Formulation Of The Convolution On The Reduced Linear Biquaternion Canonic Transformation

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Abstract. This study aims to formulate convolution for reduced biquaternion linear canonical transform (RBLCT). The research method used in this research was conducted as a library research at the PSDKU Makassar Graphic Engineering Study Program, Graphic Engineering Department, Creative Media State Polytechnic, Makassar. Research results obtained through the definition of RBLCT was obtained by determining the definition of reduced biquaternion Fourier transform (RBFT), and by analyzing the properties. The analysis of properties was conducted by replacing the FT kernel with the RBFT kernel is the definition of LCT. The proposed convolution for RBLCT are the extensions of LCT convolution over the domain of RBLCT domain. This study also reveals an evidence is much simpler since it uses the conjugation properties of RBLCT kernel. As the application of results, the RBLCT convolutions theorems are discussed to analyze the frequency-swift filter in general.

Keywords: Theorem, Convolution, Linear Canonical Transformation, Biquaternion.

1 Introduction

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Digital image processing and signal processing are objects of current discussion which are activities that are closely related to mathematical processes. The Fourier transform was first discovered by a mathematician Joseph Fourier. The Fourier transform is an extension of the Fourier series. The development of the Fourier series to the Fourier transform is because non-periodic functions are easier to analyze with the Fourier transform.

In reality, apart from being in contact with real space, there is also a complex space, so the Fourier transform is being developed in a complex space, namely the Fourier Quaternion transform (TFQ). TFQ is a generalization of the real and complex Fourier transform using quaternion algebra. Quaternion is an expansion of complex numbers which was first discovered by Hamilton. The discussion of quaternions has been widely developed on the problem of signal processing, image processing, aircraft radar and so on. However, since it is known that the multiplication rule of quaternions is not commutative, this limits the application of quaternions in signal and image processing. Moreover, in general, the convolution of the two signal quaternions f(x,y) and g(x,y) cannot be calculated by the product of the Fourier transforms F(u,v) and G(u,v) in the frequency domain.

With these weaknesses, a reduced biquatenion has been proposed which has a commutative multiplication rule [1]. This commutative property is an advantage over quaternions. A biquaternion is an eight-dimensional hypercomplex number with addition and multiplication operations similar to that of a quaternion. The collection of biquaternions forms four-dimensional (4D) algebra on complex numbers.

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Linear canonical transformation (TKL) which is a generalization of several transformations, including Fourier transform, Laplace transform, fractional Fourier transform, Fresnel transform and other transformations has an important role in many fields of optics [2] as well as processing signal [3]. TKL is more attractive in various applications due to the accuracy and efficiency of its transformation calculations [4], and many of the basic properties of these transformations are known, including shift, modulation, convolution, correlation and the uncertainty principle [5].

In previous studies, the convolution theorem for linear canonical transformation (TKL) has been introduced which is based on the properties of the convolution theorem for the Fourier Transform which is explicitly shown by the authors some important properties of the relation between TKL and convolution, and provides an alternative form of the TKL correlation theorem [6]. Likewise, the convolution for one-sided TKLQ and its important properties such as linearity, shift, modulation and so on [7].

This research was conducted using a literature review method, by first introducing the definition of the reduced Biquaternion Fourier transform (TFBT) and its important properties such as linearity, shift or dilation, scale, modulation, parseval, and plansherel. Furthermore, based on the definition of the reduced biquaternion Fourier transform (TFBT), we obtain the definition of the reduced linear biquaternion canonical transform (TKLBT) by replacing the Kernel of TF with the kernel of TFBT in the definition of TKL. And in the end it will be obtained an alternative form of the convolution for the reduced biquaternion linear canonical transformation.

2 Research Methods

2.1 **Problem identification**

Problem identification is the initial stage of research to determine the focus of the research problem.

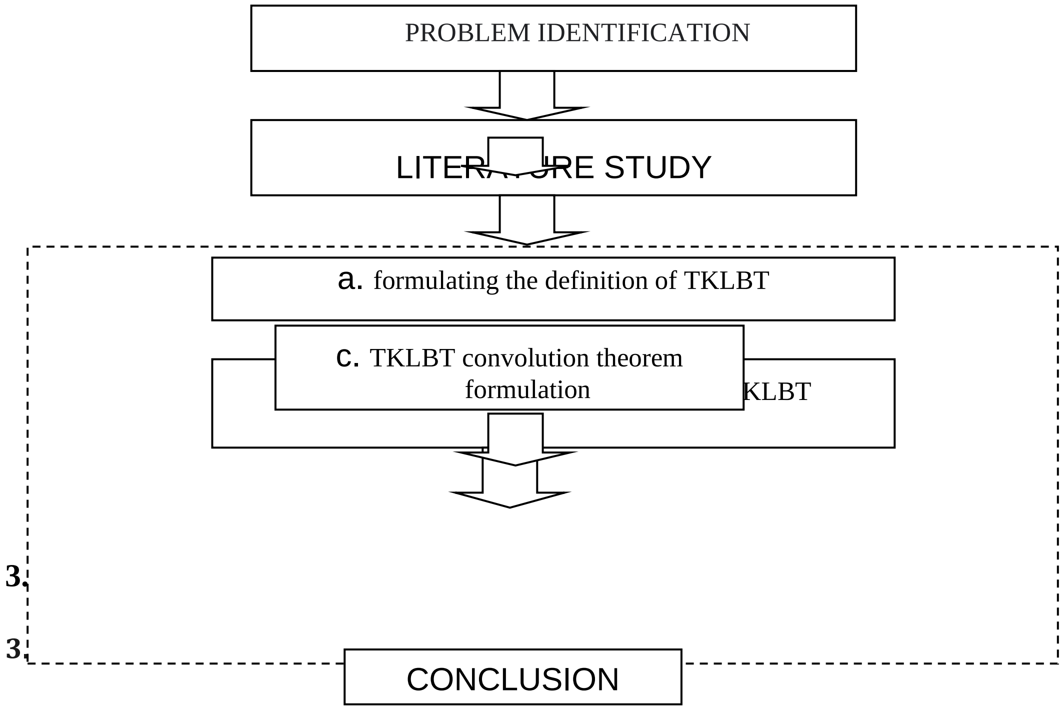
2.2 Literature Study

Literature studies were conducted on research journals related to the field of research as a stage to complete the basic knowledge of researchers for the purposes of conducting research.

a) Activities for formulating the definition of TKLBT.
b) The activity of formulating the inverse definition of TKLBT.
c) TKLBT convolution theorem formulation activities.

2.3 Research framework

After compiling the definition of TKLBT, then a framework is made based on the formulation of the problem and research objectives.
The order of the framework of thought in this research activity can be described in a flow chart below.



Based on the definition of the reduced Biquaternion Fourier transform (TFBT), the definition of TKLBT is obtained by replacing the kernel from TF with the kernel from TFBT in the definition of TKL.

Denoted by $SL\ (2,\mathbb{R})$, a special linear group of degree 2 on $\mathbb{R}$, is a group of a matrix of the order 2×2 with a determinant of one. Suppose

$$\mathrm{A_s} = (\mathrm{a_s}, \mathrm{b_s}, \mathrm{c_s}, \mathrm{d_s}) = \begin{pmatrix} \mathrm{a_s} & b_s \\ \mathrm{c_s} & d_s \end{pmatrix} \in SL\ (2,\mathbb{R}), \mathrm{s} = 1{,}2.$$

Definition 1. The reduced biquaternion linear canonical transformation (TKLBT) of the reduced biquaternion signal f is defined by

$$\boldsymbol{L}^{\boldsymbol{i},\boldsymbol{k}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = \begin{cases} \displaystyle\int_{R^2} K_{A_1}(x_1\,,\omega_1) f(\boldsymbol{x}) K_{A_2}(x_{2,}, \omega_2)\ d\boldsymbol{x}, & b_n \neq 0, n = 1{,}2 \\ \sqrt{d_1 d_2}\, e^{\boldsymbol{i}\left(\frac{c_1 d_1}{2}\right)\omega_1^2} f(d_1\omega_1, d_2\omega_2) e^{\boldsymbol{k}\left(\frac{c_2 d_2}{2}\right)\omega_2^2}, & b_1 = 0 \text{ atau } b_2 = 0 \end{cases}, \quad (1)$$

where the kernel of TKLBT is given by each.

$$K_{A_1}(x_1, \omega_1) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1} x_1^2 - \frac{2}{b_1} x_1\, \omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}, \quad (2)$$

dan

$$K_{A_2}(x_2, \omega_2) = \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2} x_2^2 - \frac{2}{b_2} x_2\, \omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}. \quad (3)$$

With $e^{\boldsymbol{i}\left(\frac{c_1 d_1}{2}\right)\omega_1^2}$ dan $e^{\boldsymbol{k}\left(\frac{c_1 d_1}{2}\right)\omega_1^2}$called Chirp signal in signal processing. Because $\boldsymbol{L}_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f\}(\boldsymbol{\omega})$ is trivial for $b_1 = 0$ or $b_2 = 0$, in this study it is always assumed that $b_n \neq 0$ for$n = 1,2$. As a special case, when $A_1 = A_2 = (a_s, b_s, c_s, d_s) = (0, 1, -1, 0)$ for $s = 1, 2$, the definition of TKLBT in equation (1) is reduced to the definition of TFBT, namely

$$L_{A_1,A_2}^{i,k}\{f\}(\boldsymbol{\omega}) = \int_{R^2} \frac{1}{\sqrt{2\pi\boldsymbol{i}}} e^{-\boldsymbol{i}\omega_1 x_1} f(\boldsymbol{x}) \frac{1}{\sqrt{2\pi\boldsymbol{k}}} e^{-\boldsymbol{k}\omega_2 x_2}\, d\boldsymbol{x}$$
$$= \frac{1}{\sqrt{2\pi\boldsymbol{i}}} \mathcal{F}_{RB}\{f\}(\boldsymbol{\omega}) \frac{1}{\sqrt{2\pi\boldsymbol{k}}}. \quad (4)$$

Theorem 1. The inverse of the reduced biquaternion linear canonical transformation is given by

$$f(\boldsymbol{x}) = \begin{cases} \iint_{R^2} K_{A_1^{-1}}(x_1, \omega_1) L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}} f(\boldsymbol{\omega}) K_{A_2^{-1}}(x_2, \omega_2)\, d\boldsymbol{\omega}, & b_n \neq 0, n = 1,2 \\ \sqrt{a_1 a_2}\, e^{-\boldsymbol{i}\left(\frac{c_1 a_1}{2}\right)x_1^2} f(a_1 x_1, a_2 x_2) e^{-\boldsymbol{k}\left(\frac{c_2 a_2}{2}\right)x_2^2}, & b_1 = 0 \text{ atau } b_2 = 0 \end{cases}, \quad (5)$$

when $A_1^{-1} = (d_1, -b_1, -c_1, a_1)$ *and* $A_2^{-1} = (d_2, -b_2, -c_2, a_2)$.

3.2 Characteristics of TKLBT

The following proposition presents some useful properties of the kernel functions $K_{A_1}(x_1, \omega_1)$ and $K_{A_2}(x_2, \omega_2)$ TKLBT, which will be used to derive the Parseval formula

Proposition 1. Given the kernels of the functions $K_{A_1}(x_1, \omega_1)$ and $K_{A_2}(x_2, \omega_2)$defined by (2) and (3). next we get:

I. $K_{A_1}(-x_1, \omega_1) = K_{A_1}(x_1, -\omega_1)$ and $K_{A_2}(-x_2, \omega_2) = K_{A_2}(x_2, -\omega_2)$;
II. $K_{A_1}(-x_1, -\omega_1) = K_{A_1}(x_1, \omega_1)$ dand $K_{A_2}(-x_2, -\omega_2) = K_{A_2}(x_2, \omega_2)$;
III. $\overline{K_{A_1}(x_1, \omega_1) K_{A_2}(x_2, \omega_2)} = K_{A_1^{-1}}(x_1, \omega_1) K_{A_2^{-1}}(x_2, \omega_2)$.

Proof 1. By using equation (2) the proof of the proposition $K_{A_1}(-x_1, \omega_1) = K_{A_1}(x_1, -\omega_1)$ is as follows

$$K_{A_1}(-x_1, \omega_1) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}(-x_1)^2 - \frac{2}{b_1}(-x_1)\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}$$
$$= \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 + \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)},$$

Similar to

$$K_{A_1}(x_1, -\omega_1) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1(-\omega_1) + \frac{d_1}{b_1}(-\omega_1)^2\right)}$$
$$= \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 + \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}$$
$$= K_{A_1}(-x_1, \omega_1).$$

If the same operation is performed in equation (3), then we get

$$K_{A_2}(-x_2,\omega_2) = \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}(-x_2)^2 - \frac{2}{b_2}(-x_2)\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}$$
$$= \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 + \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)},$$

Similar to

$$K_{A_2}(x_2,-\omega_2) = \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2(-\omega_2) + \frac{d_2}{b_2}(-\omega_2)^2\right)}$$
$$= \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 + \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}$$
$$= K_{A_2}(-x_2,\omega_2).$$

Proof 2. By using equation (2) the proof of the proposition $K_{A_1}(-x_1,-\omega_1) = K_{A_1}(x_1,\omega_1)$ is as follows

$$K_{A_1}(-x_1,-\omega_1) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}(-x_1)^2 - \frac{2}{b_1}(-x_1)(-\omega_1) + \frac{d_1}{b_1}(-\omega_1)^2\right)}$$
$$= \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}$$
$$= K_{A_1}(x_1,\omega_1).$$

while $K_{A_2}(-x_2,-\omega_2) = K_{A_2}(x_2,\omega_2)$ can be proven by using equation (3), as follows

$$K_{A_2}(-x_2,-\omega_2) = \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}(-x_2)^2 - \frac{2}{b_2}(-x_2)(-\omega_2) + \frac{d_2}{b_2}(-\omega_2)^2\right)}$$
$$= \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}$$
$$= K_{A_2}(x_2,\omega_2).$$

Proof 3. From equations (2) and (3) it can be written as follows

$$\overline{K_{A_1}(x_1,\omega_1)K_{A_2}(x_2,\omega_2)} = \overline{\frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}}$$
$$\overline{\frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}}$$
$$= \frac{1}{\sqrt{2\pi b_1(-\boldsymbol{i})}} e^{-\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} \frac{1}{\sqrt{2\pi b_2(-\boldsymbol{k})}} e^{-\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}$$
$$= \frac{1}{\sqrt{-2\pi b_1 \boldsymbol{i}}} e^{-\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1} - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} \frac{1}{\sqrt{-2\pi b_2 \boldsymbol{k}}} e^{-\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)},$$

by using the inverse TKLBT in equation (5), it can be written as follows

$$K_{A_1^{-1}}(x_1,\omega_1)K_{A_2^{-1}}(x_2,\omega_2) = \frac{1}{\sqrt{2\pi(-b_1)\boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{(-b_1)}x_1^2 - \frac{2}{(-b_1)}x_1\omega_1 + \frac{d_1}{(-b_1)}\omega_1^2\right)}$$
$$\frac{1}{\sqrt{2\pi(-b_2)\boldsymbol{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{(-b_2)}x_2^2 - \frac{2}{(-b_2)}x_2\omega_2 + \frac{d_2}{(-b_2)}\omega_2^2\right)}$$

$$= \frac{1}{\sqrt{-2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(-\frac{a_1}{b_1}x_1^2 + \frac{2}{b_1}x_1\,\omega_1 - \frac{d_1}{b_1}\omega_1^2\right)} \frac{1}{\sqrt{-2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(-\frac{a_2}{b_2}x_2^2 + \frac{2}{b_2}x_2\,\omega_2 - \frac{d_2}{b_2}\omega_2^2\right)}$$

$$= \overline{\frac{1}{\sqrt{-2\pi b_1 \boldsymbol{i}}} e^{-\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\,\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}} \frac{1}{\sqrt{-2\pi b_2 \boldsymbol{k}}} e^{-\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\,\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}$$

$$= \overline{K_{A_1}(x_1,\omega_1)K_{A_2}(x_2,\omega_2)}.$$

The following lemma describes in general the relationship between TKLBT and TFBT of a signal *f*.

Lemma 1. *TKLBT of signal f with parameter matrix* $A_1 = (a_1, b_1, c_1, d_1)$ *dan* $A_2 = (a_2, b_2, c_2, d_2)$ *can be written as signal f of TFBT which is written in the form*

$$L_{A_1,A_2}^{i,k}\{f\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}}\frac{1}{\sqrt{2\pi b_1 \boldsymbol{k}}} e^{\frac{\boldsymbol{i}d_1}{2b_1}\omega_1^2} e^{\frac{\boldsymbol{k}d_2}{2b_2}\omega_2^2} \mathcal{F}_{RB}\left\{e^{\frac{\boldsymbol{i}a_1}{2b_1}x_1^2} e^{\frac{\boldsymbol{k}a_2}{2b_2}x_2^2} f(\boldsymbol{x})\right\} \left(\frac{\omega_1}{b_1}, \frac{\omega_2}{b_2}\right). \tag{6}$$

Proof. A simple calculation using definition 1 shows that

$$L_{A_1,A_2}^{i,k}\{f\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} f(\boldsymbol{x}) \frac{1}{\sqrt{2\pi b_1 \mathrm{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}\, d\boldsymbol{x}$$

$$= \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} e^{\boldsymbol{i}\frac{a_1}{2b_1}x_1^2 - \boldsymbol{i}\frac{\omega_1}{b_1}x_1 + \boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2} f(\boldsymbol{x}) \frac{1}{\sqrt{2\pi b_1 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{a_2}{2b_2}x_2^2 - \boldsymbol{k}\frac{\omega_2}{b_2}x_2 + \boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2}\, d\boldsymbol{x}$$

$$= \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2} \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\frac{\omega_1}{b_1}x_1} \left(e^{\boldsymbol{i}\frac{a_1}{2b_1}x_1^2} e^{\boldsymbol{k}\frac{a_2}{2b_2}x_2^2} f(\boldsymbol{x})\right) \frac{1}{\sqrt{2\pi b_1 \boldsymbol{k}}} e^{-\boldsymbol{k}x_2\frac{\omega_2}{b_2}} e^{\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2}\, d\boldsymbol{x}$$

$$= \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2} \mathcal{F}_{RB}\left\{e^{\boldsymbol{i}\frac{a_1}{2b_1}x_1^2} e^{\boldsymbol{k}\frac{a_2}{2b_2}x_2^2} f(\boldsymbol{x})\right\}\left(\frac{\omega_1}{b_1}, \frac{\omega_2}{b_2}\right) \frac{1}{\sqrt{2\pi b_1 \boldsymbol{k}}} e^{\frac{\boldsymbol{k}d_2}{2b_2}\omega_2^2}. \tag{7}$$

Where the last line is obtained from the definition of TFBT in equation (4).

Furthermore, an alternative proof of the Plancherel formula for TKLBT is provided.

Theorem 2. (Plancherel TKLBT). The two reduced biquaternion functions *f* and *g* of TKLBT have the Plancherel formula, which is given as

$$\int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}d\boldsymbol{x} = \int_{\mathbb{R}^2} L_{A_1,A_2}^{i,\mathrm{k}}\{f\}(\omega)\overline{L_{A_1,A_2}^{i,\mathrm{k}}\{g\}(\omega)}\, d\boldsymbol{\omega}. \tag{8}$$

Proof. By using the TKLBT inverse in equation (5), it is given that

$$\int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\, d\boldsymbol{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} L_{A_1,A_2}^{i,\mathrm{k}}\{f\}(\boldsymbol{\omega}) K_{A_1^{-1}}(x_1,\omega_1) K_{A_2^{-1}}(x_2,\omega_2) d\boldsymbol{\omega}\overline{g(\boldsymbol{x})}\, d\boldsymbol{x}$$

By using proposition 1 part (iii), then the above equation can be written as

$$= \int_{\mathbb{R}^2} L_{A_1,A_2}^{i,\mathrm{k}}\{f\}(\boldsymbol{\omega}) \underline{\int_{\mathbb{R}^2} K_{A_1^{-1}}(x_1,\omega_1) K_{A_2^{-1}}(x_2,\omega_2)\overline{g(\boldsymbol{x})}\, d\boldsymbol{x}}\, d\boldsymbol{\omega}.$$

$$= \int_{\mathbb{R}^2} L_{A_1,A_2}^{i,\mathrm{k}}\{f\}(\boldsymbol{\omega}) \int_{\mathbb{R}^2} K_{A_1}(x_1,\omega_1) g(\boldsymbol{x}) K_{A_2}(x_2,\omega_2) d\boldsymbol{x}\, d\boldsymbol{\omega}$$

$$= \int_{\mathbb{R}^2} L^{i,\mathrm{k}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\, \overline{L^{i,\mathrm{k}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})} d\boldsymbol{\omega}. \tag{9}$$

In the first equation of equation (9) the reduced biquaternion function f has been replaced by the inverse TKLBT (7). In the second equation, the order of integration has been exchanged and in the third equation part (iii) in Proposition (1) has been applied, while in the last equation, the definition of TKLBT in equation (1) is used to complete the proof of the theorem.

A special case of the Plancherel formula for TKLBT obtained the Plancherel formula from TKLBT as follows.

Corollaries 1. (Parseval of TKLBT). *If the reduced biquaternion function is f(x) = g(x), then the Plancherel formula from TKLBT will be reduced to the Parseval formula from TKLBT, which states that*

$$\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x} = \int_{\mathbb{R}^2} \left|L^{i,\mathrm{k}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}. \tag{10}$$

Equation (10) is a statement about the energy content in the reduced biquaternion signal. It states that the total energy signal calculated in the spatial domain is the same as the total energy calculated in the TKLBT domain. Parseval's formula allows the reduced value biquaternion energy signal in either the spatial domain or the TKLBT domain and the domains can be interchanged for easy computation.

3.3 Convolution Theorem for TKLBT

Convolution plays an important role in signal processing, such as edge detection, sharpening and smoothing in image processing. It is well known that the classical convolution in the Fourier domain can be represented as the product of the Fourier transform separately. This result is similar to the convolution of the reduced Biquaternion Fourier transform (TFBT) as shown in the properties of TFBT. In this section, we will discuss the convolution theorem in the TKLBT domain, which can be more useful in practical analog filtering in the TKLBT domain and propose an alternative form of convolution for the TKLBT. This can be thought of as a continuation of the convolution in the TKL definition (see [2.30, 2.32]) to the TKLBT domain.

Definition 2. *The convolution for each reduced biquaternion function f and g, is defined by*

$$(f \odot g)(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} \int_{\mathbb{R}^2} e^{-\boldsymbol{i}\frac{a_1}{2b_1}x_1^2} e^{-\boldsymbol{k}\frac{a_2}{2b_2}x_2^2} e^{\boldsymbol{i}\frac{a_1}{2b_1}t_1^2} e^{\boldsymbol{k}\frac{a_2}{2b_2}t_2^2} f(\boldsymbol{t})$$
$$g(\boldsymbol{x}-\boldsymbol{t}) e^{\boldsymbol{i}\frac{a_1}{2b_1}(x_1-t_1)^2} e^{\boldsymbol{k}\frac{a_2}{2b_2}(x_2-t_2)^2}\, d\boldsymbol{t}. \tag{11}$$

The convolutions of two reduced biquaternions and TKLBT are given by the following theorem

Theorem 3. *Suppose f (**x**) and g (**x**) are two reduced biquaternion functions, then the following equation applies*

$$L^{\boldsymbol{i},\boldsymbol{k}}_{A_1,A_2}\{f \odot g\}(\boldsymbol{\omega}) = e^{-\boldsymbol{i}\frac{d_1\omega_1^2}{2b_1}} e^{-\boldsymbol{k}\frac{d_2\omega_2^2}{2b_2}} L^{\boldsymbol{i},\boldsymbol{k}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) L^{\boldsymbol{i},\boldsymbol{k}}_{A_1,A_2}\{g\}(\boldsymbol{\omega}). \tag{12}$$

Proof. With a direct calculation given

$$L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f \odot g\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} e^{-\boldsymbol{i}\frac{a_1}{2b_1}x_1^2}$$
$$e^{-\boldsymbol{k}\frac{a_2}{2b_2}x_2^2} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{i}\frac{a_1}{2b_1}t_1^2} e^{\boldsymbol{k}\frac{a_2}{2b_2}t_2^2} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} f(\boldsymbol{t}) g(\boldsymbol{x}-\boldsymbol{t}) e^{\boldsymbol{i}\frac{a_1}{2b_1}(x_1-t_1)^2}$$
$$e^{\boldsymbol{k}\frac{a_2}{2b_2}(x_2-t_2)^2} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)} \, d\boldsymbol{t}\, d\boldsymbol{x}.$$

Dengan mensubstitusi variabel $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{t}$, $y_1 = x_1 - t_1$, dan $y_2 = x_2 - t_2$, yang dapat dituliskan menjadi $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{t}$, $x_1 = y_1 + t_1$, dan $x_2 = y_2 + t_2$, maka diperoleh

By substituting the variables $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{t}$, $y_1 = x_1 - t_1$, dan $y_2 = x_2 - t_2$, which can be written as $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{t}$, $x_1 = y_1 + t_1$, dan $x_2 = y_2 + t_2$, we get

$$L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f \odot g\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}(y_1+t_1)^2 - \frac{2}{b_1}(y_1+t_1)\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}$$
$$e^{-\boldsymbol{i}\frac{a_1}{2b_1}(y_1+t_1)^2} e^{-\boldsymbol{k}\frac{a_2}{2b_2}(y_2+t_2)^2} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{i}\frac{a_1}{2b_1}t_1^2} e^{\boldsymbol{k}\frac{a_2}{2b_2}t_2^2} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} f(\boldsymbol{t}) g(\boldsymbol{x}-\boldsymbol{t})$$
$$e^{\boldsymbol{i}\frac{a_1}{2b_1}y_1^2} e^{\boldsymbol{k}\frac{a_2}{2b_2}y_2^2} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}(y_2+t_2)^2 - \frac{2}{b_2}(y_2+t_2)\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)} \, d\boldsymbol{t}\, d\boldsymbol{x}$$
$$= \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}(y_1^2+2y_1t_1+t_1^2) - \frac{2}{b_1}(y_1+t_1)\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} e^{-\boldsymbol{i}\frac{a_1}{2b_1}(y_1^2+2y_1t_1+t_1^2)}$$
$$e^{-\boldsymbol{k}\frac{a_2}{2b_2}(y_2^2+2y_2t_2+t_2^2)} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{i}\frac{a_1}{2b_1}t_1^2} e^{\boldsymbol{k}\frac{a_2}{2b_2}t_2^2} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} f(\boldsymbol{t}) g(\boldsymbol{x}-\boldsymbol{t}) e^{\boldsymbol{i}\frac{a_1}{2b_1}y_1^2}$$
$$e^{\boldsymbol{k}\frac{a_2}{2b_2}y_2^2} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}(y_2^2+2y_2t_2+t_2^2) - \frac{2}{b_2}(y_2+t_2)\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)} \, d\boldsymbol{t}\, d\boldsymbol{x}$$
$$= \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{a_1}{2b_1}y_1^2} e^{\boldsymbol{i}\frac{a_1}{b_1}y_1t_1} e^{\boldsymbol{i}\frac{a_1}{2b_1}t_1^2} e^{-\boldsymbol{i}\frac{1}{b_1}y_1\omega_1} e^{-\boldsymbol{i}\frac{1}{b_1}t_1\omega_1} e^{\boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2}$$
$$\frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{-\boldsymbol{i}\frac{a_1}{2b_1}y_1^2} e^{-\boldsymbol{i}\frac{a_1}{b_1}y_1t_1} e^{-\boldsymbol{i}\frac{a_1}{2b_1}t_1^2} e^{-\boldsymbol{k}\frac{a_2}{2b_2}y_2^2} e^{-\boldsymbol{k}\frac{a_2}{b_2}y_2t_2} e^{-\boldsymbol{k}\frac{a_2}{2b_2}t_2^2} e^{\boldsymbol{i}\frac{a_1}{2b_1}t_1^2} e^{\boldsymbol{k}\frac{a_2}{2b_2}t_2^2}$$
$$\frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} f(\boldsymbol{t}) g(\boldsymbol{y}) e^{\boldsymbol{i}\frac{a_1}{2b_1}y_1^2} e^{\boldsymbol{k}\frac{a_2}{2b_2}y_2^2} e^{\boldsymbol{k}\frac{a_2}{2b_2}y_2^2} e^{\boldsymbol{k}\frac{a_2}{b_2}y_2t_2} e^{\boldsymbol{k}\frac{a_2}{2b_2}t_2^2} e^{-\boldsymbol{k}\frac{1}{b_2}y_2\omega_2} e^{-\boldsymbol{k}\frac{1}{b_2}t_2\omega_2}$$
$$e^{\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2} \, d\boldsymbol{t}\, d\boldsymbol{y}$$
$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{a_1}{2b_1}t_1^2} e^{-\boldsymbol{i}\frac{1}{b_1}t_1\omega_1} e^{\boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2} f(\boldsymbol{t}) \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{a_2}{2b_2}t_2^2} e^{-\boldsymbol{k}\frac{1}{b_2}t_2\omega_2} e^{\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2}$$
$$\frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{a_1}{2b_1}y_1^2} e^{-\boldsymbol{i}\frac{1}{b_1}y_1\omega_1} g(\boldsymbol{y}) \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{a_2}{2b_2}y_2^2} e^{-\boldsymbol{k}\frac{1}{b_2}y_2\omega_2} \, d\boldsymbol{t}\, d\boldsymbol{y}$$
$$= L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f\}(\boldsymbol{\omega}) \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{a_1}{2b_1}y_1^2} e^{-\boldsymbol{i}\frac{1}{b_1}y_1\omega_1} g(\boldsymbol{y}) \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{a_2}{2b_2}y_2^2} e^{-\boldsymbol{k}\frac{1}{b_2}y_2\omega_2} \, d\boldsymbol{y}. \tag{13}$$

By multiplying both sides of the identity above by $e^{\boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2}$ and $e^{\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2}$ and using the Definition of TKLBT (1), we get

$$e^{\boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2} L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f \odot g\}(\boldsymbol{\omega}) e^{\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2} = L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f\}(\boldsymbol{\omega}) \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{a_1}{2b_1}y_1^2}$$

$$e^{-\boldsymbol{i}\frac{1}{b_1}y_1\omega_1} e^{\boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2} g(\boldsymbol{y}) \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{a_2}{2b_2}y_2^2} e^{-\boldsymbol{k}\frac{1}{b_2}y_2\omega_2} e^{\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2}\, d\boldsymbol{y}$$

$$= L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f\}(\boldsymbol{\omega}) L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{g\}(\boldsymbol{\omega}). \tag{14}$$

This is the expected result.

In the following, another way is proposed to generalize the definition of the convolution of two reduced biquaternion signals in the TKLBT domain.

Definition 3. *The convolution for each reduced biquaternion signal f and g, is defined by*

$$(f \odot g)(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} \int_{\mathbb{R}^2} e^{\boldsymbol{i}\frac{a_1}{b_1}t_1(t_1-x_1)} f(\boldsymbol{t}) g(\boldsymbol{x}-\boldsymbol{t}) e^{\boldsymbol{k}\frac{a_2}{b_2}t_2(t_2-x_2)}\, d\boldsymbol{t}. \tag{15}$$

In addition, the following important theorem explains how the convolution of two reduced biquaternion functions interacts with their TKLBT.

Theorem 4. *Let* $f(\boldsymbol{x}) = f_0(\boldsymbol{x}) + f_1(\boldsymbol{x}) + f_2(\boldsymbol{x}) + f_3(\boldsymbol{x})$ *and* $g(\boldsymbol{x}) = g_0(\boldsymbol{x}) + g_1(\boldsymbol{x}) + g_2(\boldsymbol{x}) + g_3(\boldsymbol{x})$ *are two reduced biquaternion functions, then the TKLBT of the convolution of f and g is given by*

$$L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f \odot g\}(\boldsymbol{\omega}) = e^{-\boldsymbol{i}\frac{d_1\omega_1^2}{2b_1}} e^{-\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2} L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f\}(\boldsymbol{\omega}) L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{g\}(\boldsymbol{\omega}). \tag{16}$$

This means that formally, the convolution of two reduced biquaternion signals in the TKLBT domain is written in the form

$$\{f \odot g\}(\boldsymbol{x}) = L_{A_1^{-1},A_2^{-1}}^{\boldsymbol{i},\boldsymbol{k}} \left\{ e^{-\boldsymbol{i}\frac{d_1\omega_1^2}{2b_1}} e^{-\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2} L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f\}(\boldsymbol{\omega}) L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{g\}(\boldsymbol{\omega}) \right\}(\boldsymbol{x}). \tag{17}$$

Proof. A simple calculation shows that

$$L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f \odot g\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\frac{1}{2}\boldsymbol{i}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} e^{\boldsymbol{i}\frac{a_1}{b_1}t_1(t_1-x_1)}$$

$$f(\boldsymbol{t}) g(\boldsymbol{x}-\boldsymbol{t}) \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\frac{1}{2}\boldsymbol{k}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)} e^{\boldsymbol{k}\frac{a_2}{b_2}t_2(t_2-x_2)}\, d\boldsymbol{t}\, d\boldsymbol{x}$$

$$= \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{a_1}{2b_1}x_1^2} e^{-\boldsymbol{i}\frac{2}{2b_1}x_1\omega_1} e^{\boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2} e^{\boldsymbol{i}\frac{a_1}{b_1}t_1(t_1-x_1)} f(\boldsymbol{t}) g(\boldsymbol{x}-\boldsymbol{t})$$

$$\frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{a_2}{2b_2}x_2^2} e^{-\boldsymbol{k}\frac{2}{2b_2}x_2\omega_2} e^{\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2} e^{\boldsymbol{k}\frac{a_2}{b_2}t_2(t_2-x_2)}\, d\boldsymbol{t}\, d\boldsymbol{x}.$$

$$= \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{a_1}{b_1}t_1(t_1-x_1)} e^{\boldsymbol{i}\frac{a_1}{2b_1}x_1^2} e^{-\boldsymbol{i}\frac{2}{2b_1}x_1\omega_1} e^{\boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2} f(\boldsymbol{t}) g(\boldsymbol{x}-\boldsymbol{t})$$

$$\frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{a_2}{b_2}t_2(t_2-x_2)} e^{\boldsymbol{k}\frac{a_2}{2b_2}x_2^2} e^{-\boldsymbol{k}\frac{2}{2b_2}x_2\omega_2} e^{\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2}\, d\boldsymbol{t}\, d\boldsymbol{x}. \tag{18}$$

By substituting $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{t},\ z_1 = x_1 - t_1$, and $z_2 = x_2 - t_2$, which can be written as $\boldsymbol{x} = \boldsymbol{z} + \boldsymbol{t},$ $x_1 = z_1 + t_1$, and $x_2 = z_2 + t_2$, in equation (18) we get

$$L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f \odot g\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}}\ e^{-\boldsymbol{i}\frac{a_1}{b_1}t_1 z_1}$$
$$e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}(z_1+t_1)^2 - \frac{2}{b_1}(z_1+t_1)\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} f(\boldsymbol{t}) g(\boldsymbol{z}) \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{-\boldsymbol{k}\frac{a_2}{b_2}t_2 z_2}$$
$$e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}(z_2+t_2)^2 - \frac{2}{b_2}(z_2+t_2)\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}\, d\boldsymbol{t}\, d\boldsymbol{z}. \tag{19}$$

By simplification of equation (19) we get

$$L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f \odot g\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{a_1 t_1^2}{2b_1}} e^{-\boldsymbol{i}\frac{t_1\omega_1}{b_1}} e^{\boldsymbol{i}\frac{d_1\omega_1^2}{2b_1}} e^{\boldsymbol{i}\frac{a_1 z_1^2}{2b_1}} e^{-\boldsymbol{i}\frac{z_1\omega_1}{b_1}} f(\boldsymbol{t})$$
$$g(\boldsymbol{z}) \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{a_2 t_2^2}{2b_2}} e^{-\boldsymbol{k}\frac{t_2\omega_2}{b_2}} e^{\boldsymbol{k}\frac{d_2\omega_2^2}{2b_2}} e^{\boldsymbol{k}\frac{a_2 z_2^2}{2b_2}} e^{-\boldsymbol{k}\frac{z_2\omega_2}{b_2}}\, d\boldsymbol{t}\, d\boldsymbol{z}$$
$$= \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{a_1 t_1^2}{2b_1}} e^{-\boldsymbol{i}\frac{t_1\omega_1}{b_1}} e^{\boldsymbol{i}\frac{d_1\omega_1^2}{2b_1}} f(\boldsymbol{t}) \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{a_2 t_2^2}{2b_2}} e^{-\boldsymbol{k}\frac{t_2\omega_2}{b_2}}$$
$$e^{\boldsymbol{k}\frac{d_2\omega_2^2}{2b_2}}\, d\boldsymbol{t} \int_{\mathbb{R}^2} e^{\boldsymbol{i}\frac{a_1 z_1^2}{2b_1}} e^{-\boldsymbol{i}\frac{z_1\omega_1}{b_1}} g(\boldsymbol{z})\, e^{\boldsymbol{k}\frac{a_2 z_2^2}{2b_2}} e^{-\boldsymbol{k}\frac{z_2\omega_2}{b_2}}\, d\boldsymbol{z}$$

$$= L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f\}(\boldsymbol{\omega}) \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{a_1 z_1^2}{2b_1}}\ e^{-\boldsymbol{i}\frac{z_1\omega_1}{b_1}}\ g(\boldsymbol{z}) e^{\boldsymbol{k}\frac{a_2 z_2^2}{2b_2}}\ e^{-\boldsymbol{k}\frac{z_2\omega_2}{b_2}}\ \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}}\, d\boldsymbol{z}.$$

By multiplying both sides of the identity above by $e^{\boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2}$ and $e^{\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2}$ and using the definition of TKLBT (1), it is easy to get

$$e^{\boldsymbol{i}\frac{d_1}{2b_1}\omega_1^2} L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f \odot g\}(\boldsymbol{\omega}) e^{\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2} = L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f\}(\boldsymbol{\omega}) L_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{g\}(\boldsymbol{\omega}). \tag{20}$$

This completes the proof of the TKLBT convolution theorem.

If observed, this result is in accordance with the convolution theorem for TKLBT as given in equation (12).

4. Conclusion

Based on the discussion, the reduced biquaternion linear canonical transformation (TKLBT) of the reduced biquaternion signal f is defined by

$$\boldsymbol{L}_{A_1,A_2}^{\boldsymbol{i},\boldsymbol{k}}\{f\}(\boldsymbol{\omega}) = \begin{cases} \int_{R^2} K_{A_1}(x_1, \omega_1) f(\boldsymbol{x}) K_{A_2}(x_{2,}, \omega_2)\ d\boldsymbol{x}, & b_n \neq 0, n = 1,2 \\ \sqrt{d_1 d_2}\, e^{\boldsymbol{i}\left(\frac{c_1 d_1}{2}\right)\omega_1^2} f(d_1\omega_1, d_2\omega_2) e^{\boldsymbol{k}\left(\frac{c_2 d_2}{2}\right)\omega_2^2}, & b_1 = 0 \text{ atau } b_2 = 0 \end{cases}$$

where

$$K_{A_1}(x_1,\omega_1) = \frac{1}{\sqrt{2\pi b_1 \boldsymbol{i}}} e^{\boldsymbol{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\,\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)},$$

and

$$K_{A_2}(x_2,\omega_2) = \frac{1}{\sqrt{2\pi b_2 \boldsymbol{k}}} e^{\boldsymbol{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\,\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}.$$

The inverse of the reduced biquaternion linear canonical transformation is given by $f(\boldsymbol{x}) = \begin{cases} \int_{R^2} K_{A_1^{-1}}(x_1, \omega_1) L^{\boldsymbol{i},\boldsymbol{k}}_{A_1,A_2} f(\boldsymbol{\omega}) K_{A_2^{-1}}(x_2, \omega_2)\ d\boldsymbol{\omega}, & b_n \neq 0, n = 1,2 \\ \sqrt{a_1 a_2}\, e^{-\boldsymbol{i}\left(\frac{c_1 a_1}{2}\right)x_1^2} f(a_1x_1, a_2x_2) e^{-\boldsymbol{k}\left(\frac{c_2 a_2}{2}\right)x_2^2}, & b_1 = 0 \text{ atau } b_2 = 0 \end{cases},$
where $A_1^{-1} = (d_1, -b_1, -c_1, a_1)\ dan\ A_2^{-1} = (d_2, -b_2, -c_2, a_2)$

The Plancherel and Parseval properties of the two reduced biquaternion functions f and g from TKLBT are as follows.

a) Plancherel properties

$$\int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}d\boldsymbol{x} = \int_{\mathbb{R}^2} L^{i,\mathrm{k}}_{A_1,A_2}\{f\}(\omega)\overline{L^{i,\mathrm{k}}_{A_1,A_2}\{g\}(\omega)}\, d\boldsymbol{\omega}.$$

b) Parseval Proprties

$$\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d\boldsymbol{x} = \int_{\mathbb{R}^2} \left|L^{i,\mathrm{k}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}.$$

The convolution for each function f and g in the reduced biquaternion linear canonical transformation, defined by

$$L^{\boldsymbol{i},\boldsymbol{k}}_{A_1,A_2}\{f \odot g\}(\boldsymbol{\omega}) = e^{-\boldsymbol{i}\frac{d_1\omega_1^2}{2b_1}} e^{-\boldsymbol{k}\frac{d_2}{2b_2}\omega_2^2} L^{\boldsymbol{i},\boldsymbol{k}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) L^{\boldsymbol{i},\boldsymbol{k}}_{A_1,A_2}\{g\}(\boldsymbol{\omega}).$$

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