Monte Carlo Simulation for Option Pricing with Multiple Assets

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Abstract: It is a challenged topic of option pricing with multi-asset. This paper uses Monte Carlo (MC) simulation to valuate options which possess multiple assets. Firstly, given correlative coefficients, an algorithm to generate normal distributed random variables is established. Then, MC scheme is proposed for pricing European, American, Asian and Lookback options. Numerical experiments illustrate that MC simulation is an efficient and accurate method. With MC path number 8000, the relative errors of numerical European options are less than 0.5%. The stability experiments of MC algorithm are also carried out. As an advantage, the proposed MC algorithm can be extended to more general options such as Strangles and CEV options.

Keywords: multi-asset options, Monte Carlo simulation, normal distribution, correlative coefficient

1 Introduction

In the valuation of multi-assets option pricing (MAOP), it is assumed the underlying asset prices $S^i(t)$ are modeled by Brownian motion together with a drift term under no-arbitrage assumption (see [2] and [3]), i.e.,

$$dS^i(t) = (r - D_i)S^i(t)t + \sigma_i S^i(t)dW^i(t). \tag{1}$$

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Here, d is the number of assets, r is risk-free interest, D_i are divided yields, σ_i represents the volatility and the covariance

$$Cov(dW_t^i, dW_t^j) = \rho_{ij} dt. \tag{2}$$

Equation (1) is a d-dimensional stochastic differential equation (SDE) with covariance (2). For the 1- dimensional cases, given the payoff functions on maturity date T

$$f(S^1(T), S^2(T), \ldots, S^d(T)), \tag{3}$$

Black-Scholes equation (BSE) (1) has closed form solution. However, for the case of $d > 1$ underlying assets, BSE (1) -(3) generally have no analytical solution and need to be solved by numerical methods.

Finite difference method (FDM) is a choice of valuating options. But the spatial partition of FDM is very difficult and complicated in the case of $d = 3$ (see [5] and [6]). If $d > 3$, FDMs have no possibility to implement MAOP. Binomial Tree method (BTM, see [8]) and Willow Tree method (WTM, see [7]) are successful in pricing one-asset option, but it becomes more and more difficult as the dimension d tends to be larger. The reason is that we need to calculate the very complicated transition probability from time t to time $t + \Delta t$.

In past four decades, Monte Carlo simulation becomes a powerful way to valuate some type of options (see [1] and [4]). The basic idea of MC simulation is to generate a large number of underlying price paths, then option values are taken as the discount average value of pay functions from terminal time T to time zero. Constructing random sequence satisfying pre-given correlative coefficients ρ_{ij} is the key of MC algorithm. In this paper, we focus on this method and give a full algorithm for MAOP. Algorithm-1 in Section 2 is the most important contribution of this paper. Based on this algorithm, we get normal correlative sequence $\widetilde{W}^i$ from independent random sequenceW^i.

2 Monte Carlo Simulation

2.1 Generate standard normal sequence

Assume $W^i \sim N(0,1), i = 1, 2, \ldots, d$ are standard normal distributed variables and assume they are independent on each other. We give an algorithm to generate standard normal random variables $\widetilde{W}^i \sim N(0,1), i = 1, 2, \ldots, d$ such that the covariance $Cov(\widetilde{W}^i, \widetilde{W}^j) = \rho_{ij}$ with pre-given values of ρ_{ij}. Since $\widetilde{W}^i$ are standard, we know ρ_{ij} are also the correlative coefficients between $\widetilde{W}^i$ and $\widetilde{W}^j$.

Let $\widetilde{W}^1 = W^1$ and $\widetilde{W}^2 = \lambda_{21} \widetilde{W}^1 + \lambda_{22} W^2$. The variance of $\widetilde{W}^2$ satisfies

$$D[\widetilde{W}^2] = \lambda_{21}^2 + \lambda_{22}^2 = 1. \tag{4}$$

The covariance between $\widetilde{W}^1$ and $\widetilde{W}^2$ is

$$Cov(\widetilde{W}^1, \widetilde{W}^2) = \lambda_{21} = \rho_{12}. \tag{5}$$

From (4) and (5), it is obtained that

$$\lambda_{21} = \rho_{12}, \quad \lambda_{22} = \sqrt{1-\rho_{12}^2}. \tag{6}$$

Expression (6) means that

$$\widetilde{W}^2 = \rho_{12}\widetilde{W}^1 + \sqrt{1-\rho_{12}^2}W^2 \tag{7}$$

and then $Cov(\widetilde{W}^1, \widetilde{W}^2) = \rho_{12}$.

Now, we consider the more general case of (7). Let

$$\widetilde{W}^k = \sum_{j=1}^{k-1} \lambda_{kj}\widetilde{W}^j + \lambda_{kk}W^k, \tag{8}$$

for $2 \leq k \leq d$. The variance of $\widetilde{W}^k$ is

$$D[\widetilde{W}^k] = \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} \lambda_{ki}\lambda_{kj} + \lambda_{kk}^2 = 1. \tag{9}$$

By the result (6), we get

$$\lambda_{kk}^2 = 1 - \widetilde{\lambda}^T \mathbf{A} \widetilde{\lambda}. \tag{10}$$

The definition of $\mathbf{A}$ and $\widetilde{\lambda}$ are given by

$$\mathbf{A} = [\rho_{ij}]_{(n-1)\times(k-1)}, \tag{11}$$

$$\widetilde{\lambda}^T = [\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{k,k-1}]. \tag{12}$$

The covariance is expressed by

$$Cov(\widetilde{W}^k, \widetilde{W}^j) = \sum_{i=1}^{k} \lambda_{ki} Cov(\widetilde{W}^k, \widetilde{W}^i). \tag{13}$$

Hence,

$$Cov(\widetilde{W}^k, \widetilde{W}^j) = \sum_{i=1}^{k-1} \lambda_{ki}\rho_{ki} = \rho_{kj} \tag{14}$$

Is obtained from (13). Equation (14) is a linear system, i.e.

$$\mathbf{A}\widetilde{\lambda} = \widetilde{\rho} \text{ with } \widetilde{\rho} = [\rho_{k1}, \rho_{k2}, \ldots, \rho_{k,k-1}]^T. \tag{15}$$

Therefore, from (15) we have

$$[\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{k,k-1}]^T = \mathbf{A}^{-1}\widetilde{\rho}, \tag{16}$$

$$\lambda_{kk} = \pm\sqrt{1 - \widetilde{\lambda}^T \mathbf{A} \widetilde{\lambda}}. \tag{17}$$

Using expression (8), (9) and (17), the correlative normal distributed sequence $\widetilde{W}^i$ and then underlying path $S^i(t_n)$ can be formulated. The program of generating random sequence $\widetilde{W}^i$ is created as in Algorithm 1.

Algorithm 1: Generate random variables with coefficients ρ_{ij}

Step 1 Generate standard norm distributed random variables such that each $W^i \sim N(0,1)$ for $i = 1, 2, \ldots, d$ and they are independent on each other.

Step 2 Let $\widetilde{W}^1 = W^1$.

FOR $k = 1, 2, \ldots, d$

Step 3 Create matrix **A** and vector $\widetilde{\rho}$ according to expressions (11) and (12).

Step 4 Compute λ_{kj} for $j = 1, 2, \ldots, k$ by using (16) and (17).

Step 5 Calculate random sequence $\widetilde{W}^k$ by using formula (8).

ENDFOR

Step 6 Random variables $\widetilde{W}^k, k = 1, 2, \ldots, d$ have correlative coefficients ρ_{ij}

2.2 Monte Carlo algorithm for multi-asset option pricing

Given an integer number $N > 0$, and let time mesh $\Delta t = T/N, t_0 = 0$, $t_n = n\Delta t$ $(n = 1, 2, \ldots, N)$. Denote by S_n^i the n^{th} asset price at time t_n, i.e., $S_n^i = S^i(t_n)$. Firstly, we give d standard random variables $W^i(t_n)$ at each time t_n and using Algorithm 1 to produce $\widetilde{W}^i(t_n)$. Secondly, according to SDE (1), M underlying price paths are constructed. Finally, d-asset option values are taken as the average value of payoff function at expire time $T = t_N$, and then discounted from terminal time T to time zero.

The algorithm of Monte Carlo simulation for multi-asset option pricing is described as in Algorithm 2. This algorithm can be extended to other options, such as American options and Asian options.

Algorithm 2: Monte Carlo algorithm for multi-asset option pricing

Step 1 Let initial option value $V = 0$.

Step 2 Generate $N \times d$ standard normal distributed random matrix $W \sim N(0,1)$ such that $W(:, i)$ $(i = 1, 2, \ldots d)$ are independent on each other.

Step 3 Using Algorithm-1 to generate $\widetilde{W}$ such that

$$Cov(\widetilde{W}(:, i), \widetilde{W}(:, j)) = \rho_{ij}.$$

Step 4 Given initial value of asset price S_0^i, compute underlying path,

$$S_n^i = S_{n-1}^i + rS_{n-1}^i \Delta t + \sigma_i \sqrt{\Delta t} \widetilde{W}(n, i).$$

Here, $n = 1, 2, \ldots, N$ and $i = 1, 2, \ldots, d$.

Step 5 Compute accumulation of option value on this path,

$$V \leftarrow V + f(S_N^i, S_N^i, \ldots, S_N^d).$$

Step 6 Repeat M paths and compute accumulated option value V from Step 2 to Step 5.

Step 7 The final option value is taken as

$$V = e^{-rT} V \frac{1}{M}.$$

3 Numerical examples

We use some examples to illustrate the efficiency, accuracy and stability of Monte Carlo algorithm-1 and Algorithm-2. The first one (in subsection 3.1) is European call option with exact solutions. The second (in subsection 3.2) discusses American, Asian and Lookback options, which have no analytical solutions and are compared with those solutions obtained from WTM.

3.1 European call option with exact solutions

Assume call option has geometric mean payoff functions, i.e.,

$$f(S^1, S^2, \ldots, S^d) = \left(\prod_{i=1}^{d} (S^i)^{\alpha_i} - K \right)^+$$

with $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{d} \alpha_i = 1$. Let $\tau = T - t$ and denote S $= [\mathrm{S}^1, S^2, \ldots, S^d]$, the values of d-asset option can be formulated as

$$V_E(\tau, \mathbf{S}) = e^{-\hat{q}\tau} \prod_{i=1}^{d} (S^i)^{\alpha_i} N(\hat{d}_1(\tau, \mathbf{S})) - e^{-r\tau} K \Phi(\hat{d}_2(\tau, \mathbf{S})).$$

Here, $\Phi(\cdot)$ is the standard normal distribution function,

$$\hat{d}_1(\tau, \mathbf{S}) = \frac{1}{\hat{\sigma}\sqrt{\tau}} \times \quad \left(\ln \frac{(S^1)^{\alpha_1} \ldots (S^d)^{\alpha_d}}{K} + (r - \hat{q} + \hat{\sigma}^2/2)\tau \right)$$

and parameters are set as

$$\hat{\sigma} = \sum_{i,j=1}^{d} \rho_{ij} \sigma_i \sigma_j \alpha_i \alpha_j \;, \quad \hat{d}_2(\tau, \mathbf{S}) = \hat{d}_1 - \hat{\sigma}\sqrt{\tau}.$$

In our experiments, parameters in multi-asset model are taken as follows. $r = 0.05$, $T = 1$, $K = 10, t = 0, \alpha_i = 1/d$, $\sigma_i = i\%$, $\rho_{ii} = 1$, $\rho_{ij} = 0.7$ for $i \neq j$. Table 1, Table 2 and Table 3 list some Monte Carlo solutions, errors (ERRs) and relative errors (REs). In these Tables, V_i denotes MC solutions with time partition $N = 50 \times 2^{i-1}$ and path number $M = 500 \times 2^{i-1}$. From these Tables we see (i) the REs are less than 0.5% for dimension d = 2,3,4; (ii) The ERRs and REs are more and more smaller as parameters M and N become larger. These experiments show that MC simulation has good performance for MAOP.

3.2 Options without exact solutions

Table 1: Solutions and errors of Monte Carlo simulation for 2-dimensional model

nodes	V_3	V_4	V_5	V_E
(9, 9)	*0.2888*	*0.2819*	*0.2857*	*0.2899*
(9, 10)	*0.5077*	*0.4995*	*0.4966*	*0.5027*
(9,11)	*0.7789*	*0.7748*	*0.7704*	*0.7705*
ERR	*0.0071*	*0.0052*	*0.0050*	---
RE	*0.0078*	*0.0056*	*0.0047*	---

Table 2: Solutions and errors of Monte Carlo simulation for 3-dimensional model

nodes	V_3	V_4	V_5	V_E
(9,9,9)	*0.4124*	*0.3918*	*0.4121*	*0.4157*
(9,9,10)	*0.5507*	*0.5253*	*0.5551*	*0.5556*
(9,9,11)	*0.7023*	*0.6725*	*0.7106*	*0.7083*
ERR	*0.0740*	*0.0397*	*0.0053*	---
RE	*0.0750*	*0.0404*	*0.0054*	---

Table 3: Solutions and errors of Monte Carlo simulation for 4-dimensional model

nodes	V_3	V_4	V_5	V_E
(9,9,9,9)	*0.5100*	*0.5269*	*0.5348*	*0.5382*
(9,9,9,10)	*0.6104*	*0.6328*	*0.6468*	*0.6439*
(9,9,9,11)	*0.7149*	*0.7411*	*0.7535*	*0.7519*
ERR	*0.0418*	*0.0071*	*0.00344*	---
RE	*0.0388*	*0.0066*	*0.00234*	---

Table 4: Monte Carlo solutions of European, American, Asian and Lookback opyions

options	**K=9**	**K=10**	**K=11**	**K=12**
European	*0.08287* *(0.08254)*	*0.37425* *(0.37505)*	*0.90931* *(0.90899)*	*1.65028* *(1.65048)*
American	*0.10658* *(0.10687)*	*0.44388* *(0.44692)*	*1.07225* *(1.07256)*	*2.00435* *(2.00821)*
Asian	*0.00793* *(0.00708)*	*0.19466* *(0.19430)*	*0.819114* *(0.81956)*	*1.69952* *(1.69770)*
Lookback	*0.00000* *(0.00012)*	*0.00168* *(0.00174)*	*0.224868* *(0.22475)*	*0.83107* *(0.83245)*

In this subsection, we compute four type of put options with two assets and with parameters $r = 0.05$, $T = 1$, $t = 0$, $\alpha_i = 1/d$, $\sigma_i = i\%$, $\rho_{ii} = 1$, ρ_{ij}=0.7 for $i \neq j$. The asset values at time zero are set as $S_0^i = 10$ for $i = 1, 2, \ldots, d$. These options are named "European", "American", "Asian" and "Lookback".

Table 4 presents the Monte Carlo simulation results for different values of strike price K. Options (numbers included in parentheses) obtained by Willow tree method (WTM) are also listed. Details about WTM are referred to the literature (see [7]) From this Tab., we see the Monte Carlo solutions are very consistent with those by WTM. All errors between MC solutions and the corresponding WTM solutions are about $10^{-4} \sim 10^{-3}$.

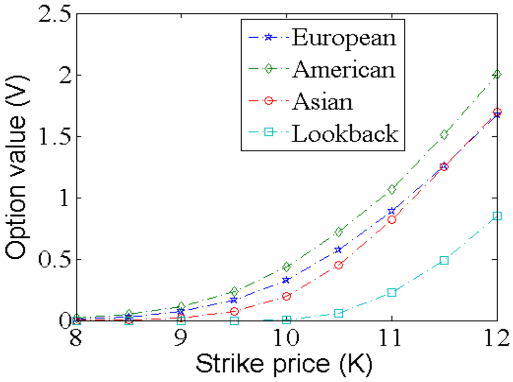

Figure 1: Shapes of four options with different strike prices

Figure 1 plots the shapes of four type options. All results are consistent with the theoretical claims as in reference (see [3]). These experiments illustrate that the proposed algorithm is effective and accurate.

Figure 2 plots Monte Carlo option values under different path numbers M. From this Fig., we see the option values become more and more stable as M goes to 6,000, which shows the Monte Carlo scheme has good stability.

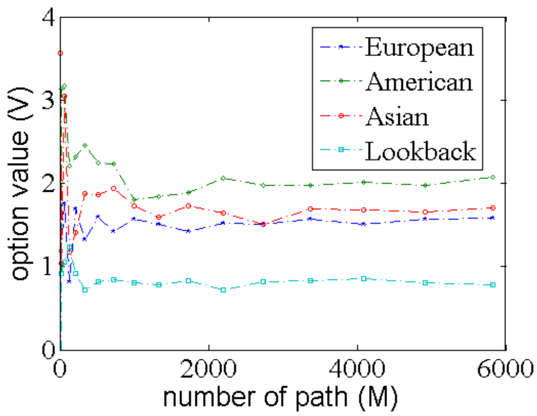

Figure 2: Monte Carlo solutions v.s different path number M

4 Conclusions

To valuate multi-asset options, finite difference method (FDM), binomial tree method (BTM), willow tree method (WTM) and MC simulation are the most popular methods. Because FDM needs to solve complicated partial differential equations (PDEs), constructing FDM scheme is a difficult task for higher dimensional problem. BTM and WTM might get trouble when calculating transition probabilities (see [7] and [8]). In addition, FDM, WTM and BTM are not easily generalized to more complex options. This paper proposes an effective and uniform algorithm to generate multi-asset paths such that they satisfy given correlative coefficients. Numerical experiments are carried out to conform that the proposed Monte Carlo method has good performance in valuating European, American, Asian and Lookback options.

As an advantage, MC algorithm can be extended to more complex options, such as Strangles options, CEV options, Parisian options and so on (see [3]). Additionally, MC simulation can be extended to Levy processes, such as VG (Variance Gamma), GH (Generalized Hyperbolic) and NIG (Normal Inverse Gamma) models. In future, convergence analysis of MC simulation will be discussed in depth.

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References

[1] Acworth, P., Broadie, M. and P. Glasserman, (1998). *The book*, A comparison of some monte carlo and quasi Monte Carlo techniques for option pricing. Springer New York.

[2] Black, F. & M. Scholes (1973). The pricing of options and corporate liabilities, Journal of Political Economy. 81, 637-654.

[3] Kwok, Y. K., (2008). *The book,* Mathematical Models of Financial Derivatives. second edn, Springer, Berlin.

[4] Poirot, J. & P. Tankov (2006). Monte Carlo Option Pricing for Tempered Stable (CGMY) Processes. Asia-Pacific Financial Markets. 13, 327-344.

[5] Zhou, Z., Ma, J. & X. Gao (2018a). Convergence analysis of iterative Laplace transform methods for a system of fractional PDEs and PIDEs. East Asian Journal on Applied Mathematics. 8, 782-808.

[6] Zhou, Z., Ma J. & H. Sun (2018b). Fast Laplace transform methods for free-boundary problems of fractional diffusion equations, Journal of Scientific Computing. 74, 49-69.

[7] Ma, J., Huang, S. and W. Xu (2020). An efficient convergent willow tree method for american and exotic option pricing under stochastic volatility models, Journal of Derivatives. 27, 75–98.

[8] Kim, K. & X. Qian (2007). Convergence of the binomial tree method for Asian optionsin jump-diffusion models. J. Math. Anal. Appl. 330, 10-23.