

Variational Iteration Method Used to Solve a SIR Epidemic Model of Tuberculosis

Gabariela Purnama Ningsi, Sudi Mungkasi
{ningsilatib5@gmail.com¹, sudi@usd.ac.id²}

Department of Mathematics Education, Universitas Sanata Dharma, Yogyakarta, Indonesia¹,
Department of Mathematics, Universitas Sanata Dharma, Yogyakarta, Indonesia²

Abstract. This paper discusses the variational iteration method for solving non-linear differential equation systems. To be specific, the system we consider is the SIR (Susceptible-Infectious-Recovered) model of tuberculosis. Based on our research results, the calculation using the variational iteration method can be done quite easily. The method can predict the exact solution accurately at an arbitrary time.

Keywords: SIR model, tuberculosis, variational iteration method.

1 Introduction

Differential equations occur not only in mathematics, but actually also in problems of everyday life. They also occur in other fields of sciences, such as, economics, physics, biology, computers science, etc. [1]-[3]. A system of ordinary differential equations consists of some ordinary differential equations that fulfill the nature that an unknown function depends only on one independent variable [3].

One of problems in everyday life and in the field of biology that can be modeled into the form of non-linear differential equations is the problem of the spread of infectious diseases. To model this problem, we can use the SIR model. In the SIR model [4]-[5], the population studied will be grouped into three groups labeled S, I, and R. Here, S(t) is a group of susceptible individuals (Susceptibles) against diseases that are not infected in time t; I(t) is a group of infected individuals (Infectives) within t, this group of infections is able to transmit the disease to vulnerable groups (Susceptibles) in case of direct contact; R(t) is a group of individuals who have been infected and recovered from the disease, and will not be infected again (permanently recovered).

In this paper, we solve the SIR model for the spread of tuberculosis (TB) using a variational iteration method. Variational iteration method is a method used to solve linear and nonlinear differential equations that will predict solutions quickly, easily, effectively, and accurately [6]-[7].

This paper is written in the following structure. In Section 2, we will write about the SIR model for the spread of tuberculosis, which is the problem we want to solve. In Section 3, we will write about the variational iteration method to solve the SIR model. Results of mathematical computation will be presented in Section 4. Conclusions will be written in Section 5.

2 Mathematical Model

Side [8] has modeled a non-linear differential equation system for the spread of tuberculosis (TB) by using the SIR model as follows:

$$\frac{dx}{dt} = \mu_h - \beta_h x - \gamma \beta_h x y - \mu_h x \quad (1a)$$

$$\frac{dy}{dt} = \beta_h x - \alpha y \quad (1b)$$

$$\frac{dz}{dt} = \gamma \beta_h x y - \eta z \quad (1c)$$

where, $x(t) = \frac{S_h}{N_h}$, $y(t) = \frac{I_h}{N_h}$, $z(t) = \frac{R_h}{N_h}$, $\alpha = \mu_h + \delta_h$, $\eta = \mu_h + \varphi_h$. (1d)

Here:

N_h is the total population at a certain time,

S_h is the number of people suspected of being susceptible to tuberculosis at any given time,

I_h is the number of people infected with TB at a certain time,

R_h is the number of people who recover from tuberculosis at a certain time,

$\varphi_h I_i$ is the number of people recovering from an epidemic,

μ_h is the rate of birth or death in the human population,

β_h is the level of suspected human being infected,

γ is the estimated level of population vulnerable to infection,

δ_h is a parameter of the number of infected populations that are cured, and

φ_h is a parameter of the number of infected populations who are cured.

3 Variational Iteration Method

Variational iteration method (VIM) was introduced by He [9]. VIM is the result of modification of the general Lagrange multiplier method which has been proven to find a solution easily from differential equations [8]-[10]. In VIM, there are 3 main concepts used to solve a differential equation, the Lagrange multiplier, the correction function and the restricted variation [3]. To solve the system (1) of differential equations (1a)-(1c) above, we will follow the writings of Rangkuti, et al. [11].

To illustrate the procedure in this variational iteration method, we consider the following differential equation:

$$L\mu_i(t) + N\mu_i(t) = g_i(t), \quad i = 1, 2, 3, \dots, n \quad (2)$$

where L is linear operator, N is nonlinear operator, and $g_i(t)$ is nonhomogeneous term of the differential equation. According to the variational iteration method, the terms of the sequence $\mu_i(t)$ are made in such a way that this finds the right solution of the mathematical model $\mu_i(t)$ can be calculated with the following correction functions:

$$\mu_{i,n+1}(t) = \mu_{i,n}(t) + \int_0^t \lambda_i [L\mu_{i,n}(s) + N\mu_{i,n}(s) - g_i(t)] ds \quad (3)$$

where, λ_i , $i=1,2,3,\dots,n$ is a Lagrange multiplier that can be identified optimally through variational theory, index n shows the n -th order approximation, $\tilde{\mu}_{i,n}(s)$ is restricted variation, that is, $\delta \tilde{\mu}_{i,n}(s) = 0$ [11].

The correction functional of the differential equation system [1] will be obtained as follows:

$$x(n+1) = x_n + \int_0^t \lambda_1 \left[\frac{dx_n}{ds} - \mu_h + \beta_h x_n + \gamma \beta_h \tilde{x}_n \tilde{y}_n + \mu_h x_n \right] ds \quad (4)$$

$$y(n+1) = y_n + \int_0^t \lambda_2 \left[\frac{dy_n}{ds} - \beta_h \tilde{x}_n + \alpha y_n \right] ds \quad (5)$$

$$z(n+1) = z_n + \int_0^t \lambda_3 \left[\frac{dz_n}{ds} - \gamma \beta_h \tilde{x}_n \tilde{y}_n + \eta z_n \right] ds \quad (6)$$

where \tilde{x}_n , \tilde{y}_n , dan \tilde{z}_n are restricted variations, that is, $\delta \tilde{x}_n = 0$, $\delta \tilde{y}_n = 0$, dan $\delta \tilde{z}_n = 0$. From equations (4-6) we will get the following equations:

$$\delta x(n+1) = \delta x_n + \delta \int_0^t \lambda_1 \left[\frac{dx_n}{ds} - \mu_h + \beta_h x_n + \gamma \beta_h \tilde{x}_n \tilde{y}_n + \mu_h x_n \right] ds,$$

$$\delta x(n+1) = \delta x_n + \delta \int_0^t \lambda_1 \left[\frac{dx_n}{ds} - \mu_h + \beta_h x_n + \mu_h x_n \right] ds; \quad (7)$$

$$\delta y(n+1) = \delta y_n + \delta \int_0^t \lambda_2 \left[\frac{dy_n}{ds} - \beta_h \tilde{x}_n + \alpha y_n \right] ds,$$

$$\delta y(n+1) = \delta y_n + \delta \int_0^t \lambda_2 \left[\frac{dy_n}{ds} + \alpha y_n \right] ds; \quad (8)$$

$$\delta z(n+1) = \delta z_n + \delta \int_0^t \lambda_3 \left[\frac{dz_n}{ds} - \gamma \beta_h \tilde{x}_n \tilde{y}_n + \eta z_n \right] ds,$$

$$\delta z(n+1) = \delta z_n + \delta \int_0^t \lambda_3 \left[\frac{dz_n}{ds} + \eta z_n \right] ds. \quad (9)$$

By using integration by parts, equations (7-9) will be:

$$\delta x(n+1) = \delta x_n + \delta \lambda_1 x_n - \int_0^t [x_n \delta \lambda_1' - \delta \lambda_1 (\beta_h + \mu_h) x_n] ds, \quad (10)$$

$$\delta x(n+1) = \delta x_n (1 + \lambda_1) - \int_0^t [(\lambda_1' - (\beta_h + \mu_h) \lambda_1) x_n \delta] ds;$$

$$\delta y(n+1) = \delta y_n + \delta \lambda_2 y_n - \int_0^t [\delta \lambda_2' y_n] ds + \int_0^t [\alpha \delta \lambda_2 y_n] ds, \quad (11)$$

$$\delta y(n+1) = \delta y_n (1 + \lambda_2) - \int_0^t [\lambda_2' - \lambda_2 \alpha] y_n \delta ds;$$

$$\delta z(n+1) = \delta z_n + \delta \lambda_3 z_n - \int_0^t [\delta \lambda_3' z_n] ds + \int_0^t [\delta \eta y_n] ds, \quad (12)$$

$$\delta z(n+1) = \delta z_n (1 + \lambda_3) - \int_0^t [(\lambda_3' - \eta) z_n \delta] ds.$$

Therefore, for each variations δx_n , δy_n , δz_n and $\delta x_n'$, $\delta y_n'$, $\delta z_n'$ we obtain stationary conditions as follows:

$$\begin{aligned} \delta x_n : (1 + \lambda_1(t))|_{s=t} &= 0 & \delta x_n : (\lambda_1' - (\beta_h + \mu_h) \lambda_1)|_{s=t} &= 0 \\ \delta y_n : (1 + \lambda_2(t))|_{s=t} &= 0 & \delta y_n : (\lambda_2' - \lambda_2 \alpha)|_{s=t} &= 0 \\ \delta z_n : (1 + \lambda_3(t))|_{s=t} &= 0 & \delta z_n : (\lambda_3' - \eta)|_{s=t} &= 0 \end{aligned} \quad (13)$$

The solution of equation (13) above is as follows:

$$\begin{aligned} \lambda_1(t) &= -1 & \lambda_1(s) &= -e^{(\beta_h + \mu_h)(s-t)} \\ \lambda_2(t) &= -1 & \lambda_2(s) &= -e^{\alpha(s-t)} \\ \lambda_3(t) &= -1 & \lambda_3(s) &= -1 + \eta(s-t) \end{aligned} \quad (14)$$

Then we obtain:

$$\begin{aligned} \lambda_1(s) &= -1 \\ \lambda_2(s) &= -1 \\ \lambda_3(s) &= -1 \end{aligned} \quad (15)$$

Substitution of the Lagrange multipliers (15) to the initial equations (4)-(6) so that it is obtained:

$$x(n+1) = x_n - \int_0^t \left[\frac{dx_n}{ds} - \mu_h + \beta_h x_n + \gamma \beta_h x_n y_n + \mu_h x_n \right] ds \quad (16)$$

$$y(n+1) = y_n - \int_0^t \left[\frac{dy_n}{ds} - \beta_h x_n + \alpha y_n \right] ds \quad (17)$$

$$z(n+1) = z_n - \int_0^t \left[\frac{dz_n}{ds} - \gamma \beta_h x_n y_n + \eta z_n \right] ds \quad (18)$$

Equations (16)-(18) will provide the estimate solution to the spread of TB disease.

4 Results and Discussion

To do the iteration, we need initial data. Suppose that the initial data is

$$x_0 = \frac{8377828}{8386763} = 0.998935, y_0 = \frac{8000}{8386763} = 0.000954,$$

$$z_0 = \frac{939}{838676} = 0.000112, \mu_h = 0.000046, \beta_h = 0.326666, \varphi_h = 0.003700 \quad [8].$$

Then the variational iteration solutions of equations (16)-(18) are as follows:

$$x_1 = 0.998935 - 0.3263563769t \quad (19)$$

$$y_1 = 0.00094 + 0.326278723t \quad (20)$$

$$z_1 = 0.000112 + 0.00003790582171t \quad (21)$$

$$x_2 = 0.9989935 - 0.6527127538t - 0.0244394405t^2 + 0.004282326039t^3 \quad (22)$$

$$y_2 = 0.000954 + 0.6525574468t - 0.1200770128t^2 \quad (23)$$

$$z_2 = 0.000112 + 0.00007581147385t + 0.0130949876t^2 - 0.004282326039t^3 \quad (24)$$

$$x_3 = 0.998935 - 0.9790691307t + 0.280588245t^2 - 0.00432526614t^3 - 0.007005728528t^4 + 0.0003392729947t^5 + 0.00002067942587t^6 \quad (25)$$

$$y_3 = 0.000954 + 0.9788361702t - 0.3602310384t^2 + 0.03550917866t^3 + 0.001398890318t^4 \quad (26)$$

$$z_3 = 0.000112 + 0.000113716564t + 0.03928496282t^2 - 0.02628096991t^3 + 0.005622682816t^4 - 0.0003392729947t^5 - 0.00002067942587t^6 \quad (27)$$

$$x_4 = 0.998935 - 1.305425507t + 0.56117649t^2 - 0.04299502079t^3 - 0.03224810867t^4 + 0.008205506472t^5 - 0.0002226690546t^6 - 0.0001312134279t^7 + 0.0000143487926t^8 + 2.092161586[10]^{-7}t^9 - 4.861773107[10]^{-8}t^{10} - 1.163378447[10]^{-9}t^{11} \quad (28)$$

$$y_4 = 0.000954 + 1.305114894t - 0.7204620768t^2 + 0.1420367147t^3 - 0.001479703929t^4 - 0.00234627912t^5 + 0.0001108289521t^6 + 0.000006755265331t^7 \quad (29)$$

$$z_4 = 0.000112 + 0.0001516226086t + 0.07856992564t^2 - 0.07942992343t^3 + 0.03237662782t^4 - 0.005937713462t^5 + 0.0001130954128t^6 + 0.0001245346764t^7 - 0.0000143487926t^8 - 2.092161586[10]^{-7}t^9 + 4.861773107[10]^{-8}t^{10} + 1.163378447[10]^{-9}t^{11} \quad (30)$$

$$\begin{aligned}
x_5 = & 0.998935 - 1.631781884t + 0.9352941499t^2 - 0.1288991823t^3 - 0.0911832581t^4 \\
& + 0.04477506009t^5 - 0.00564592224t^6 - 0.00127213563t^7 + 0.0005468029644t^8 - \\
& 0.00005903753127t^9 - 0.000005915359051t^{10} + 0.000002085411441t^{11} - \\
& 1.295993491[10]^{-7}t^{12} - 1.530653689[10]^{-8}t^{13} + 2.255697091[10]^{-9}t^{14} - \\
& 4.813819307[10]^{-12}t^{15} - 9.487319505[10]^{-12}t^{16} + 5.00824738[10]^{-14}t^{17} \\
& + 1.839325762[10]^{-14}t^{18} + 3.160547325[10]^{-16}t^{19}
\end{aligned} \tag{31}$$

$$\begin{aligned}
y_5 = & 0.000954 + 1.631392663t - 1.202075243t^2 + 0.3558122488t^3 - \\
& 0.02152945954t^4 - 0.01281807862t^5 + 0.002890480005t^6 - \\
& 0.00007066854884t^7 - 0.00004314855124t^8 + 0.000004687262683t^9 \\
& + 6.834380567[10]^{-8}t^{10} - 1.588175974[10]^{-8}t^{11} - 3.800361838[10]^{-10}t^{12}
\end{aligned} \tag{32}$$

$$\begin{aligned}
z_5 = & 0.000112 + 0.0001895282606t + 0.1309498761t^2 - 0.1771631783t^3 + \\
& 0.105656311t^4 - 0.03209270585t^5 + 0.002877753842t^6 + 0.001337781875t^7 \\
& - 0.0005044004698t^8 + 0.000054400335912t^9 + 0.000005847789345t^{10} - \\
& 0.000002069709567t^{11} + 1.299750808[10]^{-7}t^{12} + 1.530652689[10]^{-8}t^{13} - \\
& 2.255697091[10]^{-9}t^{14} + 4.813819307[10]^{-12}t^{15} + 9.487319505[10]^{-12}t^{16} - \\
& 5.00824738[10]^{-14}t^{17} - 1.839325762[10]^{-14}t^{18} - 3.160547325[10]^{-16}t^{19}
\end{aligned} \tag{33}$$

Using the Maple software, a graph of solutions of equations (31)-(33) above can be obtained in a Cartesian plane so that we can see the solution behavior of the three human populations (S, I, and R). Figure 1 is a graph of the solutions of the three human population groups for the spread of TB disease with the SIR model and solved using the variational iteration method at the fifth iteration.

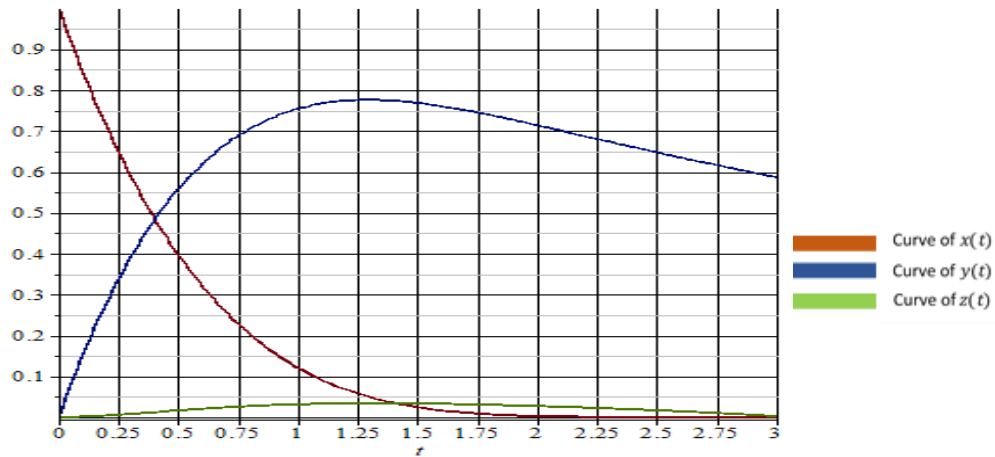


Fig. 1. Solutions to the SIR model obtained from the VIM at the fifth iteration

From Figure 1, we observe that: first, the more people infected with TB, the number of people who are vulnerable to the disease will also decrease until eventually all members of the population will be infected over time; second, the human population that is susceptible to disease

is close to zero when 3 years have passed from the time when the disease spreads; third, the rapidly infected human population to the highest level reached 77.569% of the total human population. These are results obtained using the setting we use in this paper. Different results may occur if we have different setting. However, the phenomenon will be similar. Therefore, the variational iteration method has been able to solve the SIR model.

5 Conclusion

Based on our results, the variational iteration method can estimate the solution of a nonlinear differential equation system with an initial value. The system of nonlinear differential equations studied is a system of equations of the spread of tuberculosis in the SIR epidemic model. Variational iteration methods can approximate exact solutions at any given time and the approximate solutions are continuous at all time. In addition, the formulas contained in each iteration are not complicated. Therefore, calculations carried out using variational iteration methods are easy.

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