Optimization of Portfolio Theory under the Constraint of Mean-Skewness Standardization

Empirical Research Based on China's Securities Market

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Abstract—This paper introduces the concept of skewness on the basis of the traditional mean-variance (MV) optimization model, and expands the two-dimensional optimization research into a three-dimensional mean-variance-skewness (MVS) optimization model. At the same time, under the more realistic assumption that the rate of return obeys a partial normal distribution, the Sharpe ratio is improved, and the active ratio is introduced to measure the performance of market portfolio risk-adjusted returns. On this basis, this paper analyzes the data of China's securities market and proves that the risk-adjusted return of China's securities market has a large degree of skew, and even affects the choice of investors' assets, which has a strong theoretical value for investors' asset allocation.

Keywords-component; Portfolio; Skewness; Chinese securities market; Activity

1 INTRODUCTION

In 1952, Markowitz brought linear algebra and probability theory into the field of securities investment portfolio, and later developed into modern investment portfolio theory. With the passage of time, the shortcomings of traditional modern investment portfolio theory lacking empirical research support and theoretical assumptions that do not conform to reality have gradually been exposed. Many scholars subsequently conducted in-depth research on improving modern investment portfolios. One of the directions is to increase skewness constraints to form an MVS model. As far as the domestic market is concerned, the development of China's capital market is relatively late, and there are little related theoretical studies, and most of them refer to the research results of foreign countries. Zhang Shubin et al. used the mean and variance as constraints to solve the maximum value of the objective function, and established a mean-variance-skewness asset portfolio optimization model with transaction costs [1]. Xiao Dongrong and Huang Jing proposed a fuzzy multi-objective portfolio selection model that considered mean, variance, and skewness at the same time. They believe that adding skewness considerations can improve the effectiveness of mean-variance portfolios, but due to the fact that the securities market is affected by many factors. The impact is vague [2]. Chonghui Jiang added the constraint of system skewness under the framework of mean-variance, and established a mean-variance-

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system skewness portfolio management model. Research shows that the model with skewness constraint is more perfect than the traditional effective combination [3].

On the other hand, China's securities market, as an emerging high-growth securities market, has achieved world-renowned achievements in just three decades. Modern asset portfolio theory has been widely used in asset allocation and asset portfolio in foreign securities markets. However, whether this theory is applicable to my country's securities market has been controversial. Therefore, it is of great practical significance to conduct an empirical study on the modern investment portfolio of the world's second-largest Chinese securities market. Similarly, domestic scholars have made a lot of exploration in the empirical aspects of modern investment portfolio theory. Li Jinxin et al. found that the mean-variance strategy limiting short selling and the minimum variance strategy limiting short selling had the best performance in the China's market by using the monthly data from January 1998 to December 2009, but neither mean-variance model nor mean-variance expansion model was statistically superior to the simple diversification strategy [4]. Shen Shefang used the 2011 China's stock market weekly return rate and fund return rate, and under the premise of short selling restrictions, verified the mean-variance model, and found that the theoretical performance of the portfolio is inferior to the actual fund performance, and the portfolio theory is not applicable to Chinese capital Market [5]. Zhou Ye Qin et al. randomly selected five securities from different industries to construct a portfolio, and selected the pre-weighted weekly return data from January 2015 to January 2018, it is found that the return and risk of the five portfolios outperform the market during the period, indicating that the mean-variance portfolio has good profitability and reduces the investment risk [6].

In summary, the empirical results of the mean-variance effective model in the China's securities market are different, which shows that Markowitz's mean-variance effective model is not completely suitable for the Chinese securities market. In theory, the algorithm that uses the only closed solution of the mean-variance-skew (MVS) optimization model is the most suitable. The reasons are as follows: First, many previous studies have shown that the return of China's securities market is non-normal and more in line with a partial normal distribution. Second, the uniqueness of the solution. There are few researches on the application of the only closed solution of the skewness-mean model in the Chinese market. Therefore, this article decided to use the skewness-mean model to make up for this part of the gap. Third, it is more intuitive and interpretable. The original paper model converts the effective frontier and capital allocation line in two dimensions into three-dimensional graphics, which can directly make investment decisions based on the performance of known solutions on the graphics. Fourth, the calculation is more concise, and it overcomes the shortcomings of a large amount of calculation in the mean-variance model. Although the MVS model with a unique closed solution is more complex than the solution of the MV model, compared with the non-unique or non-closed form of the MVS model, it uses non-approximate iterative calculations, giving the results at one time, and the calculations are more concise.

2 MATERIALS AND METHODS

2.1 MV Model

For the purposes of comparison, the famous solutions of the traditional MV optimization model are reformulated here. Let a random vector $X \triangleq [X_1 \; ... \; X_p]^\intercal$ be the return vector on p risky assets with a weight vector $\mathbf{w}_X$, and fulfill $X \sim \mathcal{N}(\boldsymbol{\mu}_X, \boldsymbol{\Sigma}_{XX})$. Let a random variable R denote the return on a risk-free asset with a weight $w_R \triangleq 1 - \mathbf{1}_p^\top \mathbf{w}_X$, and fulfill $R \sim \mathcal{N}(\mu_R, \sigma_{RR}^2)$. The constrained minimization problem is equivalent to

$$\begin{array}{ll} \min\limits_{\mathbf{w}} & \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \\ \text{s. t.} & \mathbf{w}^\top \boldsymbol{\mu} + w_0 r = \mathbf{w}^\top \boldsymbol{\mu} + (1 - \mathbf{w}^\top \mathbf{1}_p) r = \overline{\mu}. \end{array} \tag{1}$$

We can obtain the solution of $\mathbf{w}$:

$$\mathbf{w} = \boldsymbol{\Sigma}^{-1} \left[\frac{\boldsymbol{\mu} - r\mathbf{1}_p}{G} (\overline{\mu} - r) \right], \tag{2}$$

and solution of $\mathbf{w}$ when there only exist risky assets in the portfolio:

$$\mathbf{w} = \boldsymbol{\Sigma}^{-1} \left[\frac{C\mathbf{1}_p - B\boldsymbol{\mu}}{P} + \frac{A\boldsymbol{\mu} - B\mathbf{1}_p}{P} \overline{\mu} \right]. \tag{3}$$

After deriving $\mathbf{w}$, the corresponding effective frontier and capital market line can be obtained:

$$\mathcal{L}_{\text{EF}}: \quad -P\overline{\sigma}^2 + A\overline{\mu}^2 - 2B\overline{\mu} + C = 0 \tag{4}$$

$$\mathcal{L}_{\text{CAL}}: -G\overline{\sigma}^2 + (\overline{\mu} - r)^2 = 0. \tag{5}$$

Merging (4) and (5), the efficient frontier and CAL are tangent to each other at one tangency point $\mathcal{P}_{\text{E}\wedge\text{C}} \triangleq (\overline{\mu}_{\text{E}\wedge\text{C}}, \overline{\sigma}_{\text{E}\wedge\text{C}})$, referred to as a market portfolio, where

$$\begin{aligned} \overline{\mu}_{\text{E}\wedge\text{C}} &= \begin{cases} \dfrac{P}{A(AG-P)^{\frac{1}{2}}} + \dfrac{B}{A} = \dfrac{G}{(AG-P)^{\frac{1}{2}}} + r, & \text{if } \overline{\mu}_{\text{EF}} \geq r \\ \dfrac{P-2AG}{A(AG-P)^{\frac{1}{2}}} + \dfrac{B}{A} = -\dfrac{G}{(AG-P)^{\frac{1}{2}}} + r, & \text{otherwise} \end{cases} \\ \overline{\sigma}_{\text{E}\wedge\text{C}} &= \left(\frac{G}{AG-P}\right)^{\frac{1}{2}}. \end{aligned} \tag{6}$$

Furthermore, given $\mathcal{P}_{\text{E}\wedge\text{C}} \triangleq (\overline{\mu}_{\text{E}\wedge\text{C}}, \overline{\sigma}_{\text{E}\wedge\text{C}})$, we can define Sharpe ratio $\text{S}(\mathcal{P}_{\text{E}\wedge\text{C}})$ of return versus risk as

$$\text{S}(\mathcal{P}_{\text{E}\wedge\text{C}}) \triangleq \frac{|\overline{\mu}_{\text{E}\wedge\text{C}} - r|}{\overline{\sigma}_{\text{E}\wedge\text{C}}} = G^{\frac{1}{2}}. \tag{7}$$

2.2 MVS Model

Let a random vector $X \triangleq [X_1 \ \dots \ X_p]^\top$ be the return vector on p risky assets with a weight vector $\mathbf{w}_X$, satisfying $X \sim SN_p(\boldsymbol{\xi}_X, \boldsymbol{\Omega}_{XX}, \boldsymbol{\delta}_X)$. Let a random variable R denote the return on a risk-free asset with a weight $w_R \triangleq 1 - \mathbf{1}_p^\top \mathbf{w}_X$, satisfying $R \sim SN_1(\xi_R, \omega_{RR}^2, \delta_R)$. Then, the MVS optimization model can be re-written as

$$\begin{array}{ll} \operatorname*{Min}\limits_{\widetilde{\mathbf{w}}} & \widetilde{\mathbf{w}}^\top \widetilde{\boldsymbol{\Omega}} \widetilde{\mathbf{w}} \\ \text{s. t.} & \widetilde{\mathbf{w}}^\top \mathbf{1}_{p+1} = 1, \quad \widetilde{\mathbf{w}}^\top \widetilde{\boldsymbol{\xi}} = \overline{\xi}, \quad \widetilde{\mathbf{w}}^\top \widetilde{\boldsymbol{\delta}} = \overline{\delta}. \end{array} \tag{8}$$

We can obtain the solution of $\mathbf{w}$[7]:

$$\mathbf{w} \quad = \quad \boldsymbol{\Omega}^{-1}\left[\frac{E(\boldsymbol{\xi}-r\mathbf{1}_p)-H\boldsymbol{\delta}}{\Pi}(\overline{\xi}-r)+\frac{G\boldsymbol{\delta}-H(\boldsymbol{\xi}-r\mathbf{1}_p)}{\Pi}\overline{\delta}\right], \tag{9}$$

and solution of $\mathbf{w}$ when there only exist risky assets in the portfolio [7]:

$$\mathbf{w} \quad = \quad \boldsymbol{\Omega}^{-1}\left[\frac{R\mathbf{1}_p+W\boldsymbol{\xi}+V\boldsymbol{\delta}}{\Xi}+\frac{W\mathbf{1}_p+Q\boldsymbol{\xi}+U\boldsymbol{\delta}}{\Xi}\overline{\xi}+\frac{V\mathbf{1}_p+U\boldsymbol{\xi}+P\boldsymbol{\delta}}{\Xi}\overline{\delta}\right]. \tag{10}$$

Also from this, the effective frontier and optimal capital allocation surface (CAS) can be further introduced. The efficient frontier and CAS are tangent to each other at one partial quadratic polynomial $\mathcal{C}_{\mathrm{E\wedge C}}$. It is referred to as the curve of the tangency portfolio. Furthermore, given a point $\mathcal{P}_a \triangleq (\overline{\xi}_a, \overline{\delta}_a, \overline{\omega}_a)$ on $\mathcal{C}_{\mathrm{E\wedge C}}$, by calculating the distance from $\mathcal{P}_a$ to the variance-axis through $\mathcal{P}_{\mathrm{CAS}}$, we can define a novel ratio $\mathrm{R}(\mathcal{P}_a)$ as

$$\mathrm{R}(\mathcal{P}_a) \quad \triangleq \quad \frac{[(\overline{\xi}_a-r)^2+\overline{\delta}_a^2]^{\frac{1}{2}}}{\overline{\omega}_a} \tag{11}$$

This is similar to the Sharpe ratio, and can be used to measure the historical risk-adjusted return.

3 RESULTS AND DISCUSSION

This section uses the above algorithm, selects the daily return rate of 30 stocks from 2010 to 2020, and compares and analyzes the MV model and the MVS model under the framework of Markowitz's modern portfolio theory, and further discusses the impact of skewness.

3.1 Data Description

The individual stocks in the data set are selected from the China Securities Index 300 constituent stocks with the largest market capitalization listed before 2010, because the China Securities Index 300 includes both the Shanghai Stock Market and the Shenzhen Stock Market, and the larger the market capitalization, the more effective the stock price. In order to facilitate the comparison of the difference between the MV model and the MVS model, this paper selects four subsets from 2010 to 2020, which are January 2011 to December 2013, January 2015 to December 2017, January 2016 to December 2018 and January 2018 to December 2020. The risk-free interest rate adopts the daily average one-year SHIBOR interest rate, which varies with the time period of different subsets.

3.2 Determining the Optimization of the Portfolio

First, this paper calculates two solutions of the MV optimization model for each subset, and plots the respective results in Figures 1-4. In the first and fourth subsets, since the market portfolio $P_{E\wedge C}$ is relatively positive to the risk-free rate of return, and instructs the investors to enter into long position; in the second and third subsets, market portfolio $P_{E\wedge C}$ is relatively negative to the risk-free rate of return, and instructs the investors to enter into short position. In addition, the Sharpe ratios of the subsets $S(P_{E\wedge C})$ are -0.1722, 0.1903, 0.1587 and -0.2217 respectively. So the selling pressure of the first subset is less than that of the fourth subset, and the buying pressure of the second subset is greater than that of the third subset.

On the other hand, by calculating the multi-dimensional skew normal distribution parameters of the data set, the solution of the MVS model is obtained and the result is plotted in the second half of Figure 1-4. The active ratios of the four subsets are 0.92, 0.9294, 0.7962, and 0.7006. Because the market portfolios of the four subsets are negative relative to the risk-free interest rate, the active ratio brings selling pressure to investors, and the second subset has the greatest selling pressure. Unlike the Sharpe ratio, the pressure measured by the active ratio is not only generated by the risk-adjusted return of the market portfolio and its risks, but also the degree of activity. The degree of activity here reflects the positive or negative skewness of risk-adjusted returns.

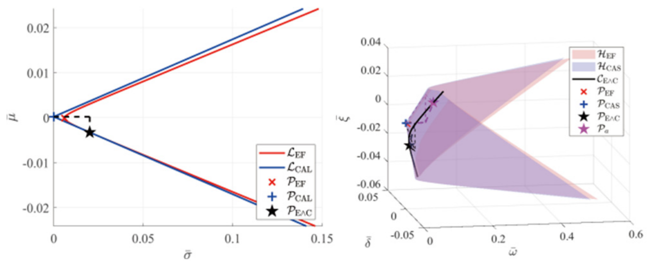

Figure 1. 2011-2013 Model comparison

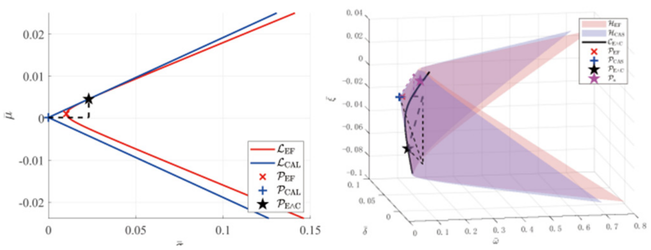

Figure 2. 2015-2017 Model comparison

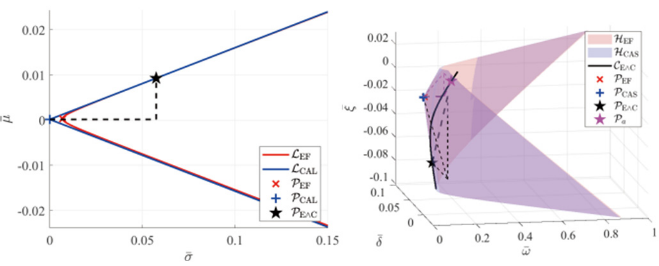

Figure 3. 2016-2018 Model comparison

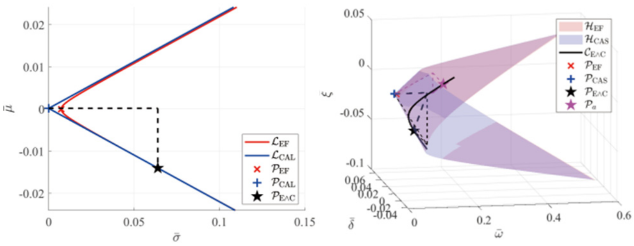

Figure 4. 2018-2020 Model comparison

The weight solutions of the MV model and MVS model of the second and third subsets are shown in Figure 5-6, respectively. In Figure 5, by observing the overall shape of the MV model and the MVS model of the second subset, it can be found that except for Kweichow Moutai, Industrial Bank and CITIC Securities, the weights of other constituent stocks are significantly different from each other. In Figure 6, it can be found that in the third subset, except for Industrial and Commercial Bank of China, PetroChina, and Bank of China, the equity weights of other constituents are also significantly different from each other. Therefore, this article believes that the degree of the portfolio activity that depends on the risk-adjusted return skewness plays an important role in the optimization of the second and third subsets of MVS models. After merging the second and third subsets, from January 2015 to December 2018, the MV model portfolio risk-adjusted return was positive. Therefore, the traditional model suggests that investors enter a long position. On the contrary, MVS The result of the model means that investors need to hold short positions for a long time because of the significant lack of activity in the market portfolio. Although the risk-adjusted return of its investment portfolio is not significantly negative, the significance of its activity reflects the positive and negative skewness of the risk-adjusted return. Therefore, the risk-adjusted return of risk assets during this time period has obvious negative bias characteristics.

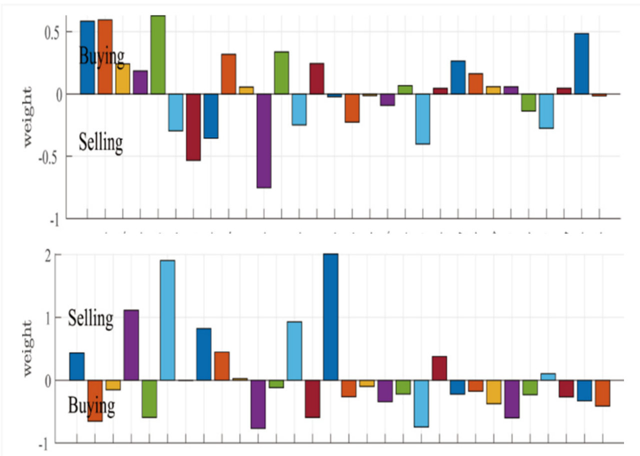

Figure 5. 2015-2017 Weight comparison

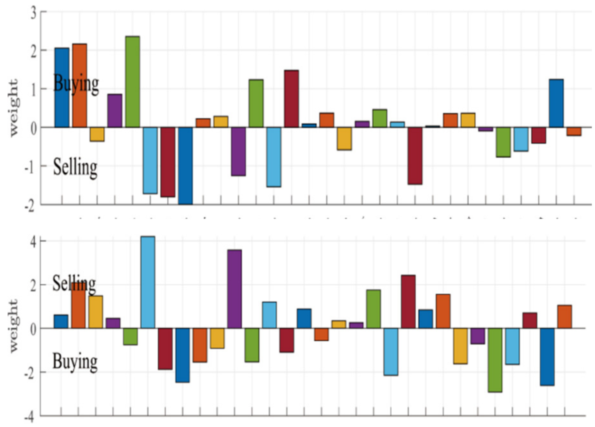

Figure 6. 2016-2018 Weight comparison

4 CONCLUSION

With the development of modern investment portfolios, skewness has gradually been incorporated into the theoretical framework of investment portfolios. Therefore, investors in the China's securities market also need to consider the impact of skewness when formulating investment strategies. Firstly this paper uses the only closed solution algorithm of the MVS model to visualize the investment portfolio model of the China's securities market in three-dimensional space, and secondly obtains the influence of skewness on the Chinese securities market by analyzing the active ratio. In order to arrive at a more effective asset allocation strategy, this article undertakes the analysis framework of Markowitz's modern investment portfolio theory and compares the traditional MV model with the MVS model, and draws the following conclusions:

First, through the active ratio, it is found that the lack of active investment portfolios in the Chinese market is generally large, and even the significant degree of lack in 2015-2018 caused the relative risk-free rate of return of the market portfolio to change from positive to negative, which led to the choices of investors for long and short positions have reversed changes.

Second, the large degree of lack of China's securities market indicates a large negative skewness of risk-adjusted returns, which means that there are a small amount of extremely negative returns and a large number of small positive returns, which are relatively risky, unfavorable to investors, and conform to the reality.

According to the above conclusions, it has the following practical significance: First, in accordance with the principle of asset selection, when the market portfolio is relatively negative, a positive weight represents the sale of risky assets, on the contrary, a negative weight represents the purchase of risky assets. Second, investors should pay attention to the extreme degree of activity. After considering the skewness, the emergence of a great degree of activity that represents a great degree of skewness often means the emergence of investment opportunities.

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