Mathematical Modeling of an Economic Cycle Model with Harmonic and Random Drivers

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Abstract. Applied mathematics has been widely applied in various fields of science and engineering. Among these applications, the study of economic cycles is a very important branch. It is because that an economic system usually develops in a repetitive manner of expansion and contraction under many deterministic and random factors. It exhibits a complex phenomenon of nonlinearity and randomness. In this paper, a nonlinear economic cycle model is proposed to consider government adjustment and market uncertainty. The periodic income adjustment from government is applied to the model as a harmonic function with income. Meanwhile, Gaussian white noise is employed to model market uncertainty. The response of the nonlinear economic cycle model is investigated on its probability density function (PDF) using a path integral method. The short-time Gaussian approximation technique is introduced to evaluated the transition PDF between the two successive time intervals, and the Gauss-Legendre integration scheme is applied to the path integration. Two cases are considered in the following analysis. One is about the high-level government adjustment. The other is about the high-level market uncertainty. The results show that the joint PDF of income and income growth rate will reach a stable state and it periodically evolve along with time. Increasing the adjustment magnitude and increasing random interference significantly reduces the peak values of income and income change rate probability distributions, increasing their dispersion.

Key words. mathematical modeling, economic cycle model, harmonic adjustment, uncertainty

1. Introduction

Economic systems exhibit complex behaviors due to nonlinearity and uncertainty, which has to be treated in a mathematical and physical manner [1]. Especially, economic systems fluctuate along with time between expansion and contraction. It is called as an economic cycle studied by economic cycle theory. There are quite a few methods and approaches to investigate various economic cycle models. By obtaining the inherent mechanisms of economic cycles, we can better understand the development of economics, effectively control economic cycle variation and keep economic development smooth.

At the early stage, economic cycle models were formulated according to the "multiplier" principle and the "accelerator" principle, respectively [2,3]. After that, in order to consider more complicated factors, the "multiplier-accelerator" principle was proposed and applied to some more complicated economic models. For example, Goodwin was one of pioneers to

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develop a class of nonlinear "multiplier-accelerator" economic cycle models [4]. The Hicks's consumption function was employed as well as an identity relationship between the consumption function and the investment function. Subsequently, Goodwin established a dynamical economic cycle model under no autonomous investment and consumption. In this direction, Puu and Sushko developed the idea of Goodwin, introduced a cubic investment function, and proposed a Samuelson-Hicks economic cycle model [5]. The bifurcation and chaotic behavior were investigated in details. They also make a further step by adding a quadratic investment function to consider asymmetric response of economics [6]. It can capture more general nonlinear effects and simulate countercyclical measures governments may take during economic downturns to yield returns with lower inputs. More recent years, the study on the economic cycle model has attracted more attentions from different aspects such as the non-stationary response of economic cycle models [7], bifurcation and chaos issues [8], the autonomous function being triangular periodic functions [9]. Based on the two-dimensional Van der Pol oscillator, the interaction was studied on its stability between economies in a business cycle model, considering the foreign capital inflow, household savings, and the gross domestic product [10]. The bifurcations were also studied analytically and numerically on a Kaldor-type business cycle model with discrete-time [11]. A general equilibrium business cycle model was proposed and studied with imperfectly observed neutral and investment-specific technology shocks [12]. The Goodwin model was further developed to considering how capital accumulation generates contradictions that may reproduce never-ending cycles of booms and slumps [13]. The effects of income distribution and economic cycles was studied in an open-economy super-multiplier model [14]. Model uncertainty was addressed on the economic development and welfare costs of business cycles [15]. A statistical Keynesian-type business-cycle model was studied on connecting heterogeneous micro investment behaviors with macro dynamics [16]. An elementary macroeconomic model with animal spirits was developed and studied with national income and aggregate investment [17]. Endogenous Keynesian business cycles was investigated on economic policy, such as income distribution, investment, aggregate demand and output [18].

In a word, the behavior of economic cycle models has attracted more attention from many scholars in recent years. Especially, the random factors play an important role to have an influence on economic cycles. However, the case simultaneously under government adjustment and market uncertainty is less addressed. In this paper, an economic cycle model is developed according to the work of Sushko et al. [6]. The government adjustment and market uncertainty are modelled, respectively. The government adjustment is modelled by parametric periodic functions on income. This can simulate the periodic adjustment of the government. On the other hand, Gaussian white noise is also employed to simulate market uncertainty. After that, a path integral method, including the short-time Gaussian approximation technique and the Gauss-Legendre integration scheme, is introduced to obtain the probability density function (PDF) of the nonlinear economic cycle models. The effects of the high-level government adjustment and the high-level market uncertainty are further studied in the numerical analysis.

2. Economic cycle model under periodic adjustment and random disturbance

The economic cycle model is introduced herein. It follows the methodology of Goodwin and Puu [7]. The investment function is a nonlinear function of the difference between two successive intervals.

$$I_t = I_{0t} + \nu(Y_{t-1} - Y_{t-2}) + \mu(Y_{t-1} - Y_{t-2})^2 - \nu(Y_{t-1} - Y_{t-2})^3 \tag{1}$$

where I_t represents current investment, I_{0t} represents spontaneous investment, ν represents the capital-output ratio ($\nu > 0$), μ is the symmetry-adjusting factor and Y_t, Y_{t-1}, Y_{t-2} represent current, previous, and pre-previous period incomes, respectively. Meanwhile, the consumption function is given below

$$C_t = C_{0t} + (1-s)Y_{t-1} + \varepsilon s Y_{t-2} \tag{2}$$

where C_t represents current consumption, C_{0t} represents spontaneous consumption, s represents the supplementary saving rate ($0 \le s \le 1$), and ε represents the savings utilization rate from two periods ago ($0 \le \varepsilon \le 1$). Herein, the economic system is assumed to be close. In such a case, all income is used for investment and consumption, i.e.,

$$Y_t = I_t + C_t \tag{3}$$

where $O^*(t) = I_{0t} + C_{0t}$ represents the total of spontaneous investment and consumption. Herein, the autonomous function *O*(t)* is given below

$$O^*(t) = \beta\xi(t) + \gamma Y\cos(\omega t) \tag{4}$$

where $\xi(t)$ is Gaussian white noise to model market uncertainty. $E[\xi(t)] = 0$, $E[\xi(t)\xi(t+\tau)] = D\delta(\tau)$, β is the random disturbance coefficient, and γ is the income adjustment coefficient. The government adjustment is modeled a parametric periodic function on income, simulating a periodic adjustment disturbance to income.

The study adopts the research approach proposed by Li et al. [9] to employ the corresponding equivalent differential equation instead of the difference equation (3). The equivalent differential equation is given below

$$\ddot{Y} + (1+s-\nu)\dot{Y} - \mu\dot{Y}^2 + \nu\dot{Y}^3 + (1-\varepsilon)sY = \beta\xi(t) + \gamma Y\cos(\omega t) \tag{5}$$

3. Path integral method

According to equation (5), the joint PDF $p(\cdot)$ of the economic cycle model on state vector $\mathbf{x}(t)$ is expressed in the following form:

$$p(\mathbf{x}^{(i)},t_i)=\int_{R_s} p(\mathbf{x}^{(i-1)},t_{(i-1)})q(\mathbf{x}^{(i)},t_i \mid \mathbf{x}^{(i-1)},t_{(i-1)})\ \mathrm{d}\mathbf{x}^{(i-1)} \tag{6}$$

The path integral method, including short-time Gaussian transition probability density approximation and Gauss-Legendre integration scheme, is employed to numerically solve equation (6). The continuous integration of $p(\cdot)$ is discretized in space and time. The path summation is used to replace continuous integration [19]. The short-time transition PDF between adjacent intervals is approximated by a short-time Gaussian probability distribution [20]. The PDF $p(\cdot)$ at each time interval is obtained step by step.

Firstly, equation (6) is discretized and evaluated by the Gauss-Legendre integration scheme. The short-time transition probability density $q(\cdot)$ between adjacent interval is approximated by a short-time Gaussian probability distribution below

$$p(x_r^i,y_s^i,t_i)=\frac{\Delta_x}{2}\frac{\Delta_y}{2}\sum_{k=1}^{2n}\sum_{l=1}^{2m} p(x_k^{(i-1)},y_l^{(i-1)},t_{(i-1)})\times q(x_r^i,y_s^i,t_i \left| x_k^{(i-1)},y_l^{(i-1)},t_{(i-1)}\right.) \tag{7}$$

where n and m represent the number of intervals discretized on the x and y axes, respectively, and Δ_x and Δ_y correspond to the interval lengths. Two Gaussian integration points at each interval are used in the Gauss-Legendre integration scheme.

Short-time Gaussian transition probability density function (PDF) is given below [20]

$$\begin{aligned} q(x_r^{(i)},y_s^{(i)},t_i \mid x_k^{(i-1)},y_l^{(i-1)},t_{i-1}) &= \frac{1}{2\pi\sigma_1(t_i)\sigma_2(t_i)\sqrt{1-\rho^2(t_i)}}\times\exp\left\{\frac{-1}{2(1-\rho^2(t_i))}\left(\frac{(x_r^{(i)}-m_{10}(t_i))^2}{\sigma_1^2(t_i)}\right.\right. \\ &\left.\left. -2\rho(t_i)\frac{(x_r^{(i)}-m_{10}(t_i))(y_s^{(i)}-m_{01}(t_i))}{\sigma_1(t_i)\sigma_2(t_i)}+\frac{(y_s^{(i)}-m_{01}(t_i))^2}{\sigma_2^2(t_i)}\right)\right\} \end{aligned} \tag{8}$$

where $\sigma_1^2(t_i)$, $\sigma_2^2(t_i)$, $\rho(t_i)$ are standard deviation and correlation coefficient, respectively. $m_{ij}=\mathrm{E}[x^i y^j]$. Short-time Gaussian transition PDFs can be obtained by using the Gaussian closure method to solve the first and second moments of the system variables.

Letting $x=Y$, $y=\dot{Y}$, equation (6) is then transformed into a system of first-order differential equations:

$$\begin{cases} \dot{x}=y \\ \dot{y}=-(1+s-\nu)y+\mu y^2-\nu y^3-(1-\varepsilon)sx+\beta\xi(t)+\gamma x\cos(\omega t) \end{cases} \tag{9}$$

Corresponding moment equations are given as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}\left[M\right]=\mathrm{E}[y\frac{\partial M}{\partial x}]+\mathrm{E}\left\{[-(1+s-\nu)y+\mu y^2-\nu y^3-(1-\varepsilon)sx+\gamma x\cos(\omega t)]\frac{\partial M}{\partial y}\right\}+\frac{1}{2}D\mathrm{E}\left\{\beta^2\frac{\partial^2 M}{\partial y^2}\right\} \tag{10}$$

where $M=x^i y^j$, (i, j =0,1,2).

The Gaussian closure method is utilized to close equation (10). The higher-order moments (above the second moment) can be expressed by the first and second moments: The resulting equations can be directly solved:

$$\begin{cases} \dot{m}_{10} = m_{01} \\ \dot{m}_{01} = -(1+s-\nu)m_{01} + \nu\mu m_{02} + 2\nu m_{01}^3 - 3\nu m_{01}m_{02} - (1-\varepsilon)sm_{10} + \gamma m_{10}\cos(\omega t) \\ \dot{m}_{20} = 2m_{11} \\ \dot{m}_{11} = m_{02} - (1+s-\nu)m_{11} + \nu\mu m_{10}m_{02} - 2\nu\mu m_{10}m_{01}^2 + 2\nu\mu m_{11}m_{01} - 3\nu m_{11}m_{02} \\ \quad +2\nu m_{10}m_{01}^3 - (1-\varepsilon)sm_{20} + \gamma m_{20}\cos(\omega t) \\ \dot{m}_{02} = -2(1+s-\nu)m_{02} + 6\nu\mu m_{01}m_{02} - 4\nu\mu m_{01}^3 - 6\nu m_{02}^2 + 4\nu m_{01}^4 - 2(1-\varepsilon)sm_{11} \\ \quad +\beta^2 D + 2\gamma m_{11}\cos(\omega t) \end{cases} \tag{11}$$

Using each Gaussian integration point coordinate as the initial solution for equation (11). The fourth-order Runge-Kutta algorithm is employed to obtain the first and second moments, which are then substituted into equation (8) to obtain the short-time Gaussian transition probability density. By iteratively solving equation (7) using Gauss-Legendre integration, the PDFs are obtained.

4. Numerical analysis

Three cases of typical economic parameters are considered. Case 1 is the benchmark example is employed for comparison. Monte Carlo simulation (MCS) is also conducted to evaluate the accuracy of the path integral method (PIS). For equation (5), s=0.2，ν=0.2，μ=0.2，ε=0.1，β=1.0. The other parameters are listed in table 1. Equation (11) has periodic functions. Therefore, each time interval is divided into four segments in an average sense, obtaining four sets of transition PDFs. They are stored for subsequent repeated intervals. Monte Carlo samples consist of 100,000 for each case.

Table 1. The values of system parameters

Case	Adjustment frequency ω	Adjustment coefficient γ	Disturbance Intensity D	Remarks
Case 1	0.2	0.1	0.2	Benchmark
Case 2	0.2	0.3	0.2	High adjustment
Case 3	0.2	0.1	0.4	High disturbance

The initial probability distribution is:

$$p^0(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{-\frac{(x-m_x)^2}{2\sigma_x^2} - \frac{(y-m_y)^2}{2\sigma_y^2}\right\} \tag{12}$$

where m_x=0, m_y=0, σ_x=0.4, σ_y=0.4.

4.1. High income cycle adjustment amplitude

This section considers increasing the amplitude of high-income cycle adjustments, with γ increasing from 0.1 to 0.3. The integration interval is [-8,8]×[-4,4], each divided into 40 grids. Figures 1(a)-1(d) show the probability density function distribution at the end of the fifth cycle, with numerical solutions obtained through the path integral method (PIS) aligning well with Monte Carlo simulation values (MCS). Compared with case 1 (PIS-1 and MCS-1), increasing the amplitude of income cycle adjustments results in a decrease in the peak of the income probability density function, from 0.45 to 0.28. Meanwhile, Figure 1(b) shows a significant increase in the dispersion of the income PDF. Examination of the income growth rate in figures 1(c)-1(d) reveals that increasing the amplitude of income cycle adjustments significantly reduces the peak of the income growth rate probability and increases its dispersion. Overall, increasing the adjustment amplitude significantly reduces the peaks of both income and income growth rate probability and increases the dispersion of their distributions.

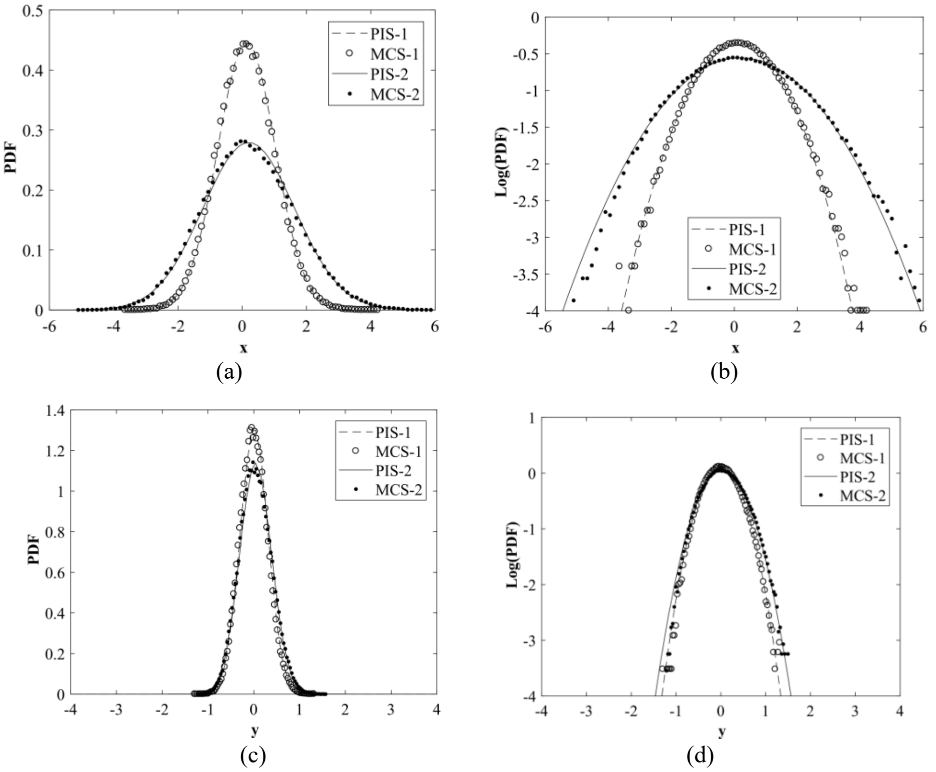

Figure 1. Comparison of PDFs in case 1: (a) PDFs of income (x); (b) logarithmic PDFs of income (x); (c) PDFs of income growth rate (y); (d) logarithmic PDFs of income growth rate (y)

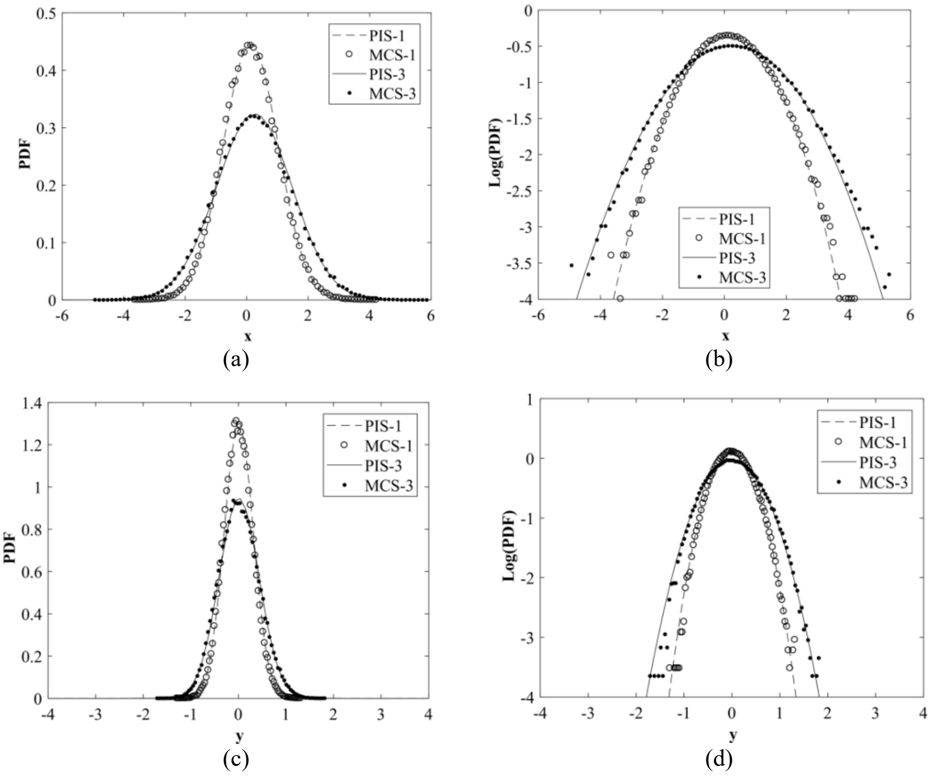

Figure 2. Comparison of PDFs in case 2: (a) PDFs of income (x); (b) logarithmic PDFs of income (x); (c) PDFs of income growth rate (y); (d) logarithmic PDFs of income growth rate (y)

4.2. High random disturbance intensity

This section considers high random disturbance intensity, increasing from 0.2 to 0.4. The integration interval remains [-8,8]×[-4,4], each divided into 40 grids. Figures 2(a)-2(d) display the probability density function distribution at the end of the fifth cycle, with numerical solutions obtained through the path integral method matching well with Monte Carlo simulation values. Compared with case 1 (PIS-1 and MCS-1), increasing the intensity of high random disturbances results in a decrease in the peak of the income probability density function, from 0.45 to 0.32. Additionally, figure 2(b) shows a significant increase in the dispersion of the income probability density function. Examination of the income growth rate in figures 4(c)-4(d) reveals that increasing the intensity of high random disturbances significantly reduces the peak of the income growth rate probability and increases its dispersion. Overall, increasing random disturbances significantly reduces the peaks of both income and income growth rate probability and leads to a more dispersed distribution of income and income growth rate probabilities.

5. Conclusions

This paper considers a nonlinear economic cycle model simultaneously under government adjustment and market uncertainty. The periodic income adjustment from government is applied to the model as a harmonic function with income. Meanwhile, Gaussian white noise is employed to model market uncertainty. A path integral method is employed to investigate the PDF of the nonlinear economic cycle model. It is observed that the joint PDF of income and income growth rate will reach a stable state after several cycles. Subsequently, it periodically evolves along with time. Furthermore, the increase of the adjustment magnitude significantly reduces the peak values of income and income growth rate probability distributions, increasing their dispersion. The increase of random disturbance also significantly reduces the peak values of income and income growth rate probability distributions, leading to further dispersion.

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