Discount Allocation for Benefit Maximization in Social Networks

Chuangen Gao[1,2], Shuyang Gu[3], Guijuan Wang[1,2]
{gaochuangen@gmail.com, s.gu@tamuct.edu, guijuan_wang@126.com}
[1]Key Laboratory of Computing Power Network and Information Security Ministry of Education, Shandong Computer Science Center,
Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, China
[2]Shandong Provincial Key Laboratory of Industrial Network and Information System Security, Shandong Fundamental Research Center for Computer Science, Jinan 250353, China
[3]Department of Computer Information Systems, College of Business Administration, Texas A&M University-Central Texas, Killeen, TX 76549, USA

Abstract. Social networks are becoming important dissemination platforms and a large body of works have been performed on viral marketing, but most works study the benefit associated with the number of active nodes. In this paper, we study the benefit related to interactions among activated nodes. Furthermore, a real advertising campaign is often conducted with discount instead of free sample, since discount has been demonstrated to be an effective method to promote the customers' purchase behavior. Motivated by the above observations, we propose a new problem named discount allocation for benefit maximization, where a few selected users are offered with discounts and hope that they promote this influence to their friends so as to maximize the benefit between all influenced users. We analyze its complexity and propose a new method which decomposes the non-submodular objective function into the difference of two submodular functions and design three algorithms which are guaranteed to monotonically increase the objective function at every step.

Keywords: Social Networks; Discount; Benefit Maximization; Non-submodular; Difference of Submodular functions.

1 Introduction

With the advancements in information science in the last two decades, social networks find important applications in viral marketing, where a few individuals are provided with free products, hoping that the product will be recursively recommended by each individual to his friends to create a large cascade of further adoptions. Kempe et al. [1] first formulate it as the influence maximization problem on independent cascade (IC) model and linear threshold (LT) model. After that, a large body of works have been performed on viral marketing to maximize the profit associated with the number of active nodes.

IIKI 2025, December 18-21, Chengdu, People's Republic of China

DOI 10.4108/eai.18-12-2025.2365301

Different from conventional online marketing, we study the benefit associated with the strength of interactions among activated nodes, instead of the number of activated nodes. For example, when multiple people participate in one game, a large amount of data are transferred to synchronize the settings of game environment and operations between players[2]. Network provider generally charges based on the amount of data used by customers. In addition, interaction behaviors between game players are always accompanied by browsing and clicking on advertisements, which can bring some advertising revenues to the network provider[3]. We use the benefit to represent all of the revenues related to the strength of interactions between players and define a corresponding benefit maximization problem.

Furthermore, mostly a real advertising campaign is often conducted with discount instead of free sample. Discount has been demonstrated to be an effective promotional method as it creates short-term excitement and therefore immediately impacts the customers' purchase behavior[4]. Hence, in online social marketing, discount enables the seller to convince the influential users who were not able to be adopters without a coupon.

Motivated by the above observations, we consider the marketing process where a few selected users are offered with discount and hope that they promote this influence to their friends so as to maximize the benefit between all influenced users. Since the total discount is constrained by a budget, an immediate question facing is to decide which set of initial users should be selected to receive the discount, and how much should the discounts be worth. To this end, we propose the discount allocation for benefit maximization problem in social networks.

For influence maximization problem, since its property of submodularity, the greedy algorithm can achieve a guaranteed approximation with $1-1/e$. But unfortunately, our discount allocation for benefit maximization problem is not submodular, thus the greedy strategy can't be directly applied to our problem to get a guaranteed approximate solution. To solve this problem, we propose a new method called decomposition strategy in which we decompose our objective function as a difference of two submodular functons. And based on the decomposition we replace them with the modular functions which are upper or lower bound of them to address the non-submodularity part of problem and design three algorithms.

The contributions of this paper are summarized as follows.

- We propose a new problem named discount allocation for benefit maximization and we analyze the modularity and complexity of the problem.
- We propose a new method for non-submodular optimization that decomposes the objective function into the difference of two submodular functions which are monotone nondecreasing.
- Based on the modular lower and upper bounds of decomposed submodular functions mentioned above, we design three algorithms to solve the discount allocation for benefit maximization problem, which are guaranteed to monotonically increase the objective function at every step and converge to a local maxima

2 Related Work

Kempe et al.[1] formulate the influence maximization problem under IC and LT models and provide a greedy algorithm with an approximation ratio. Since then, considerable work [5, 6, 7, 8, 9, 10, 11] has been devoted to extending existing models to study influence maximization and its variants. Recently, Wang et al.[12] study the activity maximization problem which finds k seeds that maximize the sum of activity strengths among the influenced users. Their objective is similar to us. But they only consider the activity between active nodes which are connected by edges, while we consider the benefits among all active nodes regardless of whether there is an edge connecting them, which can be viewed as a significant extension of it.

For discount allocation problems, [13] first propose to offer users in social networks discounts rather than free products to trigger social cascades and model it as the continuous influence maximization problem. [14] adopt a discrete function to capture the adoption probability covering both non-adaptive and adaptive settings and propose a novel adaptive policy with bounded approximation ratio. Our work is to maximize the benefit among activated users rather than the number of influenced users.

3 Problem Formulation

In this section, we formulate discount allocation for benefit maximization problem formally and prove it is neither submodular nor supermodular by counter examples. For complexity we prove it is NP-hard by a special case of the problem.

3.1 Discount Allocation for Benefit Maximization

Our model decomposes the cascade into two stages: seeding and diffusion. In the seeding stage, the initial set of users and corresponding discounts are decided for each initial user. The initial set of users who accept the discount act as seed users. In the diffusion stage, starting from seed users, the adoption propagates across the entire social network according to IC model.

In this paper, we use the directed graph $G=(V,E)$ to represent a social network, where V is a set of users and E is a set of social relations between users. Each edge $(u,v)\in E$ is assigned with a probability p_{uv} so that when u is active, v is activated by u with probability p_{uv}. Assume there are m possible discount rates $D=\{d_1\cdots d_m\}$, each user $v\in V$ is associated with an adoption probability function $p_v : d_i \to [0,1]$, which models the probability that v accepts any discount that is greater than or equal to d_i. We assume that $p_v(d_i)\geq p_v(d_j)$ for any rational user v and discounts $d_i \geq d_j$. And the benefit between nodes is represented by a nonnegative function $b : V\times V \to R^+$, in which $b(u,v)=b(v,u)$ for the unordered pair $\{u,v\}$ of node u and v. Note that for each $\{u,v\}$, we only compute once the benefit between them, i.e., $b(u,v)$ or $b(v,u)$ instead of $b(u,v)+b(v,u)$.

Denote $W\triangleq V\times D$ as the solution space, the pair $w=(v(w),d(w))\in W$ means allocate discout $d(w)\in D$ to user $v(w)\in V$. We use $C\subseteq W$ to denote a strategy for discounts allocation. Please note that it is feasible to assign multiple discounts to the same user, however, since her adoption decision only depends on the highest discount, it suffices to use the highest discount as a representative. Let $d_v(C)$ denote the highest discount assigned to user v under the strategy C, and $d_v(C)=0$ if v has not been allocated a discount. Under the allocation strategy C, the probability that a subset $S \subseteq V$ become the seed set is

$$Pr(S,C) = \prod_{v \in S} p_v(d_v(C)) \prod_{v \in V \setminus S} (1 - p_v(d_v(C))) \tag{1}$$

For any seed set S, denote by $I(S)$ the set of all active nodes at end of the diffusion process. The expected benefit would be defined as

$$B(S) = \mathbb{E}[\sum_{\{u,v\} \subseteq I(S)} b(u,v)] \tag{2}$$

where $\{u,v\} \subseteq I(S)$ denote the all unordered pair in the set $I(S)$. Note that for each unordered pair $\{u,v\}$, since $b(u,v) = b(v,u)$, we only compute once the benefit between them.

Then under the discount allocation strategy C, the expected benefit is

$$f(C) = \sum_{S \in 2^V} Pr(S,C) \cdot B(S) \tag{3}$$

In this paper, we study the following problem.

Definition 1 (Discount Allocation for Benefit Maximization Problem, DABMP)**.** *Given a social network $G = (V,E)$, a propagation probability p_{uv} for each edge (u,v) under the IC model, a benefit function $b : V \times V \to R^+$, a discount rates set $D = \{d_1 \cdots d_m\}$, and an adoption probability function $p_v : d_i \to [0,1]$, and a budget H, find a discount allocation strategy C to maximize the expected benefit between all activated users through influence propagation:*

$$\max f(C) \tag{4}$$

$$s.t. \sum_{v \in V} d_v(C) \le H \tag{5}$$

As an example of DABM problem, we use a toy social network in Fig. 1. There are three nodes $V = \{a,b,c\}$; we have discout rates $D = \{0.4, 0.6\}$; propagation probabilities are shown on the edges; adoption probabilities $p_v(0.4) = 0.2$, $p_v(0.6) = 1$, $\forall v \in \{a,b,c\}$; budget $H = 1$; benefit funciton $b(u,v) = 1$, $\forall \{u,v\} \subseteq \{a,b,c\}$. Let us consider a discount allocation strategy $C = \{(a,0.4),(b,0.6)\}$, i.e., allocating a with discount 0.4 and b with discount 0.6. Then $Pr(\{a,b\},C) = 0.2 \times 1 = 0.2$, $Pr(\{b\},C) = (1-0.2) \times 1 = 0.8$. When $\{a,b\}$ are the seeds, c is activated with probability $1-(1-0.4)\times(1-0.1) = 0.46$(i.e., $I(\{a,b\}) = \{a,b,c\}$). The probability that c is not activated is $1-0.46 = 0.54$(i.e., $I(\{a,b\}) = \{a,b\}$). So $B(\{a,b\}) = 0.46 \times \sum_{\{u,v\} \subseteq \{a,b,c\}} b(u,v) + 0.54 \times \sum_{\{u,v\} \subseteq \{a,b\}} b(u,v) = 0.46 \times 3 + 0.54 \times 1 = 1.92$. And we have $B(\{b\}) = 0.1 \times \sum_{\{u,v\} \subseteq \{b,c\}} b(u,v) = 0.1 \times 1 = 0.1$. So $f(C) = Pr(\{a,b\},C) \times B(\{a,b\}) + Pr(\{b\},C) \times B(\{b\}) = 0.2 \times 1.92 + 0.8 \times 0.1 = 0.464$.

3.2 Modularity of Objective Function

We say that $g(\cdot)$ is submodular if it satisfies a natural "diminishing returns" property: the marginal gain from adding an element to a set X is at least as high as the marginal gain from adding

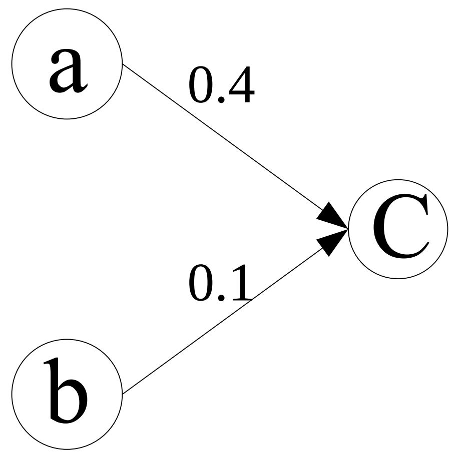

Fig. 1. A toy social network

the same element to a superset of X. Formally, For every $X, Y \subseteq V$ with $X \subseteq Y$ and every $e \in V \setminus Y$, it follows that

$$g(X \cup \{e\}) - g(X) \geq g(Y \cup \{e\}) - g(Y)$$

And it is monotone if $g(X) \leq g(Y)$ whenever $X \subseteq Y$.

Theorem 1. *$f(C)$ is neither submodular nor supermodular under IC model.*

Proof. We prove by two counter examples shown in Fig.2. We set discount rates $D = \{1\}$ and adoption probability $p_v(1) = 1, \forall v \in V$ which means each user will accept the product since it's free. The first element in the tuple tied on each edge represents the propogation probability, and the second one denote the benefit between its two end nodes. For pairs $\{u, v\}$ between which there is no edge set $b(u, v) = 0$. In Fig.2 (i), $(0, 1)$ on edge (a, b) means propogation probability $p_{ab} = 0$ and $b(a, b) = 1$, then we have $f(\{(a, 1)\}) = 0$, $f(\{(a, 1), (d, 1)\}) = 1$, $f(\{(a, 1), (b, 1)\}) = 1$ and $f(\{(a, 1), (b, 1), (d, 1)\}) = 3$. Thus, $f(\{(a, 1), (d, 1)\}) - f(\{(a, 1)\}) < f(\{(a, 1), (b, 1), (d, 1)\}) - f(\{(a, 1), (b, 1)\})$, which implies $f(\cdot)$ is not submodular. In Fig.2 (ii), we have $f(\{(a, 1)\}) = f(\{(a, 1), (b, 1)\}) = f(\{(b, 1)\}) = 4$. Thus, $f(\{(b, 1)\}) - f(\emptyset) > f(\{(a, 1), (b, 1)\}) - f(\{(a, 1)\})$ which implies $f(\cdot)$ is not supermodular. $\square$

3.3 Hardness Result

Theorem 2. *Discount allocation for benefit maximization problem is NP-hard.*

Proof. We prove by showing a special case of DABMP is NP-hard, where discount rates $D = \{1\}$ and adoption probability $p_v(1) = 1, \forall v \in V$, which means each user will accept the product since it's free. The budget equals k. Then the problem is transferred to seek k seeds that maximize the benefit between activated nodes. Now we prove by reducing from the set cover problem, which is

(i) (ii)

Fig. 2. Counter examples

NP-complete [15]. Given a ground set $U = \{u_1, u_2, \ldots, u_n\}$ and a collection of sets $\{S_1, S_2, \ldots, S_m\}$ whose union equals the ground set, the set cover problem is to decide if there exist k sets in S so that the union equals U. Given an instance of the set cover problem, we construct a corresponding graph with $m + 2n$ nodes as follows. For each set S_i we create one node p_i, and for each element u_j we create two nodes q_j and q'_j. If the S_i contains the element u_j, then we create two edges (p_i, q_j) and (p_i, q'_j). Note that each edge is live which means the probability is 1. Now we design the benefit function over pairs of nodes. For the pairs $\{q_j, q'_j\}$, the benefit equals to 1, and the other pairs equal to 0. Then the set cover problem is equivalent to deciding if there is a set S of k nodes such that the benefit of S equals to n. The theorem follows immediately. □

4 Decomposition

Since discount allocation for benefit maximization problem is not submodular, the greedy strategy can't be directly applied to our problem to get a guaranteed approximate solution. To solve this non-submodular problem, we propose a new method called decomposing strategy in which we decompose our objective function as a difference of two submodular functions. Our idea is inspired by the fact that any set function can be expressed as a difference between two submodular functions [16]. However, in general, it is conjectured to be NP-hard to do so. In our case, it is not trivial, but we successfully found a decomposition with special technique and moreover, we made obtained submodular functions computationally possible.

For seed set S, we define the $B_1(S)$ as benefit between activated users $I(S)$ and all users V, and define $B_2(S)$ as the benefit among all activated users $I(S)$ plus the benefit between the activated users

$I(S)$ and the non-activated users $V \setminus I(S)$, which are formulated as follows:

$$B_1(S) = \mathbb{E}[\sum_{u \in I(S)} \sum_{v \in V} b(u,v)]$$
$$= \mathbb{E}[\sum_{\{u,v\} \subseteq I(S)} 2 \cdot b(u,v) + \sum_{u \in I(S)} \sum_{v \in V \setminus I(S)} b(u,v)]$$

$$B_2(S) = \mathbb{E}[\sum_{\{u,v\} \subseteq I(S)} b(u,v) + \sum_{u \in I(S)} \sum_{v \in V \setminus I(S)} b(u,v)]$$

Thus we have

$$B(S) = B_1(S) - B_2(S)$$

And under discount allocation strategy C, we define the following functions

$$f_1(C) = \sum_{S \in 2^V} Pr(S,C) \cdot B_1(S) \tag{6}$$

$$f_2(C) = \sum_{S \in 2^V} Pr(S,C) \cdot B_2(S) \tag{7}$$

Then

$$f(C) = \sum_{S \in 2^V} Pr(S,C) \cdot B(S)$$
$$= \sum_{S \in 2^V} Pr(S,C) \cdot B_1(S) - \sum_{S \in 2^V} Pr(S,C) \cdot B_2(S)$$

So we have

$$f(C) = f_1(C) - f_2(C) \tag{8}$$

Actually $f(C)$ is decompose as a difference of two functions, now we prove both of them are submodular. To facilitate analysis, we introduce the following two concepts to deal with the randomness of the propagation process.

Definition 2 (Seeding realization ψ)**.** *When a node v is offered the discount d, v accepts offer and becomes the seed with probability $p_v(d)$ and rejects it with probability $1 - p_v(d)$, i.e, $w = (v,d)$ is probed. The outcome of this random event could be determined by comparing $p_v(d)$ with a random number r that is uniformly selected from $[0,1]$: if $p_v(d) \geq r$, then $w = (v,d)$ is declared to be valid under ψ, i.e, v accept the discout and become a seed.*

Definition 3 (Diffusion realization ϕ)**.** *For each edge (u,v) in the graph, a coin of bias p_{uv} is flipped at the very beginning of the process. The edges for which the coin flip indicated a successful activation are declared to be live in ϕ; the remaining edges are declared to be blocked in ϕ, which construct a live edge graph [1]. A user is influenced if it can be reached from some seed through a path consisting of live edges.*

Note that similar to the diffusion realization, where the coin can be flipped at the beginning of the process, the seeding realization can also be performed ahead of the actual seeding stage. With both seeding and diffusion realizations are performed in advance, it is easy to determine the subset of nodes that can be influenced at the end of the cascade process: given a configuration C and realization ψ, ϕ, a node u ends up influenced if and only if u is live or there is a path from some live node to u consisting entirely of live edges[14].

Theorem 3. *$f_1(C)$ is submodular and monotone under the IC model.*

Proof. By the property of seeding and diffusion realizations, $f_1(C)$ can be fomulated as

$$f_1(C) = \sum_{\psi,\phi} \Pr(\psi,\phi) \cdot f_1(C \mid \psi,\phi) \tag{9}$$

Due to the fact that a non-negative linear combination of submodular functions is also submodular, we just need to prove $f_1(C \mid \psi,\phi)$ is submodular under realization (ψ,ϕ).

According to equation 6, we have

$$f_1(C \mid \psi,\phi) = \sum_{S\in 2^V} Pr(S,C \mid \psi) \cdot B_1(S \mid \phi) \tag{10}$$

$$= \sum_{u\in I(S(C))} \sum_{v\in V} b(u,v) \tag{11}$$

$$= \sum_{u\in R_g(I(S(C)))} \sum_{v\in V} b(u,v) \tag{12}$$

Note that we denote $Pr(S,C \mid \psi)$ as the probability that nodes S become the seeds given discount allocation strategy C under seeding realization ψ. Since the realization is given which means the threshold of acceptance that a user become a seed is deterministic, $Pr(S,C \mid \psi) = 1$ or $Pr(S,C \mid \psi) = 0$, for some nodes S. Thus there is only one user set $S \in 2^V$ will become seeds under above settings and we use $S(C)$ denote the users which accept the discounts and become the seeds under seeding realization ψ. We denote $B(S \mid \phi)$ as the benefit under diffusion realization ϕ. So equation 10 follows. And g denotes that the live graph generated under realization ϕ, $R_g(S)$ denotes the set of nodes reachable from seed set S via live edges in g. So we have equation 12 from equation 11.

Let discount allocation strategy M,N be two sets such that $M \subseteq N \subseteq W$. For any node-discount pair $x \in W \setminus N$,

We have

$$f_1(M \cup \{x\} \mid \psi,\phi) - f_1(M \mid \psi,\phi) \\ = \sum_{u\in R_g(S(x))\setminus R_g(S(M))} \sum_{v\in V} b(u,v) \tag{13}$$

$$f_1(N \cup \{x\} \mid \psi,\phi) - f_1(N \mid \psi,\phi) \\ = \sum_{u\in R_g(S(x))\setminus R_g(S(N))} \sum_{v\in V} b(u,v) \tag{14}$$

Comparing all terms on the right-hand sides of 13 and 14, we have $S(M) \subseteq S(N)$, since $M \subseteq N$. And $R_g(S(x)) \setminus R_g(S(M)) \supseteq R_g(S(x)) \setminus R_g(S(N))$ follows.Through above analysis, we obtain $f_1(M \cup \{x\} \mid \psi, \phi) - f_1(M \mid \psi, \phi) \geq f_1(N \cup \{x\} \mid \psi, \phi) - f_1(N \mid \psi, \phi)$. Therefore, $f_1(C)$ is submodular.

For monotonicity, we need prove $f_1(M \mid \psi, \phi) \leq f_1(N \mid \psi, \phi)$, which is non-decreasing. According to equation 12, we have

$$f_1(M \mid \psi, \phi) = \sum_{u \in R_g(I(S(M)))} \sum_{v \in V} b(u, v) \tag{15}$$

$$f_1(N \mid \psi, \phi) = \sum_{u \in R_g(I(S(N)))} \sum_{v \in V} b(u, v) \tag{16}$$

Since the benefit function $b : V \times V \to R^+$ is non-negative and $S(M) \subseteq S(N)$, the monotonicity of $f_1(C)$ follows immediately. □

Theorem 4. *$f_2(C)$ is submodular and monotone under the IC model.*

Proof. By seeding and diffusion realizations, $f_2(C)$ can be fomulated as

$$f_2(C) = \sum_{\psi, \phi} \Pr(\psi, \phi) \cdot f_2(C \mid \psi, \phi) \tag{17}$$

Due to the fact that a non-negative linear combination of submodular functions is also submodular, we just need to prove $f_2(C \mid \psi, \phi)$ is submodular under realization (ψ, ϕ). According to equation 7, we have

$$\begin{aligned} & f_2(C \mid \psi, \phi) \\ =& \sum_{S \in 2^V} Pr(S, C \mid \psi) \cdot B_2(S \mid \phi) \\ =& \sum_{\{u,v\} \subseteq I(S(C))} b(u, v) + \sum_{u \in I(S(C))} \sum_{v \in V \setminus I(S(C))} b(u, v) \\ =& \sum_{\{u,v\} \subseteq R_g(S(C))} b(u, v) + \sum_{u \in R_g(S(C))} \sum_{v \in V \setminus R_g(S(C))} b(u, v) \end{aligned} \tag{18}$$

where $S(C)$ denote the users which accept the discounts and become the seeds under seeding realization ψ, g denotes that the live graph generated under realization ϕ, $R_g(S)$ denotes the set of nodes reachable from S via live edges in g .

Let discount allocation strategy M, N be two sets such that $M \subseteq N \subseteq W$. For any node-discount pair $x \in W \setminus N$, we have

$$\begin{aligned} & f_2(M \cup \{x\} \mid \psi, \phi) - f_2(M \mid \psi, \phi) \\ & = \sum_{\{u,v\} \subseteq R_g(S(x)) \setminus R_g(S(M))} b(u,v) + \\ & \quad \sum_{u \in R_g(S(x)) \setminus R_g(S(M))} \sum_{v \in V \setminus R_g(S(M \cup \{x\}))} b(u,v) \end{aligned} \tag{19}$$

$$\begin{aligned} & f_2(N \cup \{x\} \mid \psi, \phi) - f_2(N \mid \psi, \phi) \\ & = \sum_{\{u,v\} \subseteq R_g(S(x)) \setminus R_g(S(N))} b(u,v) + \\ & \quad \sum_{u \in R_g(S(x)) \setminus R_g(S(N))} \sum_{v \in V \setminus R_g(S(N \cup \{x\}))} b(u,v) \end{aligned} \tag{20}$$

Comparing all terms on the right-hand sides of 19 and 20, we have $S(M) \subseteq S(N)$, since $M \subseteq N$. And $R_g(S(x)) \setminus R_g(S(M)) \supseteq R_g(S(x)) \setminus R_g(S(N))$ follows. By the same way, since $M \subseteq N$, we have $S(M \cup \{x\}) \subseteq S(N \cup \{x\})$ and $V \setminus R_g(S(M \cup \{x\})) \supseteq V \setminus R_g(S(N \cup \{x\}))$. Through above analysis, we obtain $f_2(M \cup \{x\} \mid \psi, \phi) - f_2(M \mid \psi, \phi) \geq f_2(N \cup \{x\} \mid \psi, \phi) - f_2(N \mid \psi, \phi)$. Therefore, $f_2(C)$ is submodular.

For monotonicity, we need prove $f_2(M \mid \psi, \phi) \leq f_2(N \mid \psi, \phi)$, which is non-decreasing. According to equation 18, we have

$$\begin{aligned} & f_2(M \mid \psi, \phi) \\ = & \sum_{\{u,v\} \subseteq R_g(S(M))} b(u,v) + \sum_{u \in R_g(S(M))} \sum_{v \in V \setminus R_g(S(M))} b(u,v) \\ = & \sum_{\{u,v\} \subseteq R_g(S(M))} b(u,v) + \sum_{u \in R_g(S(M))} \sum_{v \in R_g(S(N)) \setminus R_g(S(M))} b(u,v) \\ & \quad + \sum_{u \in R_g(S(M))} \sum_{v \in V \setminus R_g(S(N))} b(u,v) \end{aligned}$$

$$\begin{aligned} & f_2(N \mid \psi, \phi) \\ = & \sum_{\{u,v\} \subseteq R_g(S(N))} b(u,v) + \sum_{u \in R_g(S(N))} \sum_{v \in V \setminus R_g(S(N))} b(u,v) \end{aligned}$$

Since the benefit function $b : V \times V \to R^+$ is non-negative, $S(M) \subseteq S(N)$ and $\forall (i,j) \in \{(u,v) \mid u \in R_g(S(M)), v \in R_g(S(N)) \setminus R_g(S(M))\}$, we have $i \in R_g(S(N))$, $j \in R_g(S(N))$. Thus the sum of first two items of $f_2(M)$ is less than the first item of $f_2(N)$. By the same reason, we have the third item of $f_2(M)$ is less than the second item of $f_2(N)$. Through above analysis, the monotonicity of $f_2(C)$ follows immediately. □

5 Algorithm

According to the decomposed submodular functions, we designed three heuristic algorithms based on the modular upper and lower bounds of the corresponding submodular functions. The main idea is based on the framework which is used for minimization of the difference between submodular functions[17]. Our algorithms are guaranteed to monotonically increase the objective function at every step.

5.1 Modular Upper Bound

For any submodular set function $g(\cdot)$ on ground set V, we have the following two tight modular upper bounds that are tight at a given set X [17]:

$$m^g_{X,1}(Y) \triangleq g(X) - \sum_{j \in X \setminus Y} g(j \mid X \setminus j) + \sum_{j \in Y \setminus X} g(j \mid \emptyset)$$

$$m^g_{X,2}(Y) \triangleq g(X) - \sum_{j \in X \setminus Y} g(j \mid V \setminus j) + \sum_{j \in V \setminus X} g(j \mid X)$$

5.2 Modular Lower Bound

A modular lower bound of $g(\cdot)$ is tight at a given set X can be obtained as follows ([17]). Let σ be a permutation of V and define $S^\sigma = \{\sigma(1), \sigma(2), \ldots, \sigma(i)\}$ as σ's chain containing Y, in which $S^\sigma_0 = \emptyset$ and $S^\sigma_{|X|} = X$. Define

$$h^g_{X,\sigma}(\sigma(i)) = g(S^\sigma_i) - g(S^\sigma_{i-1}).$$

Then

$$h^g_{X,\sigma}(Y) \triangleq \sum_{v \in Y} h^g_{X,\sigma}(v)$$

is a tight lower bound of $g(Y)$, i.e., $h^g_{X,\sigma}(Y) \leq g(Y)$, $\forall Y \subseteq V$, and $h^g_{X,\sigma}(X) = g(X)$.

5.3 Procedures

Algorithm 1 S-M Procedure

1: initialize $X^0 \leftarrow$ random allocation strategy C; $t \leftarrow 0$
2: **repeat**
3: $\quad X^{t+1} \leftarrow \text{argmax}_X f_1(X) - m^{f_2}_{X^t}(X)$;
4: $\quad t \leftarrow t+1$;
5: **until** *converged, i.e.,* $X^t = X^{t-1}$
6: **return** X^t or $\max X$;

For algorithm 1 named submodular-modular procedure (S-M for short), we iteratively minimized $f_1(C) - f_2(C)$ by replacing $f_2(C)$ by its modular upper bound at every iteration, which is actually a lower bound of the original function. At every iteration we maximize only a submodular function under knapsack constraint, which can be sovled by a practical greedy algorithm with $1-\frac{1}{\sqrt{e}}$ approximation factor[18]. Since we have two modular upper bounds, we can use any of the variants discussed above.

Algorithm 2 M-S Procedure

initialize $X^0 \leftarrow$ random allocation strategy C; $t \leftarrow 0$
repeat
  choose a permutation σ^t whose chain contains the set X^t;
  $X^{t+1} \leftarrow \text{argmax}_X h^{f_1}_{X^t,\sigma^t}(X) - f_2(X)$;
  $t \leftarrow t+1$;
until *converged, i.e.,* $X^t = X^{t-1}$;
return X^t;

For algorithm 2, named modular-submodular procedure (M-S for short), we iteratively maximize $f_1(C) - f_2(C)$ by replacing $f_1(C)$ by its modular lower bound at every iteration to approximate the original problem, which is actually an lower bound of the original function. At every iteration we minimize a submodular function under knapsack constraint, which is still an open problem. We use the simple greedy algorithm which is similar to [18].

Algorithm 3 M-M Procedure

initialize $X^0 \leftarrow$ random allocation strategy C; $t \leftarrow 0$
repeat
  choose a permutation σ^t whose chain contains the set X^t;
  $X^{t+1} \leftarrow \text{argmax}_X h^{f_1}_{X^t,\sigma^t}(X) - m^{f_2}_{X^t}(X)$;
  $t \leftarrow t+1$;
until *converged, i.e.,* $X^t = X^{t-1}$;
return X^t;

For algorithm 3, named modular-modular procedure (M-M for short), we use the modular lower bound of $f_1(\cdot)$ minus the modular upper bound of $f_2(\cdot)$ to approximate the original problem. At every iteration we maximize only a modular function which can be done in $O(n)$ time, so this is extremely easy. Like before, since we have two modular upper bounds, we can use any of the variants discussed in the subsection above.

5.4 Analysis

Since optimizing the difference between two submodular functions is multiplicative inapproximability unless $P = NP$. Thus, it is difficult to measure the gap between the discount allocation strategy obtained and an optimal allocaiton strategy. Note that the M-S, S-M and M-M function is actually a lower bound of the original function. This indicates that all the three algorithms are still guaranteed to monotonically increase the objective at every iteration and converge to a local maxima. Because [17] have given detailed proofs, we will not do repetitive work in this article for respect to the originality of their work. Although their work is about minimizing the difference of the submodular function, while our problem is to maximize the difference of the submodule function, in fact, the two are one-to-one correspondence.

6 Conclusion

In this paper, we investigated the discount allocation for benefit maximization problem in social networks, where benefit is derived from interactions among activated users rather than merely the number of active nodes. Unlike traditional viral marketing works that focus on free samples, our model incorporates discounts as a realistic promotion mechanism. We proved that the objective function is neither submodular nor supermodular and that the problem is NP-hard. To tackle this non-submodular optimization, we decomposed the objective into the difference of two submodular functions (DS decomposition). Leveraging this decomposition, we designed three algorithms, each guaranteed to monotonically increase the objective function at every step, converging to a local maximum. Our approach bridges the gap between theoretical non-submodularity and practical algorithm design. The results demonstrate the effectiveness of our methods in capturing interaction-based benefits under discount allocation. This work provides a new perspective on viral marketing by emphasizing user interactions over simple activation counts, and offers a tractable solution framework for non-submodular optimization problems arising in social network advertising campaigns.

Acknowledgments

This work was supported in part by the Natural Science Foundation of Shandong Province under Grant ZR2022QF010, in part by the Colleges and Universities 20 Terms Foundation of Jinan City under Grant 202228093, in part by the Major Program of Shandong Provincial Natural Science Foundation for the Fundamental Research under Grant ZR2022ZD03, in part by the National Science Foundation of China under Grants 62272256 and 62202250, in part by the Shandong Province Youth Innovation Team Project under Grant 2024KJH032, in part by the National Natural Science Foundation of China under Grant 62402254.

Declaration on Generative AI

The author(s) have not employed any Generative AI tools.

References

[1] Kempe D, Kleinberg J, Tardos E. Maximizing the Spread of Influence Through a Social Network. In: International Conference on Knowledge Discovery and Data Mining. KDD '03.

ACM; 2003. p. 137-46. Available from: http://doi.acm.org/10.1145/956750.956769.

[2] Xue Z, Wu D, He J, Hei X, Liu Y. Playing High-End Video Games in the Cloud: A Measurement Study. IEEE Transactions on Circuits and Systems for Video Technology. 2015 Dec;25(12):2013-25.

[3] Wang A, Wu W, Cui L. On Bharathi–Kempe–Salek conjecture for influence maximization on arborescence. Journal of Combinatorial Optimization. 2016;31(4):1678-84.

[4] Tong GA, Wu W, Du D. Coupon Advertising in Online Social Systems: Algorithms and Sampling Techniques. CoRR. 2018;abs/1802.06946. Available from: http://arxiv.org/abs/1802.06946.

[5] Chen W, Yuan Y, Zhang L. Scalable Influence Maximization in Social Networks under the Linear Threshold Model. In: 2010 IEEE International Conference on Data Mining; 2010. p. 88-97.

[6] Chen W, Lin T, Tan Z, Zhao M, Zhou X. Robust Influence Maximization. In: International Conference on Knowledge Discovery and Data Mining. KDD '16. New York, NY, USA: ACM; 2016. p. 795-804. Available from: http://doi.acm.org/10.1145/2939672.2939745.

[7] Lu W, Chen W, Lakshmanan LVS. From Competition to Complementarity: Comparative Influence Diffusion and Maximization. Proc VLDB Endow. 2015 Oct;9(2):60-71. Available from: http://dx.doi.org/10.14778/2850578.2850581.

[8] Tang Y, Shi Y, Xiao X. Influence Maximization in Near-Linear Time: A Martingale Approach. In: International Conference on Management of Data. SIGMOD '15. ACM; 2015. p. 1539-54. Available from: http://doi.acm.org/10.1145/2723372.2723734.

[9] Nguyen HT, Thai MT, Dinh TN. Stop-and-Stare: Optimal Sampling Algorithms for Viral Marketing in Billion-scale Networks. In: International Conference on Management of Data. SIGMOD '16. ACM; 2016. p. 695-710. Available from: http://doi.acm.org/10.1145/2882903.2915207.

[10] Tang J, Tang X, Yuan J. Towards Profit Maximization for Online Social Network Providers. CoRR. 2017;abs/1712.08963. Available from: http://arxiv.org/abs/1712.08963.

[11] Tang Y, Xiao X, Shi Y. Influence Maximization: Near-optimal Time Complexity Meets Practical Efficiency. In: International Conference on Management of Data. SIGMOD '14. ACM; 2014. p. 75-86. Available from: http://doi.acm.org/10.1145/2588555.2593670.

[12] Wang Z, Yang Y, Pei J, Chu L, Chen E. Activity Maximization by Effective Information Diffusion in Social Networks. IEEE Transactions on Knowledge and Data Engineering. 2017 Nov;29(11):2374-87.

[13] Yang Y, Mao X, Pei J, He X. Continuous Influence Maximization: What Discounts Should We Offer to Social Network Users? In: Proceedings of the 2016 International Conference on Management of Data. SIGMOD '16. New York, NY, USA: ACM; 2016. p. 727-41. Available from: http://doi.acm.org/10.1145/2882903.2882961.

[14] Yuan J, Tang SJ. Adaptive discount allocation in social networks. In: Proceedings of the 18th ACM International Symposium on Mobile Ad Hoc Networking and Computing. ACM; 2017. p. 22.

[15] Alon N, Awerbuch B, Azar Y. The Online Set Cover Problem. In: Proceedings of the Thirty-fifth Annual ACM Symposium on Theory of Computing. STOC '03. New York, NY, USA: ACM; 2003. p. 100-5. Available from: http://doi.acm.org/10.1145/780542.780558.

[16] Narasimhan M, Bilmes J. A Submodular-supermodular Procedure with Applications to Discriminative Structure Learning. In: Proceedings of the Twenty-First Conference on Uncertainty in Artificial Intelligence. UAI'05. Arlington, Virginia, United States: AUAI Press; 2005. p. 404-12. Available from: http://dl.acm.org/citation.cfm?id=3020336.3020387.

[17] Iyer R, Bilmes J. Algorithms for Approximate Minimization of the Difference Between Submodular Functions, with Applications. In: Proceedings of the Twenty-Eighth Conference on Uncertainty in Artificial Intelligence. UAI'12. Arlington, Virginia, United States: AUAI Press; 2012. p. 407-17. Available from: http://dl.acm.org/citation.cfm?id=3020652.3020697.

[18] Lin H, Bilmes J. Multi-document Summarization via Budgeted Maximization of Submodular Functions. In: Human Language Technologies: The 2010 Annual Conference of the North American Chapter of the Association for Computational Linguistics. HLT '10. Stroudsburg, PA, USA: Association for Computational Linguistics; 2010. p. 912-20. Available from: http://dl.acm.org/citation.cfm?id=1857999.1858133.