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On the Performance Analysis and Evaluation of Scaled Largest Eigenvalue in Spectrum Sensing: A Simple Form Approach

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Abstract

Scaled Largest Eigenvalue (SLE) detector stands out as the optimal single-primary-user detector in uncertain noisy environments. In this paper, we consider a multi-antenna cognitive radio system in which we aim at detecting the presence/absence of a Primary User (PU) using the SLE detector. By the exploitation of the distributions of the largest eigenvalue and the trace of the receiver sample covariance matrix, we show that the SLE could be modeled using the standard Gaussian function. Moreover, we derive the distribution of the SLE and deduce a simple yet accurate form of the probability of false alarm and the probability of detection. Hence, this derivation yields a very simple form of the detection threshold. Correlation coefficient between the largest eigenvalue and the trace is also considered as we derive a simple analytical expression. These analytical derivations are validated through extensive Monte Carlo simulations

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1. Introduction

In Cognitive Radio (CR) networks, Spectrum Sensing (SS) is the task of obtaining awareness about the spectrum usage. Mainly it concerns two scenarios of detection: (i) detecting the absence of the Primary User (PU) in a licensed spectrum in order to use it and (ii) detecting the presence of the PU to avoid interference. Hence, SS plays a major role in the performance of the CR as well as the performance of the PU networks that coexist. In this context, an extreme importance for a CR network is to have an optimal SS technique with high probability of accuracy in uncertain environments. The Scaled Largest Eigenvalue detector (SLE) is an efficient technique that is proved to be the optimal detector under Generalized Likelihood Ratio (GLR) criterion and noise uncertainty environments [1, 2].

SLE is among the detectors that use the eigenvalues of the receiver sample covariance matrix. Such detectors are known as the Eigenvalue Based Detectors (EBD) and include, in addition to SLE [1–7], other detectors like the Largest Eigenvalue detector (LE) and the Standard Condition Number detector (SCN)[8–12]. In a scenario with perfect knowledge of the noise power, the LE detector is the optimal detector [10]. However, in practical systems the noise power may not be perfectly known. In this case, the SLE and SCN detectors outperform the LE detector due to their blind nature. Moreover, the SLE is proved to be the optimal detector under GLR criterion [1, 2] and outperforms the SCN detector.

In literature, results on the statistics of the SLE, defined as the ratio of the largest eigenvalue to the normalized trace of the sample covariance matrix, are relatively limited. They are based on tools from random matrix theory [2–4] and Mellin transform [4–6]. SLE was proved, asymptotically, to follow the LE distribution (i.e. Tracy-Widom (TW) distribution) [2]. However, a non-negligible error still exists and a new form is derived based on TW distribution and its second

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derivative [3]. Using Mellin transform, The distribution of the SLE was derived by the exploitation of the distribution of LE and the distribution of the trace [4–6]. However, all the findings on SLE are too complex to be considered in real-environments and hence are no easily scalable. This is due to either a complexity in the original distributions used to model the SLE (e.g. TW distribution) or in the methods used to derive the thresholds. Hence, there is a necessity to propose novel yet simple forms in both SS cases (presence and absence of PU activity).

In this paper, we are interested in finding a simple form for the Cumulative Density Function (CDF) and Probability Density Function (PDF) of the SLE. We consider the following hypotheses: (i) $\mathcal{H}_0$: there is no primary user and the received signal is only noise; and (ii) $\mathcal{H}_1$: the primary user exists. Based on the distribution of the ratio of jointly Gaussian random variables, we show that the SLE, under both hypotheses, could be modeled using the standard Gaussian function. Accordingly simple forms for the Probability of False-alarm (P_{fa}), the Probability of detection (P_d) as well as the detection threshold could be derived. Moreover, we derive the correlation coefficient between the Largest Eigenvalue and the trace in both cases, as it is needed in the derivation of the detection analysis. In the following, we summarize the contributions of this paper:

- Derivation of the distribution of the trace of a complex sample covariance matrix for both $\mathcal{H}_0$ and $\mathcal{H}_1$ hypothesis.
- Derivation of the distribution of the SLE detector for both hypotheses.
- Derivation of a simple form for the correlation coefficient between the largest eigenvalue and the trace under both hypotheses.
- Derivation of a simple form for the probability of false-alarm, P_{fa}, the detection probability, P_d, and the threshold for detection.

The rest of this paper is organized as follows. Section 2 studies the system model. In section 3, we recall the distribution of the LE and we derive the distribution of the trace of complex sample covariance matrix. SLE is considered in section 4 as we derive its distribution. The performance probabilities and the threshold are also addressed. In section 5, we consider the correlation coefficient between the largest eigenvalue and the trace. Theoretical findings are validated by simulations in section 6 while the conclusion is drawn in section 7.

Notations. Vectors and Matrices are represented, respectively, by lower and upper case boldface. The symbols $|.|$ and $tr(.)$ indicate, respectively, the determinant and trace of a matrix while $(.)^T$, and $(.)^\dagger$ are the transpose, and Hermitian symbols respectively. $\boldsymbol{I}_n$ is the $n \times n$ identity matrix. Symbols $\sim$ stands for "distributed as", $E[.]$ for the expected value and $\|.\|$ for the Frobenius norm.

2. System Model

Consider a multi-antenna cognitive radio system and denote by K the number of received antennas. Let N be the number of samples collected from each antenna, then the received sample from antenna $k = 1 \cdots K$ at instant $n = 1 \cdots N$ under the two hypotheses is given by

$$\mathcal{H}_0 : \; y_k(n) = \eta_k(n), \tag{1}$$
$$\mathcal{H}_1 : \; y_k(n) = s(n) + \eta_k(n), \tag{2}$$

with $\eta_k(n)$ is a complex circular white Gaussian noise with zero mean and unknown variance σ_η^2 and $s(n)$ is the received signal sample including the channel effect.

After collecting N samples from each antenna, the received signal matrix, $\boldsymbol{Y}$, is given by:

$$\boldsymbol{Y} = \begin{pmatrix} y_1(1) & y_1(2) & \cdots & y_1(N) \\ y_2(1) & y_2(2) & \cdots & y_2(N) \\ \vdots & \vdots & \ddots & \vdots \\ y_K(1) & y_K(2) & \cdots & y_K(N) \end{pmatrix}, \tag{3}$$

Without loss of generality, we suppose that $K \leq N$ then the sample covariance matrix is given by $\boldsymbol{W} = \boldsymbol{Y}\boldsymbol{Y}^\dagger$. Denote the eigenvalues of $\boldsymbol{W}$ by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K > 0$.

$\mathcal{H}_0$ Analysis. Under $\mathcal{H}_0$, the received samples are complex circular white Gaussian noise with zero mean and unknown variance σ_η^2. Consequently, the sample covariance matrix is a central uncorrelated complex Wishart matrix denoted as $\boldsymbol{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K)$ where K is the size of the matrix, N is the number of Degrees of Freedom (DoF), and $\sigma_\eta^2 \mathbf{I}_K$ is the correlation matrix.

$\mathcal{H}_1$ Analysis. Under $\mathcal{H}_1$, we suppose the existence of single PU and the channel is constant during sensing time for simplicity. Consequently, the sample covariance matrix is a non-central uncorrelated complex Wishart matrix denoted as $\boldsymbol{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K, \boldsymbol{\Omega}_K)$ where $\boldsymbol{\Omega}_K$ is a rank-1 non-centrality matrix.

Denote the effective correlation matrix by $\widehat{\Sigma}_K = \sigma_\eta^2 I_K + {}^{\Omega_K}/_N$ and its vector of ordered eigenvalues by $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \cdots, \sigma_K]^T$. Accordingly, $\boldsymbol{W}$, under $\mathcal{H}_1$, could be modeled as a central semi-correlated complex Wishart matrix denoted as $\boldsymbol{W} \sim \mathcal{CW}_K(N, \widehat{\Sigma}_K)$[13]. Since $\boldsymbol{\Omega}_K$ is a rank-1 matrix, then $\widehat{\Sigma}_K$ belongs to the class of spiked population model with all but one eigenvalue of $\widehat{\Sigma}_K$ are still equal to σ_η^2 while σ_1 is given by:

$$\sigma_1 = \sigma_\eta^2 + {}^{\omega_1}/_N, \tag{4}$$

where ω_1 is the only non-zero eigenvalue of $\mathbf{\Omega}_K$. Denote the channel power by σ_h^2 and the signal to noise ratio by $r = \frac{\sigma_s^2 \sigma_h^2}{\sigma_\eta^2}$, then it can be easily shown that:

$$\omega_1 = tr(\mathbf{\Omega}_K) = NKr. \tag{5}$$

3. Distributions of the largest eigenvalue and of the trace

This section considers the distributions of the LE and of the trace under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypothesis. We prove that the LE and the trace follow Gaussian distributions for which the means and variances are formulated. Since the SLE does not depend on the noise power, we suppose, in this section, that $\sigma_\eta^2 = 1$. Based on results of this section, we derive the distribution of the SLE in the next section.

3.1. Distribution of the LE

Let λ_1 be the maximum eigenvalue of the Wishart matrix $\boldsymbol{W}$. In the following, we give its distribution for $\mathcal{H}_0$ and $\mathcal{H}_1$ cases.

$\mathcal{H}_0$ Case. Denote the centered and scaled version of λ_1 of the central uncorrelated Wishart matrix $\mathbf{W} \sim \mathcal{CW}_K(N, \mathbf{I}_K)$ by:

$$\lambda_1' = \frac{\lambda_1 - a(K,N)}{b(K,N)} \tag{6}$$

with $a(K,N)$ and $b(K,N)$, the centering and scaling coefficients respectively, are defined by:

$$a(K,N) = (\sqrt{K} + \sqrt{N})^2 \tag{7}$$

$$b(K,N) = (\sqrt{K} + \sqrt{N})(K^{-1/2} + N^{-1/2})^{\frac{1}{3}} \tag{8}$$

then, as $(K,N) \to \infty$ with $K/N \to c \in (0,1)$, λ_1' follows a Tracy-Widom distribution of order 2 (TW2) [14]. However, it was shown that, for a fixed K and as $N \to \infty$, λ_1 follows a normal distribution [15]. The mean and the variance of λ_1 could be approximated using TW2 and they are, respectively, given by :

$$\mu_{\lambda_1} = b(K,N)\mu_{TW2} + a(K,N), \tag{9}$$

$$\sigma^2{}_{\lambda_1} = b^2(K,N)\sigma^2_{TW2}, \tag{10}$$

where $\mu_{TW2} = -1.7710868074$ and $\sigma^2_{TW2} = 0.8131947928$ are, respectively, the mean and variance of TW distribution of order 2. This approximation is very efficient and it achieves high accuracy for K as small as 2 [15].

$\mathcal{H}_1$ Case. Denote the centered and scaled version of λ_1 of the central semi-correlated Wishart matrix $\mathbf{W} \sim \mathcal{CW}_K(N, \widehat{\Sigma}_K)$ by:

$$\lambda_1'' = \frac{\lambda_1 - a_2(K,N,\sigma)}{\sqrt{b_2(K,N,\sigma)}} \tag{11}$$

with $a_2(K,N)$ and $b_2(K,N)$, the centering and scaling coefficients respectively, are defined, respectively, by:

$$a_2(K,N,\sigma) = \sigma_1(N + \frac{K}{\sigma_1 - 1}) \tag{12}$$

$$b_2(K,N,\sigma) = \sigma_1^2(N - \frac{K}{(\sigma_1 - 1)^2}) \tag{13}$$

then, as $(K,N) \to \infty$ with $K/N \to c \in (0,1)$ and $r > r_c = 1/\sqrt{KN}$, λ_1'' follows a standard normal distribution ($\lambda_1'' \sim \mathcal{N}(0,1)$)[16]. Thus, λ_1 follows a normal distribution with mean and variance given by (12) and (13) respectively. However, if $r < r_c$, then λ_1 follows the same distribution as in $\mathcal{H}_0$ Case [16]. Accordingly, the PU signal has no effect on the eigenvalues and could not be detected.

3.2. Distribution of the Trace

As shown earlier, the distribution of λ_1 converges to the Gaussian distribution. On the other hand, let $T = \sum \lambda_i$ be the trace of the Wishart matrix $\boldsymbol{W}$ then the following theorem holds:

Theorem 1. Let T be the trace of $\mathbf{W} \sim \mathcal{CW}_K(N, \Sigma)$ where the vector of eigenvalues of Σ, not necessary equal, are given by $[\sigma_1, \sigma_2, \cdots, \sigma_K]$. Then, as $N \to \infty$, T follows Gaussian distribution as follows:

$$P\left(\frac{T - N\sum_{i=1}^K \sigma_i}{\sqrt{N\sum_{i=1}^K \sigma_i^2}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du, \tag{14}$$

Proof. Let $\boldsymbol{D}$ be an orthogonal matrix that diagonalizes Σ, then we write:

$$T = tr(\boldsymbol{YY}^\dagger) = tr(\boldsymbol{DD}^T\boldsymbol{YY}^\dagger) = tr(\boldsymbol{D}^T\boldsymbol{YY}^\dagger\boldsymbol{D})$$
$$= tr(\boldsymbol{ZZ}^\dagger) = \sum_{i=1}^{K}\left[\sum_{j=1}^{N}|z_{i,j}|^2\right] \tag{15}$$

with $z_{i,j}$ is the (i,j)-th element of matrix $\boldsymbol{Z} = \boldsymbol{D}^T\boldsymbol{Y}$. Let $\boldsymbol{Z} = [\boldsymbol{z}_1\ \boldsymbol{z}_2 \cdots \boldsymbol{z}_N]$ with $\boldsymbol{z}_j = [z_{1j}\ z_{2j} \cdots z_{Kj}]^T$. Since the vectors $\boldsymbol{z}_1, \boldsymbol{z}_2, \cdots \boldsymbol{z}_N$ are independent and $\boldsymbol{z}_j \sim \mathcal{CN}_K(\mathbf{0}, \boldsymbol{D}^T\Sigma\boldsymbol{D})$ then the elements z_{ij} are independent and form a circularly symmetric complex normal random variable ($z_{i,j} \sim \mathcal{CN}(0, \sigma_i)$). Accordingly, the square of the norm, $|z_{i,j}|^2$, is exponentially distributed with parameter σ_i^{-1} and hence, the mean and variance are σ_i and σ_i^2 respectively.

According to CLT, as $N \to \infty$ the term in the square bracket of (15) follows Gaussian distribution with mean and variance are $N\sigma_i$ and $N\sigma_i^2$ respectively. □

To the best of the authors' knowledge, the result in Theorem 1 is new.

Now, we consider each hypothesis as follows:

$\mathcal{H}_0$ Case. The distribution of the trace T of of $\boldsymbol{W}$ under $\mathcal{H}_0$ is given by the following Corollary:

Corollary 1. Let T be the trace of $\mathbf{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K)$. Then, as $N \to \infty$, T follows Gaussian distribution as follows:

$$P(\frac{T - NK\sigma_\eta^2}{\sqrt{NK\sigma_\eta^4}} \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du, \tag{16}$$

Proof. It follows from Theorem 1. □

$\mathcal{H}_1$ Case. The distribution of the trace T of of $\boldsymbol{W}$ under $\mathcal{H}_1$ is given by the following Corollary:

Corollary 2. Let T be the trace of $\mathbf{W} \sim \mathcal{CW}_K(N, \widehat{\Sigma}_K)$. Then, as $N \to \infty$, T follows Gaussian distribution as follows:

$$P(\frac{T - N(\sigma_1 + (K-1)\sigma_2)}{\sqrt{N(\sigma_1^2 + (K-1)\sigma_2^2)}} \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du, \tag{17}$$

Proof. All the eigenvalues of $\widehat{\Sigma}_K$ are equal except σ_1. Then, the result follows from Theorem 1. □

Normalized Trace. Let $T_n = \frac{1}{K} T$ be the normalized trace. Then T_n, following Theorem 1, is Normally distributed.

From Corollary 1, T_n is Normally distributed with mean and variance given respectively, when $\sigma_\eta^2 = 1$, by:

$$\mu_{T_n} = N, \tag{18}$$

$$\sigma^2{}_{T_n} = N/K, \tag{19}$$

From Corollary 2, T_n is Normally distributed with mean and variance given respectively, when $\sigma_\eta^2 = 1$, by:

$$\mu_{T_n} = \frac{N}{K}(\sigma_1 + K - 1), \tag{20}$$

$$\sigma^2{}_{T_n} = \frac{N}{K^2}(\sigma_1^2 + K - 1), \tag{21}$$

4. SLE Detector

Let $\boldsymbol{W}$ be the sample covariance matrix at the CR receiver, then the SLE of $\boldsymbol{W}$ is defined by:

$$X = \frac{\lambda_1}{\frac{1}{K}\sum_{i=1}^{K} \lambda_i} = \frac{\lambda_1}{T_n} \tag{22}$$

Denoting by α the decision threshold, then the false alarm probability (P_{fa}), defined as the probability of detecting the presence of PU while it does not exist, and the detection probability (P_d), defined as the probability of correctly detecting the presence of PU, are, respectively, given by:

$$P_{fa} = P(X \geq \alpha/\mathcal{H}_0) = 1 - F_0(\alpha), \tag{23}$$

$$P_d = P(X \geq \alpha/\mathcal{H}_1) = 1 - F_1(\alpha), \tag{24}$$

where $F_0(.)$ and $F_1(.)$ are the CDFs of X under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses respectively. If the expressions of the P_{fa} and/or P_d are known, then a threshold could be set according to a required error constraint. Hence, it is important to have a simple and accurate form for the distribution of X.

4.1. SLE distribution

This section provides a new formulation for the SLE distribution for $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses as follows:

$\mathcal{H}_0$ Case. Under $\mathcal{H}_0$, both the LE and the normalized trace follow the Gaussian distribution as $N \to \infty$ which is realistic in practical spectrum sensing scenarios. Herein, we show that the SLE could be formulated using standard Gaussian function as stated by the following theorem:

Theorem 2. Let X be the SLE of $\mathbf{W} \sim \mathcal{CW}_K(N, \sigma_\eta^2 \mathbf{I}_K)$. Then, for a fixed K and as $N \to \infty$, the CDF and the PDF of X are, respectively, given by:

$$F_X(x) = \Phi(\frac{x\mu_{T_n} - \mu_{\lambda_1}}{\sqrt{\sigma^2{}_{\lambda_1} - 2xc + x^2\sigma^2{}_{T_n}}}) \tag{25}$$

$$f_X(x) = \frac{\mu_{T_n}\sigma^2{}_{\lambda_1} - c\mu_{\lambda_1} + (\mu_{\lambda_1}\sigma^2{}_{T_n} - c\mu_{T_n})x}{(\sigma^2{}_{\lambda_1} - 2xc + x^2\sigma^2{}_{T_n})^{\frac{3}{2}}} \times \phi(\frac{x\mu_{T_n} - \mu_{\lambda_1}}{\sqrt{\sigma^2{}_{\lambda_1} - 2xc + x^2\sigma^2{}_{T_n}}}) \tag{26}$$

with

$$\Phi(v) = \int_{-\infty}^{v} \phi(u) du \quad \text{and} \quad \phi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \tag{27}$$

where μ_{λ_1}, μ_{T_n} and $\sigma^2{}_{\lambda_1}$, $\sigma^2{}_{T_n}$ are, respectively, the mean and the variance of λ_1 and T_n given by (9), (18) and (10), (19) respectively. The parameter c is given by $c = \sigma_{\lambda_1}\sigma_{T_n}\rho$ where ρ is the correlation coefficient between λ_1 and T_n.

Proof. Let λ_1 and T_n be two normally distributed random variables with μ_{λ_1}, μ_{T_n}, $\sigma^2{}_{\lambda_1}$ and $\sigma^2{}_{T_n}$ their means and variances and let ρ be their correlation coefficient. Denote by $g(\lambda, t)$ the joint density of λ_1 and T_n then the PDF of X is $f_X(x) = \int_{-\infty}^{+\infty} |t| g(xt, t) dt$ and the result is found in [17], however, since $\boldsymbol{W}$ is positive definite then $Pr(T_n > 0) = 1$ and the CDF of X could be written as:

$$F_X(x) = Pr(\lambda/t < x) = Pr(\lambda_1 - xt < 0) \tag{28}$$

and thus, its CDF is given by (25) and the PDF is its derivative in (26) [18]. □

$\mathcal{H}_1$ Case. Under $\mathcal{H}_1$, the normalized trace follows the Gaussian distribution as $N \to \infty$ whereas the LE follows the Gaussian distribution as $(K, N) \to \infty$ with $K/N \to c \in (0, 1)$ and $r > r_c = 1/\sqrt{KN}$. Accordingly, the distribution of the SLE is given by the following Theorem:

Theorem 3. Let X be the SLE of $\mathbf{W} \sim \mathcal{CW}_K(N, \widehat{\Sigma}_K)$. Then, as $(K, N) \to \infty$ with $K/N \to c \in (0, 1)$ and $r > r_c = 1/\sqrt{KN}$, the CDF and PDF of X are, respectively, given by (25) and (26). However, μ_{λ_1}, μ_{T_n} and $\sigma^2{}_{\lambda_1}$, $\sigma^2{}_{T_n}$ are, respectively, the mean and the variance of λ_1 and T_n given by (12), (20) and (13), (21) respectively. The parameter c is defined by $c = \sigma_{\lambda_1}\sigma_{T_n}\rho$ where ρ is the correlation coefficient between λ_1 and T_n.

Proof. Same as the proof of Theorem 2. □

4.2. Performance Probabilities and Threshold

Using (23) and (25), then P_{fa} is given by:

$$P_{fa}(\alpha) = Q\left(\frac{\alpha\mu_{T_n} - \mu_{\lambda_1}}{\sqrt{\sigma^2{}_{\lambda_1} - 2\alpha c + \alpha^2\sigma^2{}_{T_n}}}\right) \quad (29)$$

where $Q(.)$ is the Q-function. μ_{λ_1}, $\sigma^2{}_{\lambda_1}$, μ_{T_n} and $\sigma^2{}_{T_n}$ are given respectively by (9), (10), (18) and (19). P_d is derived the same way using $\mathcal{H}_1$ hypothesis.

Using P_{fa} and P_d, the threshold could be set according to a required error constraint. For example, and based on (29), we can derive a simple and accurate form for the threshold as a function of the means and variances of the *LE* and T_n and the correlation coefficient between them as well as the false alarm probability. That is, for a target false alarm probability, $\hat{P}_{fa}$, the equation of the threshold of the SLE detector will be:

$$\alpha = \frac{\mu_{12} - \beta^2\rho\sigma_{12} + \beta\sqrt{m_v - 2\rho\mu_{12}\sigma_{12} + \beta^2\sigma_{12}^2(\rho^2 - 1)}}{\mu_{T_n}^2 - \beta^2\sigma_{T_n}^2} \quad (30)$$

where $\mu_{12} = \mu_{\lambda_1}\mu_{T_n}$, $\sigma_{12} = \sigma_{\lambda_1}\sigma_{T_n}$, $m_v = \mu_{T_n}^2\sigma_{\lambda_1}^2 + \mu_{\lambda_1}^2\sigma_{T_n}^2$ and $\beta = Q^{-1}(\hat{P}_{fa})$ with $Q^{-1}(.)$ is the inverse Q-function.

5. Correlation Coefficient ρ

Theorem 2 gives the form of the distribution of the SLE as a function of the mean and the variance of λ_1 and T_n as well as the correlation coefficient between them (ρ). Consequently, P_{fa}, P_d and the threshold are a functions of these same parameters.

The mean and the variance of λ_1 and T_n are provided in Section 3. In this section, we will give a simple analytical form to calculate the correlation coefficient, ρ, between the largest eigenvalue and the trace of Wishart matrix based on the mean of the SLE. In the following, we calculate the mean of SLE in two different ways such that a simple form for ρ could be derived.

5.1. Mean of SLE using λ_1 and T_n

Under both hypotheses ($\mathcal{H}_0$ and $\mathcal{H}_1$), the mean of the SLE could be computed using the means of λ_1 and T_n as follows:

$\mathcal{H}_0$ case: using independent property. Under $\mathcal{H}_0$, the SLE and the trace of central uncorrelated Wishart matrices are proved to be independent [19]. Accordingly, and using (22), the mean of λ_1 could be written as:

$$E[\lambda_1] = E\Big[X \times T_n\Big] = E[X] \cdot E[T_n] \quad (31)$$

Recall that the mean of λ_1 and the mean of T_n are given respectively by (9) and (18), then based on (31), the mean of the SLE is given by:

$$\mu_X = \frac{\mu_{\lambda_1}}{\mu_{T_n}} = \frac{b(K, N) \cdot \mu_{TW2} + a(K, N)}{N} \quad (32)$$

$\mathcal{H}_1$ case: using Taylor series. The bi-variate first order Taylor expansion of the function $X = g(\lambda_1, T_n) = \lambda_1/T_n$ about any point $\theta = (\theta_{\lambda_1}, \theta_{T_n})$ is written as [20]:

$$X = g(\theta) + g'_{\lambda_1}(\theta)(\lambda_1 - \theta_{\lambda_1}) + g'_{T_n}(\theta)(T_n - \theta_{T_n}) + O(n^{-1}), \quad (33)$$

with $g'_{(.)}$ is the partial derivative of g over (.).

Let $\theta = (\mu_{\lambda_1}, \mu_{T_n})$, then the mean is given by [20]:

$$\mu_X = \frac{\mu_{\lambda_1}}{\mu_{T_n}} = \frac{\sigma_1(N + \frac{K}{\sigma_1 - 1})}{\frac{N}{K}(\sigma_1 + K - 1)}, \quad (34)$$

It is worth mentioning that it is more accurate to use higher order Taylor series. However, this will increase the complexity with a slightly more accurate values which is not necessary. Equation (35) provides the expression of the mean using 2^{nd} order bi-variate Taylor series so that the reader could compare the results.

$$\mu_X = \frac{\mu_{\lambda_1}}{\mu_{T_n}} - \frac{Cov(\lambda_1, T_n)}{\mu_{T_n}^2} + \frac{\sigma^2{}_{T_n}\mu_{\lambda_1}}{\mu_{T_n}^3} \quad (35)$$

5.2. Mean of SLE using variable transformation

Using SLE distribution, it is difficult to find numerically the mean of the SLE, however, it turns out that a simple and accurate approximation could be found.

An approximation of the mean of the ratio $(u + Z_1)/(v + Z_2)$ could be found when u and v are positive constants and Z_1 and Z_2 are two independent standard normal random variables. It is based on approximating formula for $E[1/(v + Z_2)]$ when $v + Z_2$ is normal variate conditioned by $Z_2 > -4$ and $v + Z_2$ is not expected to approach zero as follows [18]:

$$E\left[\frac{1}{v + Z_2}\right] = \frac{1}{1.01v - 0.2713} \quad (36)$$

By using the transformation of the general ratio of two jointly normal random variable λ_1/T_n into the ratio $(u+Z_1)/(v+Z_2)$, which has the same distribution, we have:

$$\frac{\lambda_1}{T_n} \sim \frac{1}{q}(\frac{u+Z_1}{v+Z_2})+s \tag{37}$$

with $s=\rho\frac{\sigma_{\lambda_1}}{\sigma_{T_n}}$, $v=\frac{\mu_{T_n}}{\sigma_{T_n}}$ and

$$u=\frac{\mu_{\lambda_1}-\rho\frac{\mu_{T_n}\cdot\sigma_{\lambda_1}}{\sigma_{T_n}}}{(\pm\sigma_{\lambda_1}\sqrt{1-\rho^2})} \tag{38}$$

$$q=\frac{\sigma_{T_n}}{(\pm\sigma_{\lambda_1}\sqrt{1-\rho^2})} \tag{39}$$

where one chooses the $\pm$ sign (in both u and q) so that u and v have the same sign (i.e. positive). As the left-side and the right-side of (37) must have the same mean, we can write:

$$E[\frac{\lambda_1}{T_n}]=\frac{u}{q}E[\frac{1}{v+Z_2}]+s \tag{40}$$

therefore the mean of the SLE could be approximated as follows:

$$\mu_X=\frac{\mu_{\lambda_1}+\delta\,\mu_{T_n}}{\theta}+\delta \tag{41}$$

with $\delta=\rho\frac{\sigma_{\lambda_1}}{\sigma_{T_n}}$ and $\theta=1.01\mu_{T_n}-0.2713\sigma_{T_n}$.

This practical approximation shows high accuracy; however, it could be noticed from (40) that as u increases the error due to this approximation increases.

5.3. Deduction of the Correlation coefficient ρ

Based on these results, the correlation coefficient (ρ) under $\mathcal{H}_0$ and $\mathcal{H}_1$ hypotheses is considered as follows:

$\mathcal{H}_0$ case. Using (41), then ρ, after some algebraic manipulation, is given by:

$$\rho=\frac{\sigma_{T_n}}{\sigma_{\lambda_1}}\cdot\frac{\theta\,\mu_X-\mu_{\lambda_1}}{\theta+\mu_{T_n}} \tag{42}$$

where μ_{λ_1}, μ_{T_n} and μ_X are respectively the means of the LE, the normalized trace and the SLE given by (9), (18) and (32) respectively. σ_{λ_1} and σ_{T_n} are respectively the standard deviations of the LE and the normalized trace and are the square root of (10) and (19) respectively.

$\mathcal{H}_1$ case. Under $\mathcal{H}_1$ hypothesis, results show the u increases as K or N increases because of the high correlation between λ_1 and T_n. Accordingly, results show a small error in the value of the mean of SLE with respect to the true value. Consequently, and using (41), then ρ is given by:

$$\rho=\frac{\sigma_{T_n}}{\sigma_{\lambda_1}}\cdot\frac{\theta\,(\mu_X+\epsilon)-\mu_{\lambda_1}}{\theta+\mu_{T_n}} \tag{43}$$

where μ_{λ_1}, μ_{T_n} and μ_X are respectively the means of the LE, the normalized trace and the SLE given by (12), (20) and (34) respectively. σ_{λ_1} and σ_{T_n} are respectively the standard deviations of the LE and the normalized trace and are the square root of (13) and (21) respectively. Finally, ϵ is a variable used to model the mean error.

Table 1. The Empirical and Approximated value of the correlation coefficient ρ under $\mathcal{H}_0$ hypothesis for different values of $\{K,N\}$.

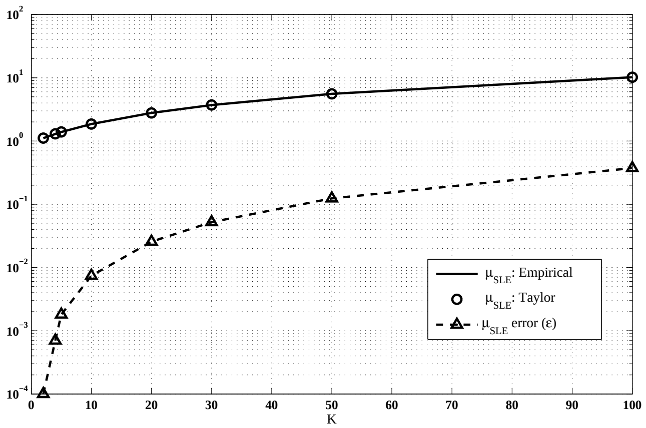
$K\times N$	2×500	4×500	2×1000	4×1000	50×1000
ρ-Emp.	0.849	0.6974	0.839	0.6915	0.3353
ρ-Ana.	0.8548	0.6957	0.8623	0.6967	0.3356

6. Numerical validation

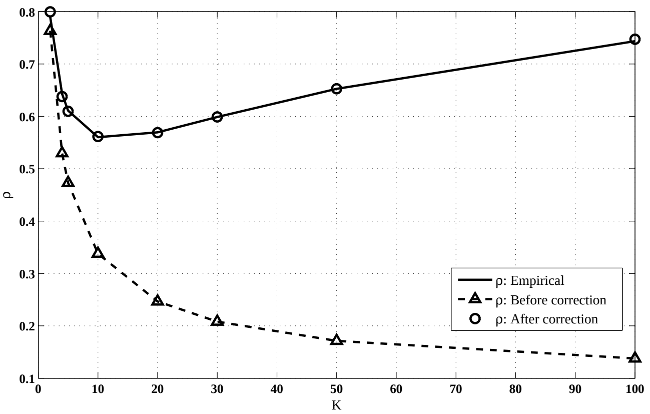
In this section, we discuss the analytical results through Monte-Carlo simulations. We validate the theoretical analysis presented in sections 3, 4 and 5. The simulation results are obtained by generating 10^5 random realizations of $\boldsymbol{Y}$.

Table 1 shows the accuracy of the analytical approximation of the correlation coefficient (ρ) of the SLE in (42). The results are shown for $K=\{2,4,50\}$ antennas and $N=\{500,1000\}$ samples per antenna. Table 1 shows that the accuracy of this approximation is higher as the number of antennas increases, however, we can also notice that we have very high accuracy even when $K=2$ antennas. Also, as expected, it is easy to notice that the correlation between the largest eigenvalue and the trace decreases as the number of antenna increases, however, this correlation could not be ignored even if the number of antennas is large.

Figure 1 shows the accuracy of the mean of the SLE as well as the correlation coefficient between the largest eigenvalue and the trace. The results are shown for different values of K where $N=500$ and $r=-10dB$. Figure 1(a) plots the empirical mean and its corresponding Taylor series approximation in (34). In addition, the figure shows the mean error (ϵ) between the Taylor approximation and the mean expression provided using variable transformation in (41). the results show a high accuracy in the approximation of the mean using Taylor series, however, it also shows a small error, ϵ, that increases as K increases. Another important point here concerns the error value epsilon. Indeed, one can easily observe the epsilon is small however its effect on correlation coefficient rho is relatively high as shown in Fig. 1(b) , hence corrective action should be taken to yield correct results. The corrected version is considered (i.e. Fig. 1(a)) then the results show high accuracy. We should mention that this is out of the scope of this paper but it is worth mentioning it for future research.



(a) Empirical and proposed approximation of μ_{SLE} with the corresponding error (ϵ)



(b) Empirical ρ and its corresponding analytical approximation before and after mean correction

Figure 1. Empirical and Analytical mean of SLE (μ_{SLE}) and correlation coefficient between largest eigenvalue and trace (ρ) under $\mathcal{H}_1$ hypothesis for different values of K where $N = 500$ sample and $r = -10dB$.

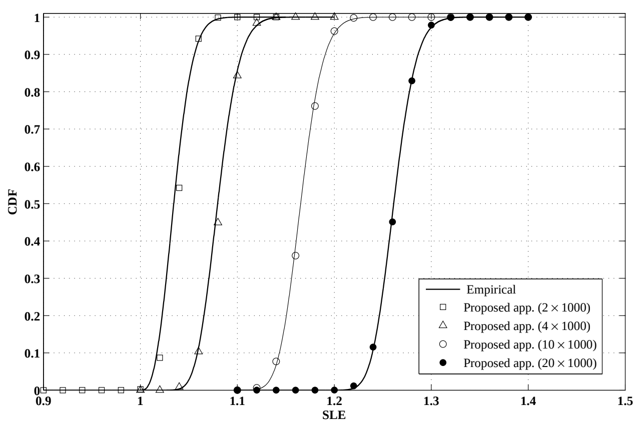


Figure 2. Empirical CDF of the SLE under $\mathcal{H}_0$ hypothesis and its corresponding Gaussian approximation for different values of K with $N = 1000$.

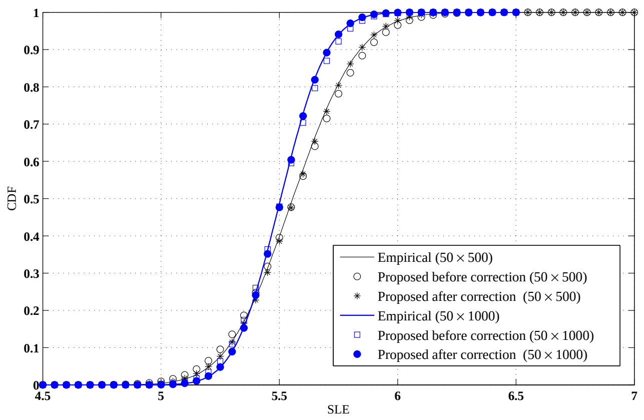


Figure 3. Empirical CDF of the SLE under $\mathcal{H}_1$ hypothesis and its corresponding proposed approximation for different values of $K = 50$ with $N = \{500, 100\}$ and $r = -10dB$.

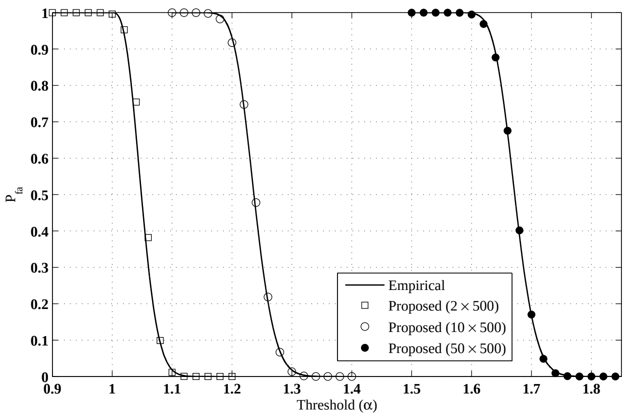


Figure 4. Empirical probability of false alarm for the SLE detector and its corresponding proposed form in (29) for different values of K with $N = 500$ samples.

Figure 2 shows the empirical CDF of the SLE and its corresponding approximation under $\mathcal{H}_0$ hypothesis given by Theorem 2. The results are shown for $K = \{2, 4, 10, 20\}$ antennas and $N = 1000$ samples per antenna. Results show a perfect match between the empirical results and our Gaussian formulation.

Figure 3 shows the empirical CDF of the SLE and its corresponding approximation (before and after mean correction) given by Theorem 3. The results are shown for $K = 50$ antennas, $N = \{500, 1000\}$ samples per antenna and $r = -10dB$. Again, the results show a perfect match between the empirical results and the proposed approximation after the mean correction in (43). However if the we consider $\epsilon = 0$, results show a slight difference between empirical and the proposed distributions in comparison with the big error in ρ (see Fig. 1(b) when $K = 50$ and $N = 500$).

Figure 4 shows the accuracy of the proposed false alarm form proposed in (29). Here, we have considered multi-antenna CR with different number of antennas and $N = 500$ samples. The considered number of antennas is as small as $K = 2$ and as large as $K = 50$. Simulation results show a high accuracy in our proposed form which increases as K increases. It is worth reminding the reader, that in addition to the accuracy, the form given in (29) is a simple Q-function equation.

7. Conclusion

In this paper, we have considered the SLE detector due to its optimal performance in uncertain environments. We proved that the SLE could be modeled using standard Gaussian function and we have derived its CDF and PDF. The false alarm probability, the detection probability and the threshold were also considered as we derived new simple and accurate forms. These forms are simple functions of the means and variances of the LE and the trace as well as the correlation function between them. The correlation between the largest eigenvalue and the trace is studied and simple expressions are provided. Simulation results have shown that the proposed simple forms achieve high accuracy. However, the approximation of the correlation coefficient under $\mathcal{H}_0$ shows high accuracy. Moreover, under $\mathcal{H}_1$ hypothesis, small mean error must be corrected to achieve high accuracy. In addition, results have shown that the correlation between the largest eigenvalue and the trace, under $\mathcal{H}_0$, decreases as the number of antenna increases but it could not be ignored even for large number of antennas.

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