

# Nash Equilibrium Seeking with Non-doubly Stochastic Communication Weight Matrix

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## Abstract

We propose a distributed Nash equilibrium seeking algorithm in a networked game, where each player has incomplete information on other players' actions. Players keep estimates and communicate over a strongly connected digraph with their neighbours according to a gossip communication protocol. Due to the asymmetric information exchange between players, a non-doubly (row)-stochastic weight matrix is defined. We prove almost-sure convergence of the algorithm to a Nash equilibrium under diminishing step-sizes. We extend the algorithm to graphical games in which players' cost functions are dependent only on their neighbouring players in an interference digraph. Given the interference digraph, a communication digraph is designed so that players exchange only their required information. The communication digraph is a subset of the interference digraph and a superset of its transitive reduction. Finally, we verify the efficacy of the algorithm via simulation on a social media behavioural case.

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**Keywords:** Nash equilibrium, communication graph, information exchange

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## 1. Introduction

The problem of finding a Nash equilibrium (NE) in a networked game has recently drawn a lot of attention. Players aim to minimize their own cost functions selfishly by making decision in response to other players' actions. Unlike a classical full-information setup, each player in the network has only access to local information of the neighbours. Due to the imperfect information available to players, they maintain an estimate of the other players' actions and communicate over a communication graph in order to exchange the estimates with local neighbours. Using this information, players update their actions and estimates.

Application scenarios range from spectrum access and internet access [1], networked Nash-Cournot competition, [2], congestion games in wireless networks, [3], ad-hoc networks [4] or peer-to-peer networks, [5], to social networks, [6]. These examples are non-cooperative in the way actions are taken (each agent

minimizes its own individual cost function), but *collaborative* in the sense that agents agree to share some information with neighbours to compensate for the lack of global information on others' decisions, as in ad-hoc or peer-to-peer networks, [5].

In many algorithms in the context of NE seeking problems, it is assumed that the communication between players is symmetric in the sense that players who are in communication can exchange their information altogether and update their estimates at the same time. This, in general, leads to a doubly stochastic communication weight matrix which preserves the global average of the estimates over time. However, there are many real-world applications in which symmetric communication is not of interest or may be an undesired feature, e.g. in applications such as sensor networks or social networks, [6], [7].

*Literature review.* Our work is related to the literature on Nash games and distributed Nash equilibrium (NE) seeking algorithms e.g. [3, 8–11]. A distributed discrete-time algorithm is proposed in [12] to compute a generalized NE when the communication graph is identical to the interference graph. In [2], an

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algorithm is provided to find an NE of *aggregative games* for a partial communication graph but complete interference graph. This algorithm is extended by [13] to a more general class of games in which players' cost functions do not necessarily depend on the aggregate of players' actions. It is further generalized to a partial (non-complete) interference graph in [14], over a partial, connected, undirected communication graph. For a *two-network zero-sum game* [15] considers a distributed algorithm for NE seeking. To find distributed algorithms for games with local-agent utility functions, a methodology is presented in [16] based on state-based *potential games*.

Gossip-based communication has been widely used in synchronous and asynchronous algorithms in consensus and distributed optimization problems [17–19]. In [17], a gossip algorithm is designed for a distributed broadcast-based optimization problem. An almost-sure convergence is provided due to the non-doubly stochasticity of the communication matrix. In [18], a broadcast gossip algorithm is proposed for computing the average of the initial measurements, which is proved to converge almost surely to a consensus.

*Contributions.* We propose an asynchronous gossip-based algorithm to find an NE of a distributed game over a partial, directed communication graph. We assume that players send/receive information to/from their local out/in-neighbours over a strongly connected digraph. Players update their own actions and estimates based on the received information. We prove an almost-sure convergence of the algorithm to the NE of the game. Furthermore, we adapt the algorithm to networked (graphical games) in which players' cost functions may depend on any subset of players' (not necessarily all) actions. The locality of cost functions is specified by an interference digraph, which marks the pair of players who interfere one with another. In this case, each player maintains an estimate of only the actions of players that interfere with him. This can greatly reduce the communication and computation overhead when the interference graph is sparse, e.g. [14]. However, due to the non-uniformity in players' estimate sizes, the communication digraph needs to be designed such that each player obtains all his required information from his communication neighbours. In this case there exists a lower bound on the communication digraph, namely it has to be a transitive reduction of the interference digraph. *Unlike in the undirected case [13, 14], due to asymmetric information exchange herein, we cannot exploit the doubly-stochastic property for the communication weight matrix. This property was critical in the convergence proof in [13, 14]. Not having it introduces technical challenges as direct properties on the weight matrix cannot be invoked.*

*Furthermore, the non-doubly stochasticity leads to the total average of players' estimates not being preserved over time.*

Our techniques are also similar to those used in the literature on distributed optimization [17], [20]. However, there are technical challenges due to the game context. In distributed optimization, all agents minimize an aggregate cost function with respect to a common optimization variable. In a game setup, each player controls only his own action, which is an element of the full decision vector. Moreover, his cost function depends on the actions of a subset of other players. This translates into *asymmetry and non-uniformity in players' data size and overall data exchange*. We circumvent these issues by introducing generalized weight matrices and exploiting their properties to prove convergence to Nash equilibrium.

A short version without proofs was presented in [21].

The paper is organized as follows. Problem statement and assumptions are given in Section 2 for games with a complete interference digraph. The algorithm is described in Section 3 and its convergence is shown in Section 4. In Section 5 we consider networked games with partial-interference digraphs; we extend the algorithm to this case in Section 6 and show its convergence in Section 7. Simulation results for a social network example are given in Section 8 and conclusions in Section 9.

## 2. Problem Statement: Game with a Complete Interference Digraph

Consider a multi-player game in a network with a set of players  $V$ . Each player  $i \in V$  has a real-valued cost function  $J_i$ , which may be affected by the actions of any number of players. In this section we consider that the interference between players' actions is represented by a complete *interference digraph*  $G(V, E)$ , with  $E$  marking the pair of players that interfere one with another. Note that for a complete digraph every pair of distinct nodes is connected by a pair of unique edges (one in each direction).

The game is denoted by  $\mathcal{G}(V, \Omega_i, J_i)$  where

- $V = \{1, \dots, N\}$ : Set of players,
- $\Omega_i \subset \mathbb{R}$ : Action set of player  $i$ ,  $\forall i \in V$  with  $\Omega = \prod_{i \in V} \Omega_i \subset \mathbb{R}^N$  the action set of all players,
- $J_i : \Omega \rightarrow \mathbb{R}$ : Cost function of player  $i$ ,  $\forall i \in V$ .

In the following we define a few notations for players' actions.

- $x = (x_i, x_{-i}) \in \Omega$ : All players actions,
- $x_i \in \Omega_i$ : Player  $i$ 's action,  $\forall i \in V$  and
- $x_{-i} \in \Omega_{-i} := \prod_{j \in V \setminus \{i\}} \Omega_j$ : All other players' actions except  $i$ .

The game is defined as a set of  $N$  simultaneous optimization problems as follows:

$$\begin{cases} \text{minimize} & J_i(y_i, x_{-i}) \\ \text{subject to} & y_i \in \Omega_i \end{cases} \quad \forall i \in V. \quad (1)$$

Each problem is run by an individual player and its solution is dependent on the solution of the other problems. The objective is to find an NE of this game which is defined as follows:

**Definition 1.** Consider an  $N$ -player game  $\mathcal{G}(V, \Omega_i, J_i)$ , each player  $i$  minimizing the cost function  $J_i : \Omega \rightarrow \mathbb{R}$ . A vector  $x^* = (x_i^*, x_{-i}^*) \in \Omega$  is called an NE of this game if

$$J_i(x_i^*, x_{-i}^*) \leq J_i(x_i, x_{-i}^*) \quad \forall x_i \in \Omega_i, \forall i \in V. \quad (2)$$

We state a few assumptions for the existence and the uniqueness of an NE.

**Assumption 1.** For every  $i \in V$ ,

- $\Omega_i$  is non-empty, compact and convex,
- $J_i(x_i, x_{-i})$  is  $C^1$  in  $x_i$ , continuous in  $x$  and convex in  $x_i$  for every  $x_{-i}$ .

The compactness of  $\Omega$  implies that  $\forall i \in V$  and  $x \in \Omega$ ,

$$\|\nabla_{x_i} J_i(x)\| \leq C, \quad \text{for some } C > 0. \quad (3)$$

Let  $F : \Omega \rightarrow \mathbb{R}^N$ ,  $F(x) := [\nabla_{x_i} J_i(x)]_{i \in V}$  be the pseudo-gradient vector of the cost functions (game map).

**Assumption 2.**  $F$  is strictly monotone,  $(F(x) - F(y))^T(x - y) > 0 \quad \forall x, y \in \Omega, x \neq y$ .

Assumption 1 and 2 imply that Nash equilibrium exists and is unique, cf. Theorem 2.3.3 in [22].

**Assumption 3.**  $\nabla_{x_i} J_i(x_i, u)$  is Lipschitz continuous in  $x_i$ , for every fixed  $u \in \Omega_{-i}$  and for every  $i \in V$ , i.e., there exists  $\sigma_i > 0$  such that

$$\|\nabla_{x_i} J_i(x_i, u) - \nabla_{x_i} J_i(y_i, u)\| \leq \sigma_i \|x_i - y_i\| \quad \forall x_i, y_i \in \Omega_i.$$

Moreover,  $\nabla_{x_i} J_i(x_i, u)$  is Lipschitz continuous in  $u$  with a Lipschitz constant  $L_i > 0$  for every fixed  $x_i \in \Omega_i, \forall i \in V$ .

In game (1), the only information available to each player  $i$  is  $J_i$  and  $\Omega$ . Thus, each player maintains an estimate of the other players actions and exchanges those estimates with the neighbours to update them. A *communication digraph*  $G_C(V, E_C)$  is defined where  $E_C \subseteq V \times V$  denotes the set of communication links between players.  $(i, j) \in E_C$  if and only if player  $i$  sends his information to player  $j$ . Note that  $(i, j) \in E_C$  does not necessarily imply  $(j, i) \in E_C$ . The set of *in-neighbours* of player  $i$  in  $G_C$ , denoted by  $N_C^{\text{in}}(i)$ , is defined as  $N_C^{\text{in}}(i) := \{j \in V | (j, i) \in E_C\}$ . The following assumption on  $G_C$  is used.

**Assumption 4.**  $G_C$  is strongly connected.

Our objective is to find an algorithm for computing an NE of  $\mathcal{G}(V, \Omega_i, J_i)$  using only imperfect information over the communication digraph  $G_C(V, E_C)$ .

### 3. Asynchronous Gossip-based Algorithm

We propose a projected gradient-based algorithm using an asynchronous gossip-based method as in [13]. The algorithm is inspired by [13] except that the communications are supposed to be directed in a sense that the information exchange is considered over a directed path. Our challenge here is to deal with the asymmetric communications between players. This makes the convergence proof dependent on a *non-doubly stochastic weight matrix*, whose properties need to be investigated and proved. The algorithm is elaborated as follows:

1- **Initialization Step:** Each player  $i$  maintains an initial *temporary* estimate  $\bar{x}^i(0) \in \Omega$  for all players. Let  $\bar{x}_j^i(0) \in \Omega_j \subset \mathbb{R}$  be player  $i$ 's initial temporary estimate of player  $j$ 's action, for  $i, j \in V$ .

2- **Gossiping Step:** At iteration  $k$ , player  $i_k$  becomes active uniformly at random and selects a communication in-neighbour indexed by  $j_k \in N_C^{\text{in}}(i_k)$  uniformly at random. Let  $\bar{x}^i(k) \in \Omega \subset \mathbb{R}^N$  be player  $i$ 's temporary estimate at iteration  $k$ . Then player  $j_k$  sends his temporary estimate  $\bar{x}^{j_k}(k)$  to player  $i_k$ . After receiving the information, player  $i_k$  constructs his final estimate of all players. Let  $\hat{x}_j^i(k) \in \Omega_j \subset \mathbb{R}$  be player  $i$ 's final estimate of player  $j$ 's action, for  $i, j \in V$ . The final estimates are computed as in the following:

- Players  $i_k$ 's final estimate:

$$\begin{cases} \hat{x}_{i_k}^{i_k}(k) = \bar{x}_{i_k}^{i_k}(k) \\ \hat{x}_{-i_k}^{i_k}(k) = \frac{\bar{x}_{-i_k}^{i_k}(k) + \bar{x}_{-i_k}^{j_k}(k)}{2}. \end{cases} \quad (4)$$

Note that  $\bar{x}_i^i(k) = x_i(k)$  for all  $i \in V$ .

- For all other players  $i \neq i_k$ , the temporary estimate is maintained, i.e.,

$$\hat{x}^i(k) = \bar{x}^i(k), \quad \forall i \neq i_k. \quad (5)$$

We use communication weight matrix  $W(k) := [w_{ij}(k)]_{i,j \in V}$  to obtain a compact form of the gossip protocol.  $W(k)$  is a *non-doubly (row) stochastic weight matrix* defined as

$$W(k) = I_N - \frac{e_{i_k}(e_{i_k} - e_{j_k})^T}{2}, \quad (6)$$

where  $e_i \in \mathbb{R}^N$  is a unit vector. Note that  $W(k)$  is different from the doubly stochastic one used in [13]. The non-doubly (row) stochasticity of  $W(k)$  means that

$$W(k)\mathbf{1}_N = \mathbf{1}_N, \quad \mathbf{1}_N^T W(k) \neq \mathbf{1}_N^T. \quad (7)$$

Let  $\bar{x}(k) = [\bar{x}^1(k), \dots, \bar{x}^N(k)]^T \in \Omega^N$  be an intermediary variable such that

$$\bar{x}(k) = (W(k) \otimes I_N) \bar{x}(k), \quad (8)$$

where  $\bar{x}(k) = [\bar{x}^1(k), \dots, \bar{x}^N(k)]^T \in \Omega^N$  is the overall temporary estimate at  $k$ . Using (6) one can combine (4) and (5) in a compact form of  $\hat{x}_{-i_k}^{i_k}(k) = \bar{x}_{-i_k}^{i_k}(k)$  and  $\hat{x}^i(k) = \bar{x}^i(k)$  for  $\forall i \neq i_k$ .

### 3- Local Step

At this moment all players update their actions according to a projected gradient-based method. Let  $\bar{x}^i = (\bar{x}_i^i, \bar{x}_{-i}^i) \in \Omega$ ,  $\forall i \in V$  with  $\bar{x}_i^i \in \Omega_i$  be the intermediary variable associated to player  $i$ . Because of imperfect information available to player  $i$ , he uses  $\bar{x}_{-i}^i(k)$  and updates his action as follows: if  $i = i_k$ ,

$$x_i(k+1) = T_{\Omega_i}[x_i(k) - \alpha_{k,i} \nabla_{x_i} J_i(x_i(k), \bar{x}_{-i}^i(k))], \quad (9)$$

otherwise,  $x_i(k+1) = x_i(k)$ . In (9),  $T_{\Omega_i} : \mathbb{R} \rightarrow \Omega_i$  is an Euclidean projection and  $\alpha_{k,i}$  are diminishing step sizes such that  $\sum_{k=1}^{\infty} \alpha_{k,i}^2 < \infty$ ,  $\sum_{k=1}^{\infty} \alpha_{k,i} = \infty \quad \forall i \in V$ . Each player uses his updated actions to update his temporary estimates as follows:

$$\bar{x}^i(k+1) = \bar{x}^i(k) + (x_i(k+1) - \bar{x}_i^i(k))e_i, \quad \forall i \in V. \quad (10)$$

At this point, players are ready to begin a new iteration from step 2.

### Algorithm 1

- 1: **initialization**  $\bar{x}^i(0) \in \Omega \quad \forall i \in V$
- 2: **for**  $k = 1, 2, \dots$  **do**
- 3:  $i_k \in V$  and  $j_k \in N_C^{\text{in}}(i_k)$  communicate.
- 4:  $W(k) = I_N - \frac{e_{i_k}(e_{i_k} - e_{j_k})^T}{2}$ .
- 5:  $\bar{x}(k) = (W(k) \otimes I_N)\bar{x}(k)$ .
- 6:  $x_{i_k}(k+1) = T_{\Omega_{i_k}}[x_{i_k}(k) - \alpha_{k,i_k} \nabla_{x_i} J_{i_k}(x_{i_k}(k), \bar{x}_{-i_k}^{i_k}(k))]$ ,  
 $x_i(k+1) = x_i(k)$ , if  $i \neq i_k$ .
- 7:  $\bar{x}^i(k+1) = \bar{x}^i(k) + (x_i(k+1) - \bar{x}_i^i(k))e_i, \quad \forall i \in V$ .
- 8: **end for**

We elaborate on the non-doubly stochasticity of  $W(k)$  from two perspectives.

1. **Design:** By the row (non-doubly) stochastic property of  $W(k)$ , the temporary estimates remain at consensus subspace once they reach there. This can be verified by (8) when  $\bar{x}(k) = \mathbf{1}_N \otimes \vec{a}$  for an  $N \times 1$  vector  $\vec{a}$ , since,

$$\bar{x}(k) = (W(k) \otimes I_N)(\mathbf{1}_N \otimes \vec{a}) = \mathbf{1}_N \otimes \vec{a}. \quad (11)$$

Equation (11), (9) and (10) imply that the consensus is maintained. On the other hand  $W(k)$  is not column-stochastic which is a critical property used in [13]. This implies that the average of temporary estimates is not equal to the average of  $\bar{x}$ . Indeed by (8),

$$\begin{aligned} \frac{1}{N}(\mathbf{1}_N^T \otimes I_N)\bar{x}(k) &= \frac{1}{N}(\mathbf{1}_N^T \otimes I_N)(W(k) \otimes I_N)\bar{x}(k) \\ &\neq \frac{1}{N}(\mathbf{1}_N^T \otimes I_N)\bar{x}(k). \end{aligned} \quad (12)$$

Equation (12), (9) and (10) imply that the average of temporary estimates is not preserved for the next iteration. Thus, it is infeasible to obtain an exact convergence to the average consensus [18]. Instead, we show an almost-sure (a.s.) convergence of the temporary estimates to an average consensus<sup>1</sup>.

2. **Convergence Proof:**  $\lambda_{\max}(W(k)^T W(k))$  is a key parameter in the proof (as in [13, 17]). Unlike [13], the non-doubly stochastic property of  $W(k)^T W(k)$  ends up in having  $\lambda_{\max}(W(k)^T W(k)) > 1$ . We resolve this issue in Lemma 2.

### 4. Convergence For Diminishing Step Sizes

In this section we prove convergence of the algorithm for diminishing step sizes. Consider a memory in which the history of the decision making is recorded. Let  $\mathcal{M}_k$  denote the sigma-field generated by the history up to time  $k-1$  with  $\mathcal{M}_0 = \{\bar{x}^i(0), i \in V\}$ .  $\mathcal{M}_k = \mathcal{M}_0 \cup \{(i_l, j_l); 1 \leq l \leq k-1\}$ ,  $\forall k \geq 2$ .

$$\mathcal{M}_k = \mathcal{M}_0 \cup \{(i_l, j_l); 1 \leq l \leq k-1\}, \quad \forall k \geq 2. \quad (13)$$

In the proof we use a well-known result on super martingale convergence, (Lemma 11, Chapter 2.2, [23]).

**Lemma 1.** Let  $V_k, u_k, \beta_k$  and  $\zeta_k$  be non-negative random variables adapted to  $\sigma$ -algebra  $\mathcal{M}_k$ . If  $\sum_{k=0}^{\infty} u_k < \infty$ ,  $\sum_{k=0}^{\infty} \beta_k < \infty$ , and  $\mathbb{E}[V_{k+1} | \mathcal{M}_k] \leq (1 + u_k)V_k - \zeta_k + \beta_k$  for all  $k \geq 0$ , then  $V_k$  converges a.s. and  $\sum_{k=0}^{\infty} \zeta_k < \infty$ .

As explained in the design challenge in Section 3, we consider a.s. convergence. Convergence is shown in two parts. First, we prove a.s. convergence of the temporary estimate vectors  $\bar{x}^i$ , to an average consensus, proved to be the vectors' average. Then we prove a.s. convergence of players' actions toward an NE.

Let  $\bar{x}(k)$  be the overall temporary estimate vector. The average of all temporary estimates at  $T(k)$  is defined as:

$$Z(k) = \frac{1}{N}(\mathbf{1}_N^T \otimes I_N)\bar{x}(k). \quad (14)$$

As mentioned in Section 3, the major difference between the proposed algorithm and the one in [13] is in using a non-doubly stochastic weight matrix  $W(k)$ , which prevents us from directly using its properties. The following lemma is used to overcome this challenge.

**Lemma 2.** Let  $Q(k) = (W(k) - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T W(k)) \otimes I_N$  and  $W(k)$  be a non-doubly (row) stochastic weight matrix defined in (6) which satisfies (7). Let also  $\gamma = \lambda_{\max}(\mathbb{E}[Q(k)^T Q(k)])$ . Then  $\gamma < 1$ .

<sup>1</sup>The same objective is followed by [17] to find a broadcast gossip algorithm (with non-doubly stochastic weight matrix) in the area of distributed optimization. However, in the proof of Lemma 2 ([17] page 1348) which is mainly dedicated to this discussion, the doubly stochasticity of  $W(k)$  is used right after equation (22) which violates the main assumption on  $W(k)$ .

*Proof.* Consider the variational characterization of  $\gamma$ . Since  $\mathbb{E}[Q(k)^T Q(k)]$  is an  $N^2 \times N^2$  symmetric matrix, we can write,

$$\gamma = \sup_{x \in \mathbb{R}^{N^2}, \|x\|=1} x^T \mathbb{E}[Q(k)^T Q(k)] x \geq 0. \quad (15)$$

Due to space limitation we drop the constraints of  $\sup(\cdot)$ . By the definition of  $Q(k)$ , we obtain,

$$\gamma = \sup_x x^T \mathbb{E} \left[ \left( W(k)^T W(k) - \frac{1}{N} W(k)^T \mathbf{1}_N \mathbf{1}_N^T W(k) \right) \otimes I_N \right] x.$$

Using (6), we expand  $\gamma$  as follows:

$$\begin{aligned} \gamma = \sup_x x^T \mathbb{E} \left[ \left( \underbrace{I_N - \frac{1}{4N} (e_{i_k} - e_{j_k})(e_{i_k} - e_{j_k})^T}_{\text{Term 1}} \right) \right. \\ \left. - \underbrace{\left( \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T + \frac{1}{2} (e_{i_k} - \frac{1}{N} \mathbf{1}_N)(e_{i_k} - e_{j_k})^T + \frac{1}{2} (e_{i_k} - e_{j_k})(e_{i_k} - \frac{1}{N} \mathbf{1}_N)^T \right)}_{\text{Term 2}} \right. \\ \left. - \frac{1}{4} (e_{i_k} - e_{j_k})(e_{i_k} - e_{j_k})^T \right) \otimes I_N \right] x. \quad (16) \end{aligned}$$

Note that  $\mathbb{E}[(\text{Term 1} - \text{Term 2}) \otimes I_N]$  is a symmetric matrix.

**Claim 1:** For all  $x \in \mathbb{R}^{N^2}, \|x\|=1$ , we have,  $x^T \mathbb{E}[\text{Term 1} \otimes I_N] x \leq 1$ . The equality only holds for  $x = \mathbf{1}_N \otimes y$  where  $y \in \mathbb{R}^N$  and  $\|y\| = \frac{1}{\sqrt{N}}$ .

**Proof of Claim 1:** Multiplying  $x^T$  and  $x$  into the argument of the expected value in (16) and using  $\|x\|=1$ , we obtain,

$$x^T \mathbb{E}[\text{Term 1} \otimes I_N] x = 1 - \frac{1}{4N} \mathbb{E} \left[ \left\| \left( (e_{i_k} - e_{j_k})^T \otimes I_N \right) x \right\|^2 \right] \leq 1.$$

Equality holds only when,

$$\begin{aligned} \mathbb{E} \left[ \left\| \left( (e_{i_k} - e_{j_k})^T \otimes I_N \right) x \right\|^2 \right] = 0 &\Leftrightarrow \left\| \left( (e_{i_k} - e_{j_k})^T \otimes I_N \right) x \right\|^2 = 0 \\ &\Leftrightarrow (e_{i_k}^T \otimes I_N) x = (e_{j_k}^T \otimes I_N) x. \end{aligned}$$

This holds for all  $k \geq 0$ ,  $i_k \in V$  and  $j_k \in N_C^{\text{in}}(i_k)$ . By the strong connectivity of  $G_C$  (Assumption 4), the foregoing becomes  $(e_i^T \otimes I_N) x = (e_j^T \otimes I_N) x$ ,  $\forall i, j \in V$  which implies that  $x = \mathbf{1}_N \otimes y$  where  $y \in \mathbb{R}^N$ . Moreover,  $\|x\|=1$  yields,

$$\|\mathbf{1}_N \otimes y\|^2 = 1 \Leftrightarrow (\mathbf{1}_N^T \otimes y^T)(\mathbf{1}_N \otimes y) = 1 \Leftrightarrow \|y\| = \frac{1}{\sqrt{N}}.$$

**Claim 2:** For  $x = (\mathbf{1}_N \otimes y) \in \mathbb{R}^{N^2}$  where  $y \in \mathbb{R}^N$  and  $\|y\| = \frac{1}{\sqrt{N}}$  we have  $x^T \mathbb{E}[\text{Term 2} \otimes I_N] x > 0$ .

**Proof of Claim 2:** For  $x = \mathbf{1}_N \otimes y$  and  $\|y\| = \frac{1}{\sqrt{N}}$  we

obtain by the mixed product property of Kronecker that,

$$\begin{aligned} x^T \mathbb{E}[\text{Term 2} \otimes I_N] x &= \mathbb{E} \left[ \left( \mathbf{1}_N^T \otimes y^T \right) (\text{Term 2} \otimes I_N) (\mathbf{1}_N \otimes y) \right] \\ &= \mathbb{E} \left[ \left( \mathbf{1}_N^T (\text{Term 2}) \mathbf{1}_N \right) \otimes y^T y \right]. \quad (17) \end{aligned}$$

It is straightforward to verify that  $\mathbf{1}_N^T (\text{Term 2}) \mathbf{1}_N = N$  because all the summands in Term 2 except the first one vanish by multiplying  $\mathbf{1}_N^T$  and  $\mathbf{1}_N$ . Having that  $y^T y = \frac{1}{N}$ , (17) implies  $x^T \mathbb{E}[\text{Term 2} \otimes I_N] x = 1 > 0$ . By Claims 1, 2 and using the fact that Terms 1, 2 are symmetric and  $\gamma \geq 0$ , (16) implies that  $\gamma < 1$ .  $\blacksquare$

We use Lemma 1 and Lemma 2 to show that  $\tilde{x}(k)$  converges a.s. to  $Z(k)$ .

**Theorem 1.** Let  $\tilde{x}(k)$  be the stack vector with all players' temporary estimates and  $Z(k)$  be its average as in (14). Let also  $\alpha_{k,\max} = \max_{i \in V} \alpha_{k,i}$ . Then under Assumptions 1,4, the following hold.

- i)  $\sum_{k=0}^{\infty} \alpha_{k,\max} \|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N) Z(k)\| < \infty$  a.s.,
- ii)  $\sum_{k=0}^{\infty} \|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N) Z(k)\|^2 < \infty$  a.s.

*Proof.* The idea of the proof is to repeatedly use Lemma 1 to show that a term is absolutely summable. While the proof follows as the proof of Theorem 1 in [13], here a critical step is in using Lemma 2.

The first step is to derive an upper bound for  $\mathbb{E} \left[ \|\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N) Z(k+1)\| \middle| \mathcal{M}_k \right]$  and apply Lemma 1 to the resulting expression.

From (10), (8), (14) and the row stochastic property of  $W(k)$  it follows that:

$$\begin{aligned} \mathbb{E} \left[ \|\tilde{x}(k+1) - (\mathbf{1}_N \otimes I_N) Z(k+1)\| \middle| \mathcal{M}_k \right] \\ \leq \underbrace{\mathbb{E} \left[ \|Q(k)(\tilde{x}(k) - (\mathbf{1}_N \otimes I_N) Z(k))\| \middle| \mathcal{M}_k \right]}_{\text{Term 1}} + \underbrace{\mathbb{E} \left[ \|R\mu(k+1)\| \middle| \mathcal{M}_k \right]}_{\text{Term 2}}, \quad (18) \end{aligned}$$

where  $\mu(k+1) = [(x_i(k+1) - \tilde{x}_i^j(k)) e_i]_{i \in V}$ ,  $Q(k) = (W(k) - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T W(k)) \otimes I_N$  (as defined in Lemma 2) and  $R = (I_N - \frac{1}{N} \mathbf{1}_N \mathbf{1}_N^T) \otimes I_N$ .

Let  $\gamma = \lambda_{\max}(\mathbb{E}[Q(k)^T Q(k)])$  be as in Lemma 2. We obtain the following upper bound for Term 1 in (18).

$$\begin{aligned} \text{Term 1} &\leq \sqrt{\mathbb{E} \left[ \|Q(k)(\tilde{x}(k) - (\mathbf{1}_N \otimes I_N) Z(k))\|^2 \middle| \mathcal{M}_k \right]} \\ &\leq \sqrt{\gamma} \|\tilde{x}(k) - (\mathbf{1}_N \otimes I_N) Z(k)\|. \quad (19) \end{aligned}$$

Note that by Lemma 2,  $\gamma < 1$ .

To bound Term 2, we use (9), the non-expansive property of projection,  $\|R\|=1$ , Assumption 1 (equation

(3)) and  $x_i(k+1) = x_i(k) = \bar{x}_i^i(k)$  for  $i \neq i_k$ . Then,

$$\begin{aligned} \text{Term 2} &\leq \mathbb{E} \left[ \|\mu(k+1)\| \middle| \mathcal{M}_k \right] = \mathbb{E} \left[ \sqrt{\sum_{i \in V} \|x_i(k+1) - \bar{x}_i^i(k)\|^2} \middle| \mathcal{M}_k \right] \\ &= \mathbb{E} \left[ \|x_{i_k}(k+1) - \bar{x}_{i_k}^{i_k}(k)\| \middle| \mathcal{M}_k \right] \\ &\leq \mathbb{E} \left[ \|x_{i_k}(k) - \bar{x}_{i_k}^{i_k}(k)\| \middle| \mathcal{M}_k \right] + \alpha_{k,\max} C \\ &= \frac{1}{2} \|\bar{x}_{i_k}^{i_k}(k) - \bar{x}_{i_k}^{i_k}(k)\| + \alpha_{k,\max} C. \end{aligned} \quad (20)$$

The last equality obtained by using  $\bar{x}_{i_k}^{i_k} = \frac{\bar{x}_{i_k}^{i_k} + \bar{x}_{i_k}^{i_k}}{2}$  which comes from (8). Using (9), (10), (8), projection's non-expansive property and (3) yields

$$\|\bar{x}_{i_k}^{i_k}(k+1) - \bar{x}_{i_k}^{i_k}(k+1)\| \leq \|\bar{x}_{i_k}^{i_k}(k) - \bar{x}_{i_k}^{i_k}(k)\| + \alpha_{k,i_k} C. \quad (21)$$

Take expected value of (21) and multiply its LHS and RHS by  $\alpha_{k+1,i_k}$  and  $\alpha_{k,i_k}$ , respectively, to yield,

$$\begin{aligned} \alpha_{k+1,i_k} \mathbb{E} \left[ \|\bar{x}_{i_k}^{i_k}(k+1) - \bar{x}_{i_k}^{i_k}(k+1)\| \middle| \mathcal{M}_k \right] \\ \leq \alpha_{k,i_k} \|\bar{x}_{i_k}^{i_k}(k) - \bar{x}_{i_k}^{i_k}(k)\| + \alpha_{k,i}^2 C, \end{aligned} \quad (22)$$

since  $\alpha_{k+1,i_k} < \alpha_{k,i_k}$ . Applying Lemma 1 for  $V_k = \alpha_{k,i_k} \|\bar{x}_{i_k}^{i_k}(k) - \bar{x}_{i_k}^{i_k}(k)\|$  and using diminishing-size steps and (3), it follows that  $\sum_{k=0}^{\infty} \alpha_{k,i_k} \|\bar{x}_{i_k}^{i_k}(k) - \bar{x}_{i_k}^{i_k}(k)\| < \infty$ . From (18), (19) and (20) it follows that

$$\begin{aligned} \mathbb{E} \left[ \|\bar{x}(k+1) - (\mathbf{1}_N \otimes I_N)Z(k+1)\| \middle| \mathcal{M}_k \right] \\ \leq \sqrt{\gamma} \|\bar{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\| + \frac{1}{2} \|\bar{x}_{i_k}^{i_k}(k) - \bar{x}_{i_k}^{i_k}(k)\| + \alpha_{k,\max} C. \end{aligned} \quad (23)$$

Multiplying the LHS and RHS of (23) by  $\alpha_{k+1,\max}$  and  $\alpha_{k,\max}$ , respectively and using  $\gamma < 1$  (Lemma 2) and diminishing-size steps, Part i) follows by applying Lemma 1.

The Proof of Part ii) is similar to that of Part i) and it is omitted due to space limitations. ■

**Corollary 1.** Under Assumptions 4-1, the following hold a.s. for players' actions  $x(k)$  and for  $\bar{x}(k)$ :

- i)  $\sum_{k=0}^{\infty} \alpha_{k,\max} \|x(k) - Z(k)\| < \infty$  a.s.,
- ii)  $\sum_{k=0}^{\infty} \|x(k) - Z(k)\|^2 < \infty$  a.s.,
- iii)  $\sum_{k=0}^{\infty} \mathbb{E} \left[ \|\bar{x}(k) - (\mathbf{1}_N \otimes I_N)Z(k)\|^2 \middle| \mathcal{M}_k \right] < \infty$  a.s.

*Proof.* The proof follows directly from Theorem 1 noting that  $x(k) = [\bar{x}_i^i(k)]_{i \in V}$  and  $\bar{x}(k) = (W(k) \otimes I_N)\bar{x}(k)$  (8).

**Theorem 2.** Let  $x(k)$  and  $x^*$  be all players' actions and the NE of  $\mathcal{G}$ , respectively. Under Assumptions 4-3, the sequence  $\{x(k)\}$  generated by the algorithm converges to  $x^*$ , almost surely.

*Proof.* The proof follows based on Theorem 1 and Corollary 1, and is similar to the proof of Theorem 2 in [13].

Theorem 2 verifies that the actions of all players converge a.s. toward the NE using the fact that the actions converge to a consensus subspace (Corollary 1).

## 5. Game With a Partial Interference Digraph

We extend the game defined in Section 2 to the case with partially coupled cost functions, such that the cost functions may be interfered by the actions of any subset of players. The game is denoted by  $\mathcal{G}(V, G_I, \Omega_i, J_i)$  where  $G_I(V, E_I)$  is an interference digraph with  $E_I$  marking players whose actions interfere with the other players' cost functions. We denote by  $N_I^{\text{in}}(i) := \{j \in V \mid (j, i) \in E_I\}$ , the set of *in-neighbours* of player  $i$  in  $G_I$  whose actions affect  $J_i$  and  $\tilde{N}_I^{\text{in}}(i) := N_I^{\text{in}}(i) \cup \{i\}$ .

The following assumption is considered for  $G_I$ .

**Assumption 5.**  $G_I$  is strongly connected.

The cost function of player  $i$ ,  $J_i$ ,  $\forall i \in V$ , is defined over  $\Omega^i \rightarrow \mathbb{R}$  where  $\Omega^i = \prod_{j \in \tilde{N}_I^{\text{in}}(i)} \Omega_j \subset \mathbb{R}^{|\tilde{N}_I^{\text{in}}(i)|}$  is the action set of players interfering with the cost function of player  $i$ . A few notations for players' actions are given:

- $x^i = (x_i, x_{-i}^i) \in \Omega^i$ : All players' actions which interfere with  $J_i$ ,
- $x_{-i}^i \in \Omega_{-i}^i := \prod_{j \in \tilde{N}_I^{\text{in}}(i)} \Omega_j$ : Other players' actions which interfere with  $J_i$ .

Given  $x_{-i}^i$ , each player  $i$  aims to minimize his own cost function selfishly,

$$\begin{cases} \text{minimize} & J_i(y_i, x_{-i}^i) \\ & y_i \\ \text{subject to} & y_i \in \Omega_i \end{cases} \quad \forall i \in V. \quad (24)$$

Known parameters to player  $i$  are as follows: 1) Cost function of player  $i$ ,  $J_i$  and 2) Action set  $\Omega^i$ . Note that this game setup is similar to the one in [14] except for a directed  $G_C$  used for asymmetric communication. We assume that each player maintains an estimate of only his interfering players' actions according to  $G_I$ , and that players exchange information over a communication digraph  $G_C(V, E_C)$ , which is a subset of the interference digraph  $G_I$ . As no unnecessary data needs exchanged, this can greatly reduce the communication and computation cost when  $G_I$  is sparse, see [14].

Our first objective is to design an assumption on  $G_C$  such that all required information is communicated by players after sufficiently many iterations. In other words, we want to ensure that player  $i$ ,  $\forall i \in V$  receives information on all players whose actions interfere with his cost function.

**Definition 2.** Transitive reduction: A digraph  $H$  is a transitive reduction of  $G$  if it is obtained as follows: for all three vertices  $i, j, l$  in  $G$  such that edges  $(i, j), (j, l)$  are in  $G$ ,  $(i, l)$  is removed from  $G$ .

**Assumption 6.** The following holds for the communication graph  $G_C$ :  $G_{TR} \subseteq G_C \subseteq G_I$ , where  $G_{TR}$  is a transitive reduction of  $G_I$ .

The lower bound is required because each player does not maintain estimates of all other players as in Section 3, or as in distributed optimization, [17], but only of those interfering with him. It can be shown via a counter example that if players communicate via a path of length greater than 2, they may lose some information. Note that a transitive reduction is different from a *maximal triangle-free spanning subgraph* which is used in Assumption 2 in [14]. In simple terms, a transitive reduction of a digraph is a digraph without the parallel paths between the vertices. Based on Assumption 6 we show next that each player  $i \in V$  can obtain his necessary information about players in  $N_I^{\text{in}}(i)$  from his neighbours in  $N_C^{\text{in}}(i)$ .

**Lemma 3.** Let  $G_I$  and  $G_C$  satisfying Assumptions 5 and Assumption 6, respectively. Then,  $\forall i \in V$ ,

$$\bigcup_{j \in N_C^{\text{in}}(i)} (N_I^{\text{in}}(i) \cap \tilde{N}_I^{\text{in}}(j)) = N_I^{\text{in}}(i). \quad (25)$$

*Proof.* The proof is similar to the proof of Lemma 2 in [14], but modified to adapt for the directed graph. We need to show  $N_I^{\text{in}}(i) \subseteq \bigcup_{j \in N_C^{\text{in}}(i)} \tilde{N}_I^{\text{in}}(j) \forall i \in V$  from which it is straightforward to deduce (25).

For the case when  $G_C = G_I$ , we obtain,

$$\bigcup_{j \in N_C^{\text{in}}(i)} \tilde{N}_I^{\text{in}}(j) = \bigcup_{j \in N_I^{\text{in}}(i)} \tilde{N}_I^{\text{in}}(j) \supseteq \bigcup_{j \in N_I^{\text{in}}(i)} \{j\} = N_I^{\text{in}}(i). \quad (26)$$

In (26), we used  $\{j\} \subseteq \tilde{N}_I^{\text{in}}(j)$  by the definition of  $\tilde{N}_I^{\text{in}}(j)$ .

Now assume that  $G_{TR} \subseteq G_C \subset G_I$ . To prove (25), it is sufficient to show that  $N_I^{\text{in}}(i) \subseteq \bigcup_{j \in N_C^{\text{in}}(i)} \tilde{N}_I^{\text{in}}(j)$ , where  $N_{TR}^{\text{in}}(i)$  is the set of in-neighbours of player  $i$  in  $G_{TR}$  and  $\tilde{N}_{TR}^{\text{in}}(i)$  in addition to  $N_{TR}^{\text{in}}(i)$  contains  $\{i\}$ . In other words we need to show that any in-neighbour of player  $i$  (any vertex with an incoming edge to  $i$ ) in  $G_I$  is either an in-neighbour or “in-neighbour of an in-neighbour” of player  $i$  (a vertex with an incoming path of at most length 2 to  $i$ ) in  $G_{TR}$ . If an incoming edge to  $i$  exists both in  $G_I$  and  $G_{TR}$ , the corresponding in-neighbour of  $i$  in  $G_I$  is an in-neighbour of  $i$  in  $G_{TR}$ . Otherwise, if there exists an incoming edge to  $i$  in  $G_I$  that is missing in  $G_{TR}$ , according to Definition 2, there exists a directed path of length 2 parallel to the missing edge in  $G_{TR}$ . So the corresponding in-neighbour of player  $i$  in  $G_I$  is an in-neighbour of an in-neighbour of player  $i$  in  $G_{TR}$ . ■

The assumptions for existence and uniqueness of an NE are Assumptions 1-3 with the cost functions adapted to  $G_I$ .

Our second objective is to find an algorithm for computing an NE of  $\mathcal{G}(V, G_I, \Omega_i, J_i)$  over  $G_C(V, E_C)$  with partially coupled cost functions as described by the directed graph  $G_I(V, E_I)$ .

## 6. Asynchronous Gossip-based Algorithm adapted to $G_I$

The structure of the algorithm is similar to the one in Section 3. The steps are elaborated in the following:

### 1- Initialization Step:

- $\bar{x}^i(0) \in \Omega^i$ : Player  $i$ 's initial temporary estimate.

### 2- Gossiping Step:

- $\hat{x}_j^i(k) \in \Omega_j \subset \mathbb{R}$ : Player  $i$ 's temporary estimate of player  $j$ 's action at  $k$ .
- $\hat{x}_j^i(k) \in \Omega_j \subset \mathbb{R}$ : Player  $i$ 's final estimate of player  $j$ 's action at  $k$ , for  $i \in V, j \in \tilde{N}_I^{\text{in}}(i)$ .
- Final estimate construction:

$$\hat{x}_l^i(k) = \begin{cases} \frac{\hat{x}_l^i(k) + \hat{x}_l^j(k)}{2}, & l \in (N_I^{\text{in}}(i_k) \cap \tilde{N}_I^{\text{in}}(j_k)) \\ \hat{x}_l^i(k), & l \in \tilde{N}_I^{\text{in}}(i_k) \setminus (N_I^{\text{in}}(i_k) \cap \tilde{N}_I^{\text{in}}(j_k)). \end{cases} \quad (27)$$

For  $i \neq i_k, j \in \tilde{N}_I^{\text{in}}(i)$ ,

$$\hat{x}_j^i(k) = \bar{x}_j^i(k) \quad (28)$$

We suggest a compact form for gossip protocol by using  $W^I(k)$  defined as,

$$W^I(k) := I_m - \sum_{l \in (N_I^{\text{in}}(i_k) \cap \tilde{N}_I^{\text{in}}(j_k))} \frac{e_{s_{ij}l} (e_{s_{ik}l} - e_{s_{jk}l})^T}{2}, \quad (29)$$

where  $e_i \in \mathbb{R}^m$  is a unit vector and  $s_{ij}$  is an index of the estimate vector element associated with player  $i$ 's estimate of player  $j$ 's action. Note that  $W^I(k)$  (29) is different from the doubly stochastic one used in [14].

- $\bar{x}(k) := [\bar{x}^1(k), \dots, \bar{x}^N(k)]^T$ : Stack vector of all temporary estimates,
- $\bar{x}(k) := W^I(k)\bar{x}(k)$ : Intermediary variable.

Using the intermediary variable, one can construct the final estimates as follows:

$$\hat{x}_{-i}^i(k) = [\bar{x}_{s_{ij}}(k)]_{j \in N_I^{\text{in}}(i)}. \quad (30)$$

3- Local Step: Player  $i$  updates his action as follows:

If  $i = i_k$ ,

$$x_i(k+1) = T_{\Omega_i} \left[ x_i(k) - \alpha_{k,i} \nabla_{x_i} J_i(x_i(k), [\bar{x}_{s_{ij}}(k)]_{j \in N_I^{\text{in}}(i)}) \right], \quad (31)$$

otherwise,  $x_i(k+1) = x_i(k)$ .

Then he updates his temporary estimates as:

$$\bar{x}_j^i(k+1) = \begin{cases} \bar{x}_{s_{ij}}(k), & \text{if } j \neq i \\ x_i(k+1), & \text{if } j = i. \end{cases} \quad (32)$$

At this point, players are ready to begin a new iteration from step 2.

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### Algorithm 2

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- 1: **initialization**  $\bar{x}^i(0) \in \Omega^i \quad \forall i \in V$
  - 2: **for**  $k = 1, 2, \dots$  **do**
  - 3:  $i_k \in V$  and  $j_k \in N_C^{\text{in}}(i_k)$  communicate.
  - 4:  $W^I(k) := I_m - \sum_{l \in (\tilde{N}_I^{\text{in}}(i_k) \cap \tilde{N}_I^{\text{in}}(j_k))} \frac{e_{s_{ik}l}(e_{s_{ik}l} - e_{s_{jk}l})^T}{2}$ .
  - 5:  $\bar{x}(k) = W^I(k)\bar{x}(k)$ .
  - 6:  $x_i(k+1) = T_{\Omega_i} [x_i(k) - \alpha_{k,i} \nabla_{x_i} J_i(x_i(k), [\bar{x}_{s_{ij}}(k)]_{j \in N_I^{\text{in}}(i)})]$  if  $i = i_k$ , otherwise,  $x_i(k+1) = x_i(k)$ .
  - 7:  $\bar{x}^i(k+1) = \bar{x}^i(k) + (x_i(k+1) - \bar{x}_i^i(k))e_i, \quad \forall i \in V$ .
  - 8: **end for**
- 

## 7. Convergence of the algorithm adapted to $G_I$

Similar to Section 4, the convergence proof is split into two steps:

1. First, we prove a.s. convergence of  $\bar{x}(k) \subset \mathbb{R}^m$  to an average consensus which is shown to be the augmented average of all temporary estimate vectors. Let

- $m_i^{\text{out}} := \deg_{G_I}^{\text{out}}(i) + 1$ , where  $\deg_{G_I}^{\text{out}}(i)$  is the out-degree of vertex  $i$  in  $G_I$ ,
- $1./\mathbf{m}^{\text{out}} := [\frac{1}{m_1^{\text{out}}}, \dots, \frac{1}{m_N^{\text{out}}}]^T$ ,
- $H := [\sum_{i:1 \in N_I^{\text{in}}(i)} e_{s_{i1}}, \dots, \sum_{i:N \in N_I^{\text{in}}(i)} e_{s_{iN}}] \in \mathbb{R}^{m \times N}$  (33)

where  $i : j \in N_I^{\text{in}}(i)$  is all  $i$ 's such that  $j \in N_I^{\text{in}}(i)$ . The augmented average of all temporary estimates is denoted by  $Z^I(k) \in \mathbb{R}^m$  and defined as follows:

$$Z^I(k) := H \text{diag}(1./\mathbf{m}^{\text{out}}) H^T \bar{x}(k) \in \mathbb{R}^m \quad (34)$$

2. Secondly, we prove almost-sure convergence of players' actions to an NE.

The proof depends on some key properties of  $W^I$  and  $H$  given in Lemma 4 and Lemma 5.

**Lemma 4.** Let  $W^I(k)$  and  $H$  be defined in (29) and (33). Then,  $W^I(k)H = H$ .

*Proof.* The proof is similar to the proof of Lemma 3 in [14] but adapted for the different  $W^I$  here. Using the definitions of  $H$  and  $W^I(k)$  (33), (29), we expand  $W^I(k)H$  as  $W^I(k)H = H - \frac{1}{2} \sum_{l \in (\tilde{N}_I^{\text{in}}(i_k) \cap \tilde{N}_I^{\text{in}}(j_k))} e_{s_{ik}l} \left[ \sum_{i:1 \in N_I^{\text{in}}(i)} (e_{s_{ik}l} - e_{s_{jk}l})^T e_{s_{i1}} \dots + \sum_{i:N \in N_I^{\text{in}}(i)} (e_{s_{ik}l} - e_{s_{jk}l})^T e_{s_{iN}} \right]$ . Note that  $\sum_{i:j \in N_I^{\text{in}}(i)} (e_{s_{ik}l} - e_{s_{jk}l})^T e_{s_{ij}} = 0$  for all  $j \in V$  because  $e_{s_{ik}l}^T e_{s_{ij}} = 1$  for  $i = i_k, j = l$  and  $e_{s_{ik}l}^T e_{s_{ij}} = 0$ , otherwise. Similarly,  $e_{s_{jk}l}^T e_{s_{ij}} = 1$  for  $i = j_k, j = l$  and  $e_{s_{jk}l}^T e_{s_{ij}} = 0$ , otherwise. This completes the proof. ■

Lemma 4 can be interpreted as a generalized row-stochastic property of  $W^I(k)$ . Note that the generalized non-doubly stochasticity of  $W^I(k)$  is translated into  $H^T W^I(k) \neq H^T$ .

**Lemma 5.** Let  $Q^I(k) := W^I(k) - H \text{diag}(1./\mathbf{m}^{\text{out}}) H^T W^I(k)$ , where  $W^I(k)$  and  $H$  are defined in (29) and (33), and  $\gamma^I = \lambda_{\max}(\mathbb{E}[Q^I(k)^T Q^I(k)])$ . Then  $\gamma^I < 1$ .

*Proof.* As suggested in (15), we employ the variational characterization of  $\gamma$ ,  $\gamma^I = \sup_{x \in \mathbb{R}^m, \|x\|=1} x^T \mathbb{E}[Q^I(k)^T Q^I(k)] x = \sup_{x \in \mathbb{R}^m, \|x\|=1} x^T \mathbb{E} \left[ \left( W^I(k)^T - W^I(k)^T H \text{diag}(1./\mathbf{m}^{\text{out}}) H^T \right) \cdot \left( W^I(k) - H \text{diag}(1./\mathbf{m}^{\text{out}}) H^T W^I(k) \right) \right] x = \sup_{x \in \mathbb{R}^m, \|x\|=1} x^T \mathbb{E} \left[ W^I(k)^T (I - H \text{diag}(1./\mathbf{m}^{\text{out}}) H^T) W^I(k) \right] x$ .

For the last equality, we used  $H^T H = \text{diag}(\mathbf{m}^{\text{out}})$  which is straightforward to verify. We expand  $\gamma^I$  and split the terms as follows (Let  $l \in \tilde{N}_I^{\text{in}}(i_k) \cap \tilde{N}_I^{\text{in}}(j_k)$ ):

$$\begin{aligned} \gamma^I &= \sup_x x^T \mathbb{E} \left[ \underbrace{\left( I_m - \frac{1}{4} \sum_l (e_{s_{ik}l} - e_{s_{jk}l}) e_{s_{ik}l}^T H \text{diag}(1./\mathbf{m}^{\text{out}}) \right)}_{\text{Term 1}} \right. \\ &\quad \cdot \underbrace{H^T \sum_l e_{s_{ik}l} (e_{s_{ik}l} - e_{s_{jk}l})^T}_{\text{Term 1}} \left. - \underbrace{\left( H \text{diag}(1./\mathbf{m}^{\text{out}}) H^T \right)}_{\text{Term 2}} \right. \\ &\quad \left. - \frac{1}{4} \sum_l (e_{s_{ik}l} - e_{s_{jk}l}) e_{s_{ik}l}^T \sum_l e_{s_{ik}l} (e_{s_{ik}l} - e_{s_{jk}l})^T \right. \\ &\quad \left. + \frac{1}{2} \underbrace{\left( I_m - H \text{diag}(1./\mathbf{m}^{\text{out}}) H^T \right) \sum_l e_{s_{ik}l} (e_{s_{ik}l} - e_{s_{jk}l})^T}_{\text{Term 2}} \right. \\ &\quad \left. + \frac{1}{2} \underbrace{\left( \sum_l (e_{s_{ik}l} - e_{s_{jk}l}) e_{s_{ik}l}^T \right) \left( I_m - H \text{diag}(1./\mathbf{m}^{\text{out}}) H^T \right)}_{\text{Term 2}} \right] x. \end{aligned}$$



We aim to prove that  $x^T \mathbb{E}[\text{Term 1}]x \leq 1$ . Multiplying by  $x^T$  and  $x$  into Term 1, we arrive at,

$$x^T \mathbb{E}[\text{Term 1}]x = 1 - \frac{1}{4} \mathbb{E} \left[ \left\| \text{diag}(1./\sqrt{\mathbf{m}^{\text{out}}}) H^T \sum_l e_{s_{ikl}} (e_{s_{ikl}} - e_{s_{jkl}})^T x \right\|^2 \right] \leq 1.$$

Equality holds for all  $x$ 's that satisfy  $H^T \sum_l e_{s_{ikl}} (e_{s_{ikl}} - e_{s_{jkl}})^T x = 0$ . After a few manipulations, by the strong connectivity of  $G_C$  for  $i \in V$ ,  $j \in N_C^{\text{in}}(i)$  and  $l \in (\tilde{N}_I^{\text{in}}(i_k) \cap \tilde{N}_I^{\text{in}}(j_k))$  we obtain,

$$x_{s_{il}} = x_{s_{jl}}. \quad (35)$$

To complete the proof we need to show  $x^T \mathbb{E}[\text{Term 2}]x > 0$  for all  $x$ 's satisfy (35) and  $\|x\| = 1$ . After some manipulations we obtain,  $x^T \mathbb{E}[\text{Term 2}]x = x^T H \text{diag}(1./\mathbf{m}^{\text{out}}) H^T x = \|\text{diag}(1./\sqrt{\mathbf{m}^{\text{out}}}) H^T x\|^2 \geq 0$ . The rest of the proof is straightforward by verifying that for all  $x$ 's which satisfy (35) and  $\|x\| = 1$ ,  $H^T x \neq 0$ . ■

**Theorem 3.** Let  $\tilde{x}(k)$  be the stack vector with all players' temporary estimates and  $Z^I(k)$  be its average as in (34). Let also  $\alpha_{k,\max} = \max_{i \in V} \alpha_{k,i}$ . Then under Assumptions 5, 6, 1', the following hold.

- i)  $\sum_{k=0}^{\infty} \alpha_{k,\max} \|\tilde{x}(k) - Z^I(k)\| < \infty$  a.s.
- ii)  $\sum_{k=0}^{\infty} \|\tilde{x}(k) - Z^I(k)\|^2 < \infty$  a.s.

*Proof.* A similar argument as in the proof of Theorem 1 is used, this time based on using Lemma 4 and Lemma 5. The proof is similar to the proof Theorem 1 in [14] and is omitted due to space constraints.

**Corollary 2.** Let  $z^I(k) := \text{diag}(1./\mathbf{m}^{\text{out}}) H^T \tilde{x}(k) \in \mathbb{R}^N$  be the average of all players' temporary estimates. Under Assumptions 5, 6, 1', the following hold for players' actions  $x(k)$  and  $\tilde{x}(k)$ :

- i)  $\sum_{k=0}^{\infty} \alpha_{k,\max} \|x(k) - z^I(k)\| < \infty$  a.s.
- ii)  $\sum_{k=0}^{\infty} \|x(k) - z^I(k)\|^2 < \infty$  a.s.,
- iii)  $\sum_{k=0}^{\infty} \mathbb{E} \left[ \|\tilde{x}(k) - Z^I(k)\|^2 \middle| \mathcal{M}_k \right] < \infty$  a.s.

*Proof.* The proof follows by taking into account  $x(k) = [\tilde{x}_i^i(k)]_{i \in V}$ ,  $Z^I(k) = H z^I(k)$ ,  $\tilde{x}(k) = W^I(k) \tilde{x}(k)$  and using Theorem 3.

**Theorem 4.** Let  $x(k)$  and  $x^*$  be all players' actions and the NE of  $\mathcal{G}$ , respectively. Under Assumptions 1'-3', 5, 6, the sequence  $\{x(k)\}$  generated by the algorithm converges to  $x^*$ , almost surely.

*Proof.* The proof uses Theorem 3 and is similar to the proof of Theorem 2 in [14].

## 8. Simulation Results

### 8.1. Social Media behaviour

In this example we aim to investigate a social networking media example and users' behaviour. In

such media like Facebook, Twitter and Instagram users are allowed to follow (or be friend with) other users and post status updates, photos and videos or share links and events, [6], [7]. Depending on the type of social media, the way of communication is defined. For instance, in Instagram, friendship is defined unidirectional in the sense that either side could be only a follower and/or being followed. Recently, researchers at Microsoft have been studying the behavioural attitude of the users of Facebook as a giant and global network [24]. This study can be useful in many areas e.g. business (posting advertisements) and politics (posting for the purpose of presidential election campaign).

Generating new status usually comes with a payoff (utility) for users, while also incurring some cost; if there is no benefit in posting status, the users don't bother to generate new ones. In any social media drawing others' attention is one of the most important motivation/stimulation to post status [25]. Our objective is to find the optimal rate of posting status for each user to draw more attention in his network. In the following, we use an information/attention model of a generic social media [25] and define a communication between users ( $G_C$ ) and an interference graph between them ( $G_I$ ).

Consider a social media network of  $N$  users. Each user  $i$  produces  $x_i$  unit of information that the followers can see in their news feeds. The users' communication network is defined by a strongly connected digraph  $G_C$  in which  $\textcircled{i} \rightarrow \textcircled{j}$  means  $j$  is a follower of  $i$  or  $j$  receives  $x_i$  in his news feed. We also assume a strongly connected interference digraph  $G_I$  to show the influence of users on the others. We assume that each user  $i$ 's utility/cost function is not only affected by the users he follows, but also by the users that his followers follow.

The cost function of user  $i$  is denoted by  $J_i$  and consists of three parts: 1)  $C_i(x_i) := h_i x_i$ ,  $h_i > 0$  which is a cost that user  $i$  incurs to produce  $x_i$  unit of information. 2)  $f_i^1(x) := L_i \sqrt{\sum_{j \in N_C^{\text{in}}(i)} q_{ji} x_j}$ ,  $L_i > 0$  which is a differentiable, increasing and concave utility function of user  $i$  obtained from receiving information from his news feed, where  $f_i^1(\mathbf{0}) = 0$  and  $q_{ji}$  represents follower  $i$ 's interest in user  $j$ 's information and  $L_i$  is a user-specific parameter. 3)  $f_i^2(x) := \sum_{l: i \in N_C^{\text{in}}(l)} L_l \left( \sqrt{\sum_{j \in N_C^{\text{in}}(l)} q_{jl} x_j} - \sqrt{\sum_{j \in N_C^{\text{in}}(l) \setminus \{i\}} q_{jl} x_j} \right)$  which is an incremental utility function that each user obtains from receiving attention in his network with  $f_i^2(x)|_{x_i=0} = 0$ . Specifically, this function targets the amount of attention that each follower pays to the information of other users in his news feed. The total cost function for user  $i$  is then  $J_i(x) = C_i(x_i) - f_i^1(x) - f_i^2(x)$ .

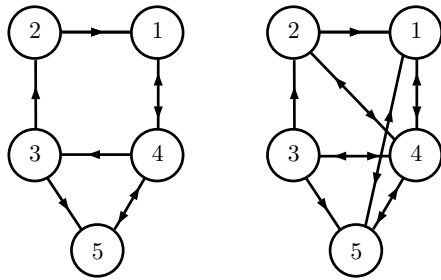


Figure 1. (a)  $G_C$  and (b)  $G_I$

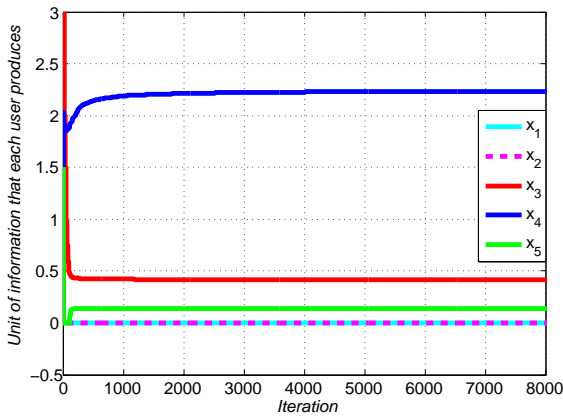


Figure 2. Convergence to a NE for the unit of information that each user produces over  $G_C$ .

For this example, we consider 5 users in the social media whose network of followers  $G_C$  is given in Fig. 1. (a). From  $G_C$  and taking  $J_i$  into account, one can construct  $G_I$  (Fig. 1. (b)) in a way that the interferences among users are specified.

Note that this is a reverse process of the one discussed in Section 5 because  $G_C$  is given as the network of followers and  $G_I$  is constructed from  $G_C$ . For the particular networks in Fig. 1. (a,b), Assumptions 5, 6 hold. We then employ the algorithm in Section 6 to find an NE of this game for  $h_i = 2$  and  $L_i = 1.5 \forall i \in V$ , and  $q_{41} = q_{45} = 1.75$ ,  $q_{32} = q_{43} = 2$  and the rest of  $q_{ij} = 1$ . The result is shown in Fig. 2. To analyze the NE  $x^* = [0, 0, 0.42, 2.24, 0.14]^T$ , note that one can realize from  $G_C$  that user 4 has 3 followers (users 1, 3 and 5), user 3 has 2 followers (users 2 and 5) while the rest have only 1 follower. Thus, it is straightforward to predict that users 4 and 3 could draw more attention and produce more information.

## 9. Conclusions

We proposed an asynchronous gossip-based algorithm to find an NE of a networked game with a complete interference digraph, over a partial, connected communication digraph. We extended our algorithm to the case of graphical games. We specified the locality of cost functions using a (partial) interference digraph and we showed almost-sure convergence to the NE of the game under an assumption on the communication digraph and diminishing step-sizes.

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