# On The Composite Ideal

Denik Agustito, Puguh Wahyu Prasetyo, Esti Harini, Istiqomah, Tri Astuti Arigiyati, Irham Taufiq
{rafaelagustito@gmail.com[1], puguh@pmat.uad.ac.id[2], estiharini@ustjogja.ac.id[3]}

Universitas Sarajanawiyata Tamansiswa, Yogyakarta, Indonesia[1,3,4,5,6], Universitas Ahmad Dahlan Yogyakarta, Indonesia[2]

**Abstract.** This paperwork in the commutative ring and general results of my research are: ideal I is called composite if and only if R/I is a commutative ring with zero-divisor. And then we have I and J are the composite ideals of R such that I and J is coprime then $I \cap J$ is a composite ideal of R. If we have $I \subseteq J$ are two ideas of R and J is a composite ideal of R then J/I is a composite ideal of R/I.

**Keywords:** Composite, Coprimely, Ideal

## 1 Introduction

Let R be a commutative ring and a is a non-zero element of R is called not zero-divisor if there exist non-zero element b of R such that $ab \neq 0$. The commutative ring R is called ring without zero-divisor if for any $a, b \in R$ such that $a, b \neq 0$ then $ab \neq 0$. This is equivalent to saying that the commutative ring R is called without zero-divisor if-then $a = 0$ or $b = 0$. From this idea, we have a property from trivial ideal $\{0\}$ is if $ab \in \{0\}$ then $a \in \{0\}$ or $b \in \{0\}$. In a study of Commutative Algebra, the property of trivial ideal $\{0\}$ is generalized into non-trivial ideal as follow: if $ab \in I$ then $a \in I$ or $b \in I$, and then the ideal I is called a prime ideal [1].

Some properties from prime ideal in the study of Commutative Algebra is as follows: Let R be a commutative ring with the unit then P is a prime ideal of R if and only if R/P is a domain integral (Hungerford : 1974). And then we introduce a prime element p of R is a non-zero and non-unit element and $p|ab$ then $p|a$ or $p|b$ [2]. If p is a prime element in commutative ring R then the principal ideal $(p) = \{pr | r \in R\}$ is a prime ideal of R. And then if we have $P_1$ and $P_2$ is called prime ideals, then $P_1 \cap P_2$ is so.

In a commutative ring R, the non-zero element is denoted by a is called zero-divisor if there exist non-zero element b such that $ab = 0$ [3]. The commutative ring R is called ring with zero-divisor if there exist two elements $a, b \in R$ where $a, b \neq 0$ but $ab = 0$. From this idea we have a property from trivial ideal $\{0\}$ is $a, b \notin \{0\}$ but $ab \in \{0\}$. From the property of this trivial ideal, we can generalize into non-trivial ideal is as follow: If there exist $a, b \in R$ with $a, b \notin I$ but $ab \in I$ then I is called the composite ideal. The purpose of this paper is to find out the characters that must be possessed by a composite ideal in a commutative ring. Referring to the composite ideal

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Idea in a commutative ring, we get an idea of the ideal properties of a composite that is analogous to a composite number. Some things related to the development of composite numbers can be found in the following paper [4] and [5].

## 2 Main Results

In a study of Commutative Algebra, the ideal P is called prime ideal of a commutative ring if-then $a \in P$ or $b \in P$. Then we introduce a new ideal with property anti-prime is as follow:

### 2.1. Definition

Let R be a commutative ring dan I is a proper ideal of R. Ideal I si called composite ideal of R if there exist $a, b \in R, a, b \notin I$ but $ab \in I$.

### 2.2. Example

In the commutative ring $\mathbb{Z}$, the ideal $6\mathbb{Z}$ is a composite ideal, because we have $4,9 \notin 6\mathbb{Z}$ but $36 = 4.9 \in 6\mathbb{Z}$.

### 2.3. Example

In the commutative ring $\mathbb{Z}_{20}$, the ideal $J = \{\overline{0}, \overline{10}\}$ is a composite ideal, because we have $\overline{2}, \overline{5} \notin J = \{\overline{0}, \overline{10}\}$ but $\overline{10} = \overline{2}.\overline{5} \in J = \{\overline{0}, \overline{10}\}$.

### 2.4. Theorem

Let $n$ be a non-zero and non-unit element of the commutative ring of an integer number $\mathbb{Z}$. The number n is called composite number if and only if $n\mathbb{Z}$ is a composite ideal of $\mathbb{Z}$.

Proof.

Since n be a composite number, then n is not a prime number. So we have there existed $a, b \notin \mathbb{Z}$ where $n|ab$ but $n \nmid a$ and $n \nmid b$. Since there exist $a, b \notin \mathbb{Z}$ where $n|ab$ but $n \nmid a$ and $n \nmid b$, then we have there existed $a, b \notin \mathbb{Z}$ where $ab \in n\mathbb{Z}$ but $a \notin n\mathbb{Z}$ and $b \notin n\mathbb{Z}$. So $n\mathbb{Z}$ is a composite ideal of $\mathbb{Z}$.

Conversely, we have $n\mathbb{Z}$ is a composite ideal of $\mathbb{Z}$. It si clear that there exist $a, b \notin \mathbb{Z}$ where $ab \in n\mathbb{Z}$ but $a \notin n\mathbb{Z}$ and $b \notin n\mathbb{Z}$. So we have there existed $a, b \notin \mathbb{Z}$ where $n|ab$ but $n \nmid a$ and $n \nmid b$. So n is not a prime number, and then n is a composite number.

From the Theorem 1.4, we can generalized a notion about a composite element of the commutative ring as follow:

### 2.5. Definition

Let R be a more commutative ring and $q \in R$. Element q is called a composite element of R if q is non-zero and non-unit and there exist $a, b \in R$ where $q \nmid a$ and $q \nmid b$ but $q|ab$.

### 2.6. Theorem

Let R be a commutative ring and $q \in R$. Element q is called a composite element of R if and only if the principal ideal $(q) = \{qr|r \in R\}$ is a composite ideal of R.

Proof.

Since q be a composite element, then q is not a prime number. So we have there existed $a, b \notin R$ where q$|ab$ but q $\nmid a$ and q $\nmid b$. Since there exist $a, b \notin R$ where $q|ab$ but $q \nmid a$ and $q \nmid b$, then we have there existed $a, b \notin R$ where $ab \in (q)$ but $a \notin (q)$ and $b \notin (q)$. So $(q)$ is a composite ideal of $R$.

Conversely, we have $(q)$ is a composite ideal of $R$. It is clear that there exist $a, b \notin R$ where $ab \in (q)$ but $a \notin (q)$ and $b \notin (q)$. So we have there existed $a, b \notin R$ where $n|ab$ but $n \nmid a$ and $n \nmid b$. So n is not a prime element, and then n is a composite element.

The next fact is about the relationship between composite ideal I of a commutative ring with their factor ring.

### 2.7. Theorem

Let R be a commutative ring and I is an ideal of R. Factor ring R/I is a ring with zero-divisor if and only if I is a composite ideal.

Proof.

Since factor ring R/I is a ring with zero-divisor, then there exist $a + I, b + I \in R/I$ where $a + I \neq I$ and $b + I \neq I$ but $ab + I = I$. So we have there existed $a, b \in R$ where $a, b \notin I$ but $ab \in I$. So ideal I is a composite ideal.

Conversely, since ideal I is a composite ideal, then there exist $a, b \in R$ where $a, b \notin I$ but $ab \in I$. So we have there existed $a + I, b + I \in R/I$ where $a + I \neq I$ and $b + I \neq I$ but $ab + I = I$. It is clear that factor ring R/I is a ring with zero-divisor.

### 2.8. Corrolary

The natural number n is a composite number if and only if $\mathbb{Z}_n$ is not an integral domain.

Proof.

Since n is a composite number, based on Theorem 1.4, it is clear that $n\mathbb{Z}$ is the composite ideal of $\mathbb{Z}$. Based on Theorem 1.7, so we have $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$ is not an integral domain.

Conversely, since $\mathbb{Z}_n$ is not an integral domain, it is clear $\mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$ is not an integral domain. Since $\mathbb{Z}/n\mathbb{Z}$ is not an integral domain, based on Theorem 1.7, we have $n\mathbb{Z}$ is the composite ideal of $\mathbb{Z}$. Based on Theorem 1.4, we have n is a composite number.

Let R be a commutative ring and I, J are ideals of R and $I \subset J$. The ideal I is called composite on J if there exist $a, b \in J$ where $a, b \notin I$ but $ab \in I$. If $J = R$ then I become a composite on R; in another hand I is a composite ideal of R.

**2.9. Lemma**

Let R be a commutative ring and I, I are ideals of R such that $I \subset J$. If I is composite on J then I is composite on R.

Proof.

Since I is composite ideal on J, it is clear that there exist $a, b \in J$ where $a, b \notin I$ but $ab \in I$. Since J is an ideal of R, then we have I is the composite ideal of R. So I is composite on R.

**2.10. Definition**

Let R be a commutative ring and I, J are ideals of R. I and J is said coprime if $I + J = R$.

**2.11. Lemma**

Let R be a commutative ring. If I and J are ideals of R then there exist ring isomorphism $I/(I \cap J) \cong (I + J)/J$.

Some results of the relationship between coprime between ideals and ring isomorphism theorem (Lemma 1.11) is used to show that intersection between composite ideals is a composite ideal.

**2.12. Theorema**

Let R be a commutative ring and I, J ideal of R is coprime. If J is composite on R then $I \cap J$ is the composite ideal of R.

Proof.

Since ideal, I and J of R is coprime, then $I + J = R$. By Lemma 1.11, we have $I/(I \cap J) \cong (I + J)/J = R/J$. Since J is composite on R, then $R/J$ is not an integral domain, and consequently $I/(I \cap J)$ is not an integral domain. By Theorem 1.7, we have $I \cap J$ is composite on I. By Lemma 1.9, we have $I \cap J$ is the composite ideal of R.

**2.13. Corrolary**

Let R be a commutative ring and I, J are ideals of R is coprime. If I and J are composite ideals then $I \cap J$ is the composite ideal of R.

Proof.

By Theorem 1.12, since I and J are ideals of R is coprime and I and J are composite ideals, then it is clear that $I \cap J$ is the composite ideal of R.

### 2.14. Lemma

Let R be a commutative ring and I, J, K are ideals of R. If I and J is coprime, J and K is coprime, and I and K is coprime, then I and $J \cap K$ is coprime.

Proof.

Since J and K is coprime, then $JK = J \cap K$ [6]. Since I and J is coprime and I and K is coprime, then I and $JK = J \cap K$ is coprime [6]. So we have I and $J \cap K$ is coprime.

### 2.15. Theorem

Let R be a commutative ring and I, J, K are ideals of R. If I and J is coprime, J and K is coprime, and I and K is coprime, then $I \cap J \cap K$ is a composite ideal of R.

Proof.

Since I, J and K are composite ideals of R, I and J is coprime, J and K is coprime, and I and K is coprime, then based on Corrolary 1.13 we have $I \cap J$, $I \cap K$ and $J \cap K$ are composite ideals of R. Based on Lemma 1.14, we have I and $J \cap K$ is coprime. Since I is a composite ideal of R, based on Corollary 1.13, we have $I \cap J \cap K$ is a composite ideal of R.

### 2.16. Lemma

Let R be a commutative ring and I and J are ideals of R. If $I \subseteq J$ then $(R/I)/(J/I) \cong R/J$.

### 2.17. Theorem

Let R be a commutative ring and I and J are ideals of R. If $I \subseteq J$ and J is the composite ideal of R then J/I is a composite ideal of factor ring R/I.

Proof.

Since $I \subseteq J$ and J is the composite ideal of R, based on Lemma 1.16 and Theorem 1.7 we have $(R/I)/(J/I) \cong R/J$ has a zero-divisor. It is clear that $(R/I)/(J/I)$ has a zero-divisor and based on Theorem 1.7, we have J/I is a composite ideal of factor ring R/I.

## 3 Conclusion

Main results in this paper as follow: on the commutative ring R, ideal I of R is called composite ideal if and only if the factor ring R/I is ring with zero-divisor. And then if I and J are ideals of R is coprime then $I \cap J$ is the composite ideal of R. If $I \subseteq J$ and J is the composite ideal of R then J/I is the composite ideal of factor ring R/I.

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