A Comparative Study of Heston Pricing Model and BS Pricing Model–Based on 50 ETF Call Options

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Abstract. The Heston model is an extension of the BS model. Among them, the volatility is no longer assumed to be constant and the variance follows a stochastic process. This paper compares the theoretical basis of the Heston model with the BS model, and options are priced and contrasted with ideal prices by using these two pricing models to find out which is better. The empirical analysis shows that Heston model has smaller errors for option pricing.

Keywords: Heston model, BS model, Pricing of options

1 Introduction

The Heston Model, introduced by Steven Heston in 1993, stands as a pivotal advancement in options pricing theory, particularly in capturing the dynamics of financial markets with greater fidelity than the Black-Scholes Model. Where the Black-Scholes Model assumes constant volatility, an assumption at odds with empirical observations, the Heston Model innovatively incorporates stochastic volatility, a key feature that reflects the fluctuating nature of market volatility.

At its core, the Heston Model articulates a dynamic relationship between asset prices and their corresponding volatilities, allowing for a more nuanced understanding of options pricing dynamics. Unlike the simplistic assumption of constant volatility in the Black-Scholes Model, the Heston Model posits that both asset prices and volatilities evolve stochastically over time, acknowledging the inherent uncertainty and variability in financial markets.

Central to the Heston Model's formulation are the stochastic differential equations (SDEs) governing the evolution of asset prices and volatilities. While the asset price dynamics in the Heston Model resemble those of the Black-Scholes Model, following a geometric Brownian motion, the inclusion of a stochastic process for volatility sets it apart. Volatility in the Heston Model adheres to a mean-reverting process, characterized by a square-root diffusion, wherein volatility tends to gravitate towards a long-term average, reflecting observed market behavior more accurately.

Furthermore, the Heston Model considers the correlation between asset price changes and changes in volatility, recognizing the intertwined nature of these two factors in driving options prices. This correlation feature allows the model to capture the co-movement between asset

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prices and volatilities, enhancing its predictive power and applicability in various market conditions.

In contrast to the Black-Scholes Model's simplicity, the Heston Model's incorporation of stochastic volatility imbues it with greater flexibility and robustness. By accounting for time-varying volatility, the Heston Model better accommodates the complexities of real-world market dynamics, making it a preferred choice for options pricing and risk management, especially in volatile or turbulent market environments.

In summary, while the Black-Scholes Model laid the groundwork for options pricing theory, the Heston Model represents a significant leap forward by embracing stochastic volatility. Through its nuanced depiction of asset price and volatility dynamics, the Heston Model offers a more comprehensive and realistic framework for understanding and pricing financial derivatives, underscoring its enduring relevance in modern finance.

2 Literature Review

2.1 Ito Lemma[1]

Ito Lemma was put forward by Japanese mathematician Ito in 1944. Ito Lemma is an important result describing the differential operation rules of stochastic processes. If there is a stochastic process X_t, which can be described by the stochastic differential equation $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$, where $\mu(t, X_t)$ is the drift term, $\sigma(t, X_t)$ is the volatility term, and dW_t is the differential of a Brownian motion. Then, the Ito lemma states that if $f(t,x)$ is a function with continuous partial derivatives, then the differential form of the process $Y_t = f(t, X_t)$ is given by:

$$dY_t = (\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2})dt + \sigma\frac{\partial f}{\partial x}dW_t$$

Here, $\frac{\partial f}{\partial t}$ represents the partial derivative of f with respect to time, $\frac{\partial f}{\partial x}$ represents the partial derivative of f with respect to x, and $\frac{\partial^2 f}{\partial x^2}$ represents the second partial derivative of f with respect to x.

2.2 Option-Pricing Model

2.2.1 BS Model[2]

In 1973, American mathematician Black and economist Scholes proposed the BS formula of European stock option pricing without paying dividends under some idealized assumptions, that is:

$$C(S,t) = S\phi(d_1) - Ke^{-r(T-t)}\phi(d_2)$$

Where $d_1 = \frac{\ln(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$, S is the stock price, $\phi(U)$ is the cumulative distribution function of normal distribution. Tis the maturity date, t is the present time, r is a risk-free interest rate. The premise of their conclusion is that the underlying asset price obeys the geometric Brownian motion: $ds_t = \mu s_t dt + \sigma s_t dW_t$, where W_t is a Brownian motion. However, the hypothesis in the BS model: the return rate of the underlying asset obeys the normal distribution, the risk-free interest rate is constant, and the volatility rate is constant, which is obviously inconsistent with the actual situation. In addition, a fatal flaw of the BS formula is that it can only be priced for European options. The BS formula is directly used to price off options and other exotic options.

2.2.2 CIR Model[3]

Cox, Intersoll and Ross(1985)abandoned the assumption that the risk-free interest rate is constant in BS model and proposed CIR Model:

$$dr_t = \alpha(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

Where α , σ , θ are positive constants, $2\alpha\theta > \sigma^2$. The interest rate determined by r_t this equation is positive, thus avoiding the occurrence of negative interest rate. From the model, we can see that the volatility of interest rate is a monotonically increasing function of $\sigma\sqrt{r_t}$, which increases with the increase of r_t , is no longer a constant, and is closer to the heteroscedasticity of the real interest rate. However, the problem of CIR model is that it is difficult to use the model to estimate the parameters, and it can not explain the term structure of interest rate well.

2.2.3 Heston Model [4]

Heston (1993) obtained the following stochastic differential equation model by using the CIR model:

$$\begin{cases} dS(t) = \mu S dt + \sigma S dw_1(t) \\ d\sigma(t) = -\beta\sigma(t)dt + \delta dw_2(t) \end{cases}$$

Where $dw_i(t)$, $i = 1,2$ is the Brownian motion under the realistic measure. The variance therefore follows the following SDE:

$$dv(t) = \left[\delta^2 - 2\beta v(t)\right]dt + 2\delta\sqrt{v(t)}dw_2(t)$$

We believe that $Cov(dw_1(t), dw_2(t)) = \rho$ then any derivative $U(s,v,t)$ satisfies the following PDE:

$$\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2}+\rho\sigma vS\frac{\partial^2 U}{\partial S\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}+rs\frac{\partial U}{\partial S}+\{\kappa[\theta-v(t)]-\lambda(s,v,t)\}-ru+\frac{\partial U}{\partial T}=0$$

$\lambda(s,v,t)=\lambda v$ is risk of volatility, λ is a constant. We have the following formula:

$$U(s,v,T)=\max\{0,S-K\},U(0,v,T)=0$$

Call option price:

$$C(s,v,t)=SP_1-KP(t,T)P_2$$

Let $x=\ln S$, substitute the above solution into the PDE, we have:

$$\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2}+\rho\sigma v\frac{\partial^2 P_j}{\partial x\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 Pj}{\partial v^2}+(r+u_j v)\frac{\partial P_j}{\partial x}+(a_j-b_j)v\frac{\partial P_j}{\partial v}+\frac{\partial P_j}{\partial t}=0$$

$$u_1=\frac{1}{2},u_2=-\frac{1}{2},a=\kappa\theta,b_1=\kappa+\lambda-\rho\sigma,b_2=\kappa+\lambda$$

Where P_j satisfies the boundary value condition as follows:

$$P_j=(x,v,T;\ln K)=I_{\{x\geq\ln K\}}$$

$x(t)$ satisfies the following SDE:

$$dx(t)=\left[r+u_j v\right]dt+\sqrt{v(t)}dw_1(t)$$

$$dv=(a_j-b_j v)dt+\sigma\sqrt{v(t)}dw_2(t)$$

Then P_j satisfies the following conditional probability form:

$$P_j(x,v,T;\ln K)=P_r(x(T)\geq\ln K\mid x(t)=x,v(t)=v)$$

Solving the conditional probability is transformed into solving its characteristic function, and its characteristic function is defined as $f_j(x,v,T;\phi)$, which satisfies the following final value conditions:

$$f_j(x,v,T;\phi)=e^{C(T-t;\phi)+D(T-t)\phi v+i\phi x}$$

$$C(\tau,\phi)=r\phi i\tau+\frac{a}{\sigma^2}\left\{(b_j-\rho\sigma\phi i+d)\tau-2\ln\frac{1-ge^{d\tau}}{1-g}\right\}$$

$$D(\tau,\sigma)=\frac{b_j-\rho\sigma\phi i+d}{\sigma^2}\frac{1-e^{d\tau}}{1-ge^{d\tau}}$$

And

$$g=\frac{b_j-\rho\sigma\phi i+d}{b_j-\rho\sigma\phi i-d}$$

$$d = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)}$$

From this, we use the inverse Fourier transform and get the result:

$$P_j(x, v, T; \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi \ln K} f_j(x, v, T; \phi)}{i\phi}\right] d\phi$$

We can also use the two-dimensional Girsanov theorem to derive the SDE of the stock price S_t and the volatility v_t under the risk-neutral measure, which is not expanded here[5].

The Heston model greatly enriches and develops the stochastic volatility model, which can better characterize the market volatility. In addition, the model derivation has a theoretical basis, which considers both the mean reversion process of volatility and the wave. Volatility, stock price correlation, in line with market observations. The literature shows that the model has European option pricing. The semi-explicit solution of the stochastic volatility model is found for the first time, which is more suitable than the past stochastic volatility model. Use value [6].

3 Empirical Analysis

3.1 Data Selection

This article selects 50 ETF subscription options that expire in March, April, June, and September 2024. Their theoretical option prices are calculated by BS model and Heston model and compared with the market price (March 8, 2024), and the calculation errors under the two methods are obtained. The daily closing price of the Shanghai 50 ETF from January 4, 2016 to January 4, 2023 is selected as the underlying asset price, and the logarithmic yield is calculated. The daily closing price of the option is the actual market price of the option.

3.2 Descriptive Statistical Analysis

We perform descriptive statistics on the above data and the results are shown in the table below.

Table 1: Descriptive Statistics

mean value	variance	standard deviation	skewness	kurtosis	jarque-bera test
0.000473	0.000234	0.0153	-0.4768	9.7367	2982.3468

Conduct descriptive statistical analysis on the daily logarithmic returns of the Shanghai Stock Exchange 50ETF and use Ljung Box Q test for data autocorrelation. The logarithmic return is $R_t = \ln P_t - \ln P_{t-1}$, where P_t is the daily closing price on day t. As shown in Table 1, the average daily return of the Shanghai Stock Exchange 50ETF approaches 0, with a kurtosis value of 9.7367 greater than 3, showing a significant peak feature; The skewness is -0.4768 and is less than 0, showing a left skewness with left trailing characteristics; The statistical value of the J-B test is much greater than 0, which is significantly inconsistent with the normal distribution.

3.3 Calculation of the Theoretical Price of Options

The 240-day closing price of ETF50 subscription options on March 8 is selected to calculate its volatility. The calculation formula is

$$\sigma = \sqrt{240 * Var(\ln(S_{t+1}) - \ln(S_t))}$$

The result is 14.32% . The SVT model is established for the return on underlying assets, and Bayesian estimation of volatility in BS model is obtained by MCMC algorithm[7]. We use simulated annealing algorithm to estimate the parameters of the Heston model [8]. According to the data given by shibor, we select the risk-free interest rate of 2 weeks, 3 months, 1 month and 6 months. Next, we price all of the March 8 50ETF call options using the BS model and the Heston model, respectively, and the results are presented in the Appendix. According to the above data, the following line chart is drawn. As shown in the following figure. (See Figure 1)

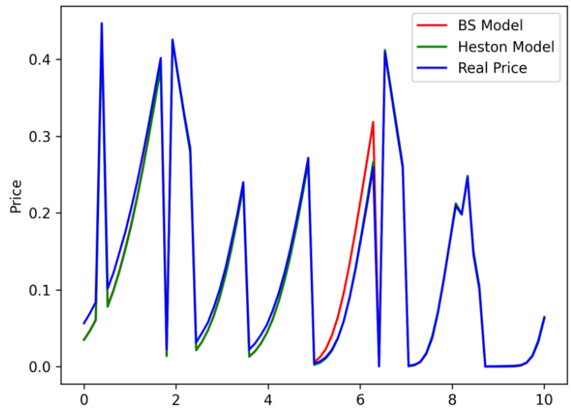

Figure 1: Different Price Model.

3.4 Calculation of the Error

Let C_R be the true market price of 50 ETF call option, C_B is the theoretical option price obtained from the BS model, and C_H is the theoretical option price obtained from the Heston model. We compare the pricing errors of the two model prices with the market prices by means of the mean absolute error(MAE) and the mean relative error(MRE). Let δ_{MAE} be the MAE and δ_{MRE} be the MRE. Then:

$$\delta_{MAE(B)} = \frac{\sum_{i=1}^{n} |C_B - C_R|}{n}$$

$$\delta_{MAE(H)} = \frac{\sum_{i=1}^{n} |C_H - C_R|}{n}$$

$$\delta_{MRE(B)} = \frac{\sum_{i=1}^{n} |C_B - C_R|}{n * C_R} * 100\%$$

$$\delta_{MRE(H)} = \frac{\sum_{i=1}^{n} |C_H - C_R|}{n * C_R} * 100\%$$

Using Matlab software, their δ_{MAE} and δ_{MRE} were calculated for the BS model and the Heston model, respectively. The statistical results are shown in the table below. (See Table 2)

Table 2: Sample Error

$\delta_{MAE(B)}$	$\delta_{MAE(H)}$	$\delta_{MRE(B)}$	$\delta_{MRE(H)}$
0.01106	0.0072	0.2504	0.1848

By comparing the difference between the 50 ETF subscription option price and the market price calculated by the Heston model and the difference between the 50 ETF subscription option price and the market price calculated by the BS model, it is found that both MAE and MRE are small, indicating that the pricing effect of the Heston model and the BS model are great. However, whether it is the MAE or the MRE, the Heston model is significantly lower than the BS model, indicating that the pricing effect of the Heston model is better than that of the BS model.

4 Conclusion

The daily logarithmic return rate of Shanghai 50 ETF shows significant peak characteristics and does not obey the normal distribution. The pricing effects of the two models on call options are good, and the pricing effect of the Heston model is better than that of the B-S model. The pricing effect of the B-S model lies in the accuracy of the volatility estimation, while the pricing effect of the Heston model depends on the selection of the algorithm and the accuracy of the parameter estimation.

It is normal to have a certain degree of deviation between the actual price and the theoretical price. In the practical application of the model, we can combine the advantages and disadvantages of the model and different situations to choose, or we can use the two models comprehensively to make the calculation results more reference.

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