, 101, and 110. Here, in each node, there is a difference of exactly 1 bit. The symmetrical binary relation in section 3.2 of example 3, it is proving that the vector connectivity, i.e., a $R\ b$ and b $R\ a$, is similar to the relation in the addresses of nodes in the Interconnection Network. Both vectors of the hypercube and the addressing of nodes of the hypercube show tautology; hence, the lemma holds.

3.2.1 Symmetry in topology

In network topology [14], Symmetry refers to the availability of self-similar structures or ornaments inside a network where predictable nodes and edges are functionally equivalent because of shared structural aspects. The overall structure of the hypercube is symmetrical, which allows for potent routing and data transfer due to certain connectivity ornaments as shown in Table 1.

3.2.2. Importance of symmetry in an interconnection network [9]

The Symmetrical networks facilitate simpler routing algorithms for better load balancing and improved fault catholicity. The importance of symmetry in an interconnection network is given below:

- Uniform Connectivity: All node has the same degree (number of connections) to the other nodes.
- No positions are special: No node has a particular induction compared to others.
- Simplified routing: With symmetric nodes, finding the shorter path between any two nodes becomes easier due to the predictable structure.
- Productive load balancing: Symmetrical networks can disseminate workload evenly across nodes, improving overall performance.
- Enhanced Fault catholicity: If any node fails, the network can often re-route data through other nodes because of the steady connectivity.

3.3 Antisymmetric relationship

Anti-symmetric means relating to a relation that implies equality of any two quantities for which it holds in both directions. It means the communication between a and b holds in both directions [16].

Let $a = 001$ and $b = 001$ be binary addresses of two nodes, then antisymmetric represented as:

$$(a \wedge b) \wedge (b \wedge a) \Rightarrow (a = y) \quad (11)$$

In the context of the Hamming distance between 000 and 001 there is exactly one bit difference in the addressing of nodes.

3.4 Transitive relationship

In an interconnection network, a "transitive" refers to the addresses of the node that, if connected to another node which is further connected to a third node, implies a connection between the first and third node as well [6] [7] [17]. The addresses of the node $000, 001$ & 100 are transitive means the tripod is transitive.

Let $a, b,$ and c are the nodes of a Hypercube then transitivity can be defined as:

$$(a \wedge b) \wedge (b \wedge c) \Rightarrow (a \wedge c) \quad (12)$$

The above explanation shows that the tripod is a lattice because the architecture shows the properties of a binary relation.

- The tripods are modular lattices due to the properties of reflexivity, symmetry & transitivity holding.
- The tripods pursue the properties of an equivalence relation because the properties of reflexivity, Anti-symmetry, and transitivity hold.
- The property of connex is also pursued by tripods.

The substantiation of the binary relation between vectors of the connectivity matrix and addresses of nodes proves that the architecture contains properties of a modular lattice [12] [13].

4 Conclusion

In this study, the connectivity of nodes is shown in the connectivity matrix of the hypercube. We found that the binary relation of addressing of the tripod in Hamming distance and the binary relation of vectors of the connectivity matrix remain the same. The tripod is a modular lattice in the addressing of hypercube and binary relations have been proven by applying binary operations, and proves that the shortest path between two nodes in a hypercube is equal to the Hamming distance between the addresses of the nodes.

5 Future Scope

This study will allow us to investigate the reliability of the system in convoluted computational environments and help to develop competent, cost-effective algorithms. In the future, with the help of this study and by embedding various AI methods and machine learning applications, the parallel execution of computation can be predicted earlier, which will efficiently reduce the processing and communication complexities. Various new interconnection network properties are expected, which may be useful for further research.

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