Generalized Hölder's inequality in Orlicz sequence spaces

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Abstract. Orlicz spaces is one of Banach space were firstly initiated by Birnbaum and W. Orlicz. There are two categories of Orlicz spaces, i.e. continuous Orlicz spaces and Orlicz sequence spaces. Many authors have discussed about continuous Orlicz spaces, especially Hölder's inequality in these spaces. In this paper, we present a sufficient condition for generalized Hölder's inequality in Orlicz sequence spaces. One of the keys to finding our results is to use the norm of the characteristic sequence of the balls in $\mathbb{Z}$.

Keywords: Hölder's Inequality, Sufficient Condition, Orlicz Sequence Spaces

1 Introduction

Orlicz spaces is one of Banach space were firstly initiated by Birnbaum and W. Orlicz in [1]. We can consider Orlicz spaces as a generalizations of Lebesgue spaces. There are two types of Orlicz spaces which are 'continuous' Orlicz spaces denoted by L_Φ and sequence Orlicz spaces denoted by ℓ_Φ.

We called a function $\Phi: [0,\infty) \to [0,\infty)$ as a Young function if Φ is a convex, left-continuous, $\lim_{t\to 0}\Phi(t) = 0 = \Phi(0)$, and $\lim_{t\to\infty}\Phi(t) = \infty$. Let Φ be a Young function, we define $\Phi^{-1}(r) := \inf\{s \geq 0: \Phi(s) > r\}$ for every $r \geq 0$.

Let us firstly recall the definition of Orlicz spaces and Orlicz sequence spaces. For Φ is a Young function, continuous Orlicz space $L_\Phi(\mathbb{R}^n)$ is a set of all measurable functions $f: \mathbb{R}^n \to \mathbb{R}$ such that

$$\|f\|_{L_\Phi(\mathbb{R}^n)} \coloneqq \inf\left\{b > 0: \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{b}\right) dx \leq 1\right\}$$

is finite.

Now, let Φ be a Young function. The Orlicz sequence spaces $\ell_\Phi(\mathbb{Z})$ is the set of all sequences $x \coloneqq (x_n)_{n\in\mathbb{Z}}$ such that

$$\|x\|_{\ell_\Phi(\mathbb{Z})} = \inf\left\{b > 0: \sum_{k=1}^{\infty} \Phi\left(\frac{|x_k|}{b}\right) \leq 1\right\} < \infty.$$

MSCEIS 2019, October 12, Bandung, Indonesia

DOI 10.4108/eai.12-10-2019.2296537

If we take $\Phi(t) = t^p$ for every $t \geq 0$ and $1 \leq p < \infty$ then we have $\ell_\Phi(\mathbb{Z}) = \ell_p(\mathbb{Z})$ is p-summable sequence spaces. Furthermore, $\ell_\Phi(\mathbb{Z})$ is Banach spaces.

Some expert have discussed about Orlicz spaces (see [2-7], etc.). They have revealed sufficient and essential condition. One of them, Ifronika *et al.* [8] propose to generalized Hölder's inequality in Orlicz spaces. In addition, Masta *et al.* suggest to generalized Hölder's in p-summable sequence spaces. There for, in this paper will be discussed about the generalized Hölder's inequality in sequence Orlicz spaces and in weak sequence Orlicz spaces.

In the process of finding result, we involve the Young function, Luxemburg norm in sequence Orlicz spaces, and some lemmas as follows.

Lemma 1. [9] *Let Φ be a Young function and $\Phi^{-1}(s) := inf\{r \geq 0 : \Phi(r) > s\}$, then we have*

1. $\Phi^{-1}(0) = 0$.
2. $\Phi^{-1}(s_1) \leq \Phi^{-1}(s_2)$ for $s_1 \leq s_2$.
3. $\Phi(\Phi^{-1}(s)) \leq s \leq \Phi^{-1}(\Phi(s))$ *for* $0 \leq s < \infty$.

Lemma 2. *If $x := (x_k) \in \ell_\Phi(\mathbb{Z})$ and $\|x\|_{\ell_\Phi(\mathbb{Z})} \neq 0$, then $\sum_{k=1}^{\infty} \Phi\left(\frac{|x_k|}{\|x\|_{\ell_\Phi(\mathbb{Z})}}\right) \leq 1$.*

2 Results and Discussions

Now, we have already known the properties of continuous Orlicz spaces. For getting the result, we will apply these properties to Orlicz sequence spaces. First, we present several lemmas in the following. We will also prove that the mapping on Orlicz sequence spaces define a norm on $\ell_\Phi(\mathbb{Z})$.

Theorem 3. Let Φ_1, Φ_2 and Φ_3 be Young functions and satisfy the condition $\Phi_1^{-1}(t)\Phi_2^{-1}(t) \leq \Phi_3^{-1}(t)$ for every $t \geq 0$. If $X := (x_n) \in \ell_{\Phi_1}(\mathbb{Z})$ and $Y := (y_n) \in \ell_{\Phi_2}(\mathbb{Z})$ then $XY \in \ell_{\Phi_3}(\mathbb{Z})$ with

$$\|XY\|_{\ell_{\Phi_3}(\mathbb{Z})} \leq 2\|X\|_{\ell_{\Phi_1}(\mathbb{Z})}\|Y\|_{\ell_{\Phi_2}(\mathbb{Z})}$$

.

Proof. Let $t_1, t_2 \geq 0$. Without loss of generality, suppose that $\Phi_1(t_1) \leq \Phi_2(t_2)$. By Lemma 1, we have

$$t_1 t_2 \leq \Phi_1^{-1}(\Phi_1(t_1))\Phi_2^{-1}(\Phi_2(t_2)) \leq \Phi_1^{-1}(\Phi_2(t_2))\Phi_2^{-1}(\Phi_2(t_2)) \leq \Phi_3^{-1}(\Phi_2(t_2))$$

Hence

$$\Phi_3(t_1 t_2) \leq \Phi_3\left(\Phi_3^{-1}(\Phi_2(t_2))\right) \leq \Phi_2(t_2) \leq \Phi_1(t_1) + \Phi_2(t_2)$$

Since, Φ is convex function, we have

$$\sum_{k=1}^{\infty} \Phi_3\left(\frac{|x_k y_k|}{2\|X\|_{\ell_{\Phi_1}(\mathbb{Z})}\|Y\|_{\ell_{\Phi_2}(\mathbb{Z})}}\right) \leq \frac{1}{2}\sum_{k=1}^{\infty} \Phi_3\left(\frac{|x_k||y_k|}{\|X\|_{\ell_{\Phi_1}(\mathbb{Z})}\|Y\|_{\ell_{\Phi_2}(\mathbb{Z})}}\right)$$

.

Furthermore, using Lemma 2 we get

$$\sum_{k=1}^{\infty} \Phi_3\left(\frac{|x_k||y_k|}{\|X\|_{\ell_{\Phi_1}(\mathbb{Z})}\|Y\|_{\ell_{\Phi_2}(\mathbb{Z})}}\right) \leq \sum_{k=1}^{\infty} \Phi_1\left(\frac{|x_k|}{\|X\|_{\ell_{\Phi_1}(\mathbb{Z})}}\right) + \sum_{k=1}^{\infty} \Phi_2\left(\frac{|y_k|}{\|Y\|_{\ell_{\Phi_1}(\mathbb{Z})}}\right) \leq 2.$$

Whenever $X := (x_n) \in \ell_{\Phi_1}(\mathbb{Z})$ and $Y := (y_n) \in \ell_{\Phi_2}(\mathbb{Z})$. Based on definition of $\|\cdot\|_{\ell_{\Phi_3}(\mathbb{Z})}$ we have $\|XY\|_{\ell_{\Phi_3}(\mathbb{Z})} \leq 2\|X\|_{\ell_{\Phi_1}(\mathbb{Z})}\|Y\|_{\ell_{\Phi_2}(\mathbb{Z})}$, as desired. ■

Now, we present the sufficient condition for generalized Hölder's inequality in sequence Orlicz spaces.

Theorem 4. For $m \geq 2$. Let Φ_i, Φ be Young functions for $i = 1,2,3,\ldots,m$. If

$$\prod_{i=1}^{m} \Phi_i^{-1}(\mathrm{t}) \leq \Phi^{-1}(\mathrm{t})$$

for $t > 0$, then for every $X_i \in \ell_{\Phi_i}(\mathbb{Z})$, $i = 1,2,3,\ldots,m-1$ we have $\prod_{i=1}^{m} X_i \in \ell_{\Phi}(\mathbb{Z})$ with

$$\left\|\prod_{i=1}^{m} X_i\right\|_{\ell_{\Phi}(\mathbb{Z})} \leq \mathrm{m}\prod_{i=1}^{m} \| X_i \|_{\ell_i(\mathbb{Z})}.$$

Proof. . Let $\prod_{i=1}^{m} \Phi_i^{-1}(\mathrm{t}) \leq \Phi^{-1}(\mathrm{t})$ holds for $t > 0$. By Lemma 1, we have

$$t_i \leq \Phi_i^{-1}(\Phi_i(t_i)) \leq \Phi_i^{-1}\left(\sum_{i=1}^{m} \Phi_{\mathrm{i}}(t_i)\right)$$

for $i = 1,2,3,\ldots,m$. Hence we have

$$\prod_{i=1}^{m} t_i \leq \prod_{i=1}^{m} \Phi_i^{-1}\left(\sum_{i=1}^{m} \Phi_{\mathrm{i}}(t_i)\right) \leq \Phi^{-1}\left(\sum_{i=1}^{m} \Phi_{\mathrm{i}}(t_i)\right)$$

Since, Φ_{m} is an increasing function, we have

$$\Phi\left(\prod_{i=1}^{m} t_i\right) \leq \Phi\left(\Phi_m^{-1}\left(\sum_{i=1}^{m} \Phi_{\mathrm{i}}(t_i)\right)\right) \leq \sum_{i=1}^{m} \Phi_{\mathrm{i}}(t_i).$$

Now, let $X_i \in \ell_{\Phi_i}(\mathbb{Z})$, $i = 1,2,3,\ldots,m$. Since Φ convex function and using Lemma 2, we have

$$\sum_{k=1}^{\infty} \Phi\left(\frac{1}{m}\prod_{i=1}^{m}\frac{|x_k^{(i)}|}{\|X\|_{\ell_{\Phi_i}(\mathbb{Z})}}\right) \leq \frac{1}{m}\sum_{k=1}^{\infty} \Phi\left(\prod_{i=1}^{m}\frac{|x_k^{(i)}|}{\|X\|_{\ell_{\Phi_i}(\mathbb{Z})}}\right) \leq \frac{1}{m}\sum_{i=1}^{m}\sum_{k=1}^{\infty} \Phi\left(\frac{|x_k^{(i)}|}{\|X\|_{\ell_{\Phi_i}(\mathbb{Z})}}\right) \leq 1.$$

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Whenever $X_i := \left(x_k^{(i)}\right) \in \ell_{\Phi_i}(\mathbb{Z})$. By using the definition of $\|\cdot\|_{\ell_{\Phi}(\mathbb{Z})}$ we have

$$\left\|\prod_{i=1}^{m} X_i\right\|_{\ell_{\Phi}(\mathbb{Z})} \leq \mathrm{m}\prod_{i=1}^{m} \| X_i \|_{\ell_i(\mathbb{Z})},$$

as desired. ■

3 Conclusion

In this short paper, We have shown the sufficient condition for generalized H$o$lder's inequality in of sequence Orlicz spaces. From Theorem 2.1 in [10], we can state that the condition $\prod_{i=1}^{m} \Phi_i^{-1}(\mathrm{t}) \leq \Phi^{-1}(\mathrm{t})$ for $t > 0$ is sufficient condition for generalized H$o$lder's inequality in continuous Orlicz spaces and Orlicz sequence spaces.

Acknowledgment. This research was made possible by the following grants supports: Hibah Penguatan Kompetensi UPI and Hibah Afirmasi dan Pembinaan Dosen UPI 2019.

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