A Note on Generalized Hölder's Inequality in P-summable Sequence Spaces

Al Azhary Masta[1], Siti Fatimah[2], Eka Rahma Kurniasi[3], Rajab Vebrian[4], Iis Juniati Lathifah[5]
{alazhari.masta@upi.edu[1], sitifatimah@upi.edu[2], eka.rachmakurniasi@stkipmbb.ac.id[3]}

Department of Mathematics Education, Universitas Pendidikan Indonesia, Jl. Dr. Setiabudi 229, Bandung 40154, INDONESIA[1, 2], Department of Mathematics Education, STKIP Muhammadiyah, Jl. KH. Ahmad Dahlan 51, Kabupaten Bangka Tengah, Bangka Belitung, 33684, INDONESIA[3]

Abstract. In this paper, we present the sufficient and necessary conditions for generalized Hölder's inequality in weak p-summable sequence spaces which complete the Masta, et al. results in 2018. One of the keys to prove our results is to use the norm of the characteristic sequence of the balls in $\mathbb{Z}$.

Keywords: Hölder's Inequality, Sufficient and Necessary Conditions, P-summable Sequence Spaces

1 Introduction

In mathematics, function spaces are one of the important topics, particularly in real and functional analysis. Lebesgue space is one of the spaces that are often studied in various fields such as statistics, applied mathematics and etc. There are two kinds of Lebesgue spaces which are 'continuous' Lebesgue spaces denoted by L_p and p-summable sequence spaces denoted by ℓ_p. Many researchers have studied Lebesgue spaces and its generalization over a few decades [1-12]. For example, in 2016, discussed generalized Hölder's inequality in 'continuous' Lebesgue spaces and in 'continuous' Orlicz – Morrey spaces [7]. In 2018, also presented sufficient and necessary conditions for generalized Hölder's inequality in Morrey spaces and in their weak type [13]. Recently, obtained the sufficient and necessary conditions for generalized Hölder's inequality in p-summable sequence spaces [14].

Motivated by these results, we are interested to obtain the sufficient condition for generalized Hölder's inequality in weak p-summable sequence spaces.

Let us recall definition of p-summable sequence spaces and weak p-summable sequence spaces. Let $1 \leq p < \infty$, the p-summable sequence space $\ell_p\,(\mathbb{Z})$ is the set of sequences $X := (x_n)_{n\in\mathbb{Z}}$ such that Equation (1).

$$\|X\|_{\ell_p(\mathbb{Z})} \coloneqq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < \infty. \quad (1)$$

Now, let $1 \leq p < \infty$, the weak p-summable sequence $w\ell_p(\mathbb{Z})$ spaces is the set of sequences $X := (x_n)_{n\in\mathbb{Z}}$ such that $\|X\|_{w\ell_p(\mathbb{Z})} \coloneqq \sup_{\gamma>0} \gamma |\{n \in \mathbb{N} : |x_n| > \gamma\}|^{\frac{1}{p}} < \infty$.

MSCEIS 2019, October 12, Bandung, Indonesia

DOI 10.4108/eai.12-10-2019.2296444

Note that, space $w\ell_p(\mathbb{Z})$ is quasi-Banach spaces equipped with quasi-norm $\|\cdot\|_{w\ell_p(\mathbb{Z})}$.

The rest of this paper is organized as follows. In Section 2, we presented some lemmas which useful to obtain our results. The main results are presented in Section 3. In Section 3, we recall the sufficient and necessary conditions for generalized Hölder's inequality in p-summable sequence spaces and proved the sufficient and necessary conditions for generalized Hölder's inequality in weak p-summable sequence spaces.

2 Method

To obtain the sufficient and necessary conditions for generalized Hölder's inequality in weak p-summable sequence spaces, we use similar method and some lemmas as in the following [14].

Lemma 1. *Let* $m \in \mathbb{Z}$*and* $N \in \{0,1,2,3,\dots\}$, *write* $S_{m,N} := \{m-N, \dots m, \dots, m+N\}$ [3]. *Let* Equation (2)

$$\xi_k^{m,N} := \begin{cases} 1, if\ k \in S_{m,N} \\ 0, otherwise \end{cases} \tag{2}$$

then there exists $C > 0$ *(independent of* m*and* N*) such that* like Equation (3)

$$(2N+1)^{1/p} \leq \left\|\xi_k^{m,N}\right\|_{\ell_p(\mathbb{Z})} \leq \left\|\xi_k^{m,N}\right\|_{w\ell_p(\mathbb{Z})} \leq C(2N+1)^{1/p} \tag{3}$$

for every $N \in \{0,1,2,3,\dots\}$.

Lemma 2. *Let* $x_i > 0$ *for* $i = 1,2,3,\dots,m$. *If* $1 \leq p_1, p_2, \dots, p_m, p < \infty$ *satisfy the condition* $\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{p}$, *then we have* Equation (4) [13]

$$\prod_{i=1}^{m} x_i \leq \sum_{i=1}^{m} \frac{p}{p_i} x_i^{\frac{p_i}{p}}. \tag{4}$$

Corollary 3. *Let* $1 \leq p_1, p_2, p < \infty$ *satisfy the condition* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $X = (x_n) \in \ell_{p_1}(\mathbb{Z})$ *and* $Y = (y_n) \in \ell_{p_2}(\mathbb{Z})$. *If* $\sum_{n=1}^{\infty} |x_n|^{p_1} = \sum_{n=1}^{\infty} |y_n|^{p_2} = 1$, *then* $\sum_{n=1}^{\infty} |x_n y_n|^p \leq 1$. [13].

Theorem 4. If $1 \leq p_1 \leq p_2 < \infty$, then we have $\|\cdot\|_{\ell_{p_2}(\mathbb{Z})} \leq \|\cdot\|_{\ell_{p_1}(\mathbb{Z})}$ [15].

3 Results and Discussions

First, we recall the sufficient and necessary conditions for Hölder's inequality in $\ell_p(\mathbb{Z})$ space in the following theorem.

Theorem 5. Let $1 \leq p_1, p_2, p_3, \dots, p_m, p < \infty$ [14].

(1) *If* $\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{p}$, *then* $\|\prod_{i=1}^{m} \boldsymbol{X_i}\|_{\boldsymbol{\ell_p}(\mathbb{Z})} \leq \prod_{i=1}^{m} \|\boldsymbol{X_i}\|_{\boldsymbol{\ell_{p_i}}(\mathbb{Z})}$, *for every* $\boldsymbol{X_i} \in \boldsymbol{\ell_{p_i}}(\mathbb{Z})$.

(2) *If* $\|\prod_{i=1}^{m} \boldsymbol{X_i}\|_{\boldsymbol{\ell_p}(\mathbb{Z})} \leq \prod_{i=1}^{m} \|\boldsymbol{X_i}\|_{\boldsymbol{\ell_{p_i}}(\mathbb{Z})}$, *for every* $\boldsymbol{X_i} \in \boldsymbol{\ell_{p_i}}(\mathbb{Z})$, then $\sum_{i=1}^{m} \frac{1}{p_i} \geq \frac{1}{p}$.

Proof

(1) For convenience, we recall the proof of Theorem 5. Let $\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{p}$, $X_i = (x_{n,i}) \in \ell_{p_i}(\mathbb{Z})$ for $i = 1,2,3,\dots,m$. First, suppose that $\sum_{n=1}^{\infty} |x_{n,i}|^{p_i} = A_i$ for every $i = 1,2,\dots,m$. By setting $x'_{n,i} = \frac{x_{n,i}}{A_i^{1/p_i}}$, we have $\sum_{n=1}^{\infty} |x'_{n,i}|^{p_i} = \frac{\sum_{n=1}^{\infty} |x_{n,i}|^{p_i}}{A_i} = 1$. Using Lemma 4, we obtain Equation (5).

$$\sum_{n=1}^{\infty} \left|\prod_{i=1}^{m} x_{n,i}\right|^{p} \leq \frac{p}{p_1} \sum_{n=1}^{\infty} |x_{n,1}|^{p_1} + \frac{p}{p_2} \sum_{n=1}^{\infty} |x_{n,2}|^{p_2} + \dots + \frac{p}{p_m} \sum_{n=1}^{\infty} |x_{n,m}|^{p_m} = \sum_{i=1}^{m} \frac{p}{p_i} = 1. \quad (5)$$

So we have Equation (6),

$$1 \geq \sum_{n=1}^{\infty} \left|\prod_{i=1}^{m} x'_{n,i}\right|^{p} = \sum_{n=1}^{\infty} \left|\prod_{i=1}^{m} \frac{x_{n,i}}{A_i^{\frac{1}{p_i}}}\right|^{p} = \frac{1}{\prod_{i=1}^{m} A_i^{p}} \sum_{n=1}^{\infty} \left|\prod_{i=1}^{m} x_{n,i}\right|^{p}. \quad (6)$$

Since $1 \geq \prod_{i=1}^{m} A_i^{-\frac{p}{p_i}} \left(\sum_{n=1}^{\infty} \left|\prod_{i=1}^{m} x_{n,i}\right|^{p}\right)$ is equivalent to $\sum_{n=1}^{\infty} \left|\prod_{i=1}^{m} x_{n,i}\right|^{p} \leq \prod_{i=1}^{m} A_i^{\frac{p}{p_i}}$, we obtain Equation (7)

$$\left(\sum_{n=1}^{\infty} \left|\prod_{i=1}^{m} x_{n,i}\right|^{p}\right)^{1/p} \leq \prod_{i=1}^{m} \left(\sum_{n=1}^{\infty} |x_{n,i}|^{p_i}\right)^{\frac{1}{p_i}}. \quad (7)$$

So, we have Equation (8)

$$\|\prod_{i=1}^{m} X_i\|_{\ell_p(\mathbb{Z})} \leq C \|\prod_{i=1}^{m} X_i\|_{\ell_p(\mathbb{Z})} \leq C \prod_{i=1}^{m} \|X_i\|_{\boldsymbol{\ell}_{p_i}(\mathbb{Z})}. \quad (8)$$

(2) Now, assume that $\|\prod_{i=1}^{m} X_i\|_{\ell_p(\mathbb{Z})} \leq \prod_{i=1}^{m} \|X_i\|_{\ell_{p_i}(\mathbb{Z})}$ holds *for every* $X_i \in \ell_{p_i}(\mathbb{Z})$. Take $X_i = \xi_k^{m,N}$ for every $i = 1,2,3,\dots,m$, by using Lemma 1, we have Equation (9)

$$(2N+1)^{1/p} \leq \left\|\xi_k^{m,N}\right\|_{\ell_p(\mathbb{Z})} \leq \prod_{i=1}^{m} \left\|\xi_k^{m,N}\right\|_{\ell_{p_i}(\mathbb{Z})} \leq C^m (2N+1)^{\sum_{i=1}^{m} \frac{1}{p_i}} \quad (9)$$

or $(2N+1)^{\frac{1}{p} - \left(\sum_{i=1}^{m} \frac{1}{p_i}\right)} \leq C^m$ for every $N \in \{0, 1, 2, 3, \dots\}$. Hence, we can conclude that $\frac{1}{p} \leq \sum_{i=1}^{m} \frac{1}{p_i}$.

Now, we come into sufficient and necessary conditions for generalized Hölder's inequality in weak p-summable spaces as presented in the following theorem.

Theorem 6. Let $1 \leq p_1, p_2, p_3, \dots, p_m, p < \infty$.

(1) *If* $\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{p}$, then $\|\prod_{i=1}^{m} \boldsymbol{X_i}\|_{w\ell_p(\mathbb{Z})} \leq \prod_{i=1}^{m} \|\boldsymbol{X_i}\|_{w\ell_{p_i}(\mathbb{Z})}$, *for every* $\boldsymbol{X_i} \in \boldsymbol{w\ell_{p_i}}(\mathbb{Z})$.

(2) *If* $\|\prod_{i=1}^{m} \boldsymbol{X_i}\|_{\boldsymbol{w\ell_p}(\mathbb{Z})} \leq \boldsymbol{m} \prod_{i=1}^{m} \|\boldsymbol{X_i}\|_{w\ell_{p_i}(\mathbb{Z})}$, *for every* $\boldsymbol{X_i} \in \boldsymbol{w\ell_{p_i}}(\mathbb{Z})$, then $\sum_{i=1}^{m} \frac{1}{p_i} \geq \frac{1}{p}$.

Proof.

(1) Suppose that $\sum_{i=1}^{m}\frac{1}{p_i}=\frac{1}{p}$ holds for every $1 \le p_1, p_2, p_3, \dots, p_m, p < \infty$. Let $X_i \in w\ell_{p_i}(\mathbb{Z})$, where $i = 1, \dots, m$. For an arbitrary $n \in \mathbb{N}$ and $\gamma > 0$, let Equation (10)

$$A(n,\gamma) := \left[\gamma^p \left|\left\{n \in \mathbb{N} : \prod_{i=1}^{m} \left|\frac{x_n^{(i)}}{\|X_i\|_{w\ell_{p_i}(\mathbb{Z})}}\right| > \gamma\right\}\right|\right]^{\frac{1}{p}}. \tag{10}$$

Observe that, by using Young's inequality for products (Lemma 2), we have Equation (11,12)

$$A(B,\gamma) \le \left[\gamma^p \left|\left\{n \in \mathbb{N} : \sum_{i=1}^{m} \frac{p}{p_i}\left|\frac{x_n^{(i)}}{\|X_i\|_{w\ell_{p_i}(\mathbb{Z})}}\right|^{\frac{p_i}{p}} > \gamma\right\}\right|\right]^{\frac{1}{p}} \tag{11}$$

$$\le \left[\sum_{i=1}^{m} \gamma^p \left|\left\{n \in \mathbb{N} : \frac{p}{p_i}\left|\frac{x_n^{(i)}}{\|X_i\|_{w\ell_{p_i}(\mathbb{Z})}}\right|^{\frac{p_i}{p}} > \frac{\gamma}{m}\right\}\right|\right]^{\frac{1}{p}}. \tag{12}$$

Note that $\frac{p}{p_i}\left|\frac{x_n^{(i)}}{\|X_i\|_{w\ell_{p_i}(\mathbb{Z})}}\right|^{\frac{p_i}{p}} > \frac{\gamma}{m}$ is equivalent to $|x_n^{(i)}| > \left(\frac{p_i\gamma}{mp}\right)^{\frac{p}{p_i}} \| X_i \|_{w\ell_{p_i}(\mathbb{Z})} =: \gamma_i$. Hence we obtain Equation (13-15)

$$A(B,\gamma) \le \left[\sum_{i=1}^{m} \left(\frac{mp.}{p_i}\right)^p \left(\frac{\gamma_i}{\|X_i\|_{w\ell_{p_i}(\mathbb{Z})}}\right)^{p_i} \left|\left\{n \in \mathbb{N} : |x_n^{(i)}| > \gamma_i\right\}\right|\right]^{\frac{1}{p}} \tag{13}$$

$$= m\left[\sum_{i=1}^{m} \left(\frac{p}{p_i}\right)^p \frac{{\gamma_i}^{p_i}|\{x\in B : |f_i(x)|>\gamma_i\}|}{\|X_i\|_{w\ell_{p_i}(\mathbb{Z})}{}^{p_i}}\right]^{\frac{1}{p}} \tag{14}$$

$$\le m\left[\sum_{i=1}^{m} \left(\frac{p}{p_i}\right)^p\right]^{\frac{1}{p}} = m. \tag{15}$$

We then take the supremum of $A(n,\gamma)$ over $n \in \mathbb{N}$ and $\gamma > 0$ to obtain Equation (16)

$$\|\prod_{i=1}^{m} X_i\|_{w\ell_p(\mathbb{Z})} \le m\prod_{i=1}^{m}\|X_i\|_{w\ell_{p_i}(\mathbb{Z})} \tag{16}$$

(2) Now, let $\|\prod_{i=1}^{m} X_i\|_{w\ell_p(\mathbb{Z})} \le m\prod_{i=1}^{m}\|X_i\|_{w\ell_{p_i}(\mathbb{Z})}$ holds for every $X_i \in w\ell_{p_i}(\mathbb{Z})$. Take $X_i = \xi_k^{m,N}$ for every $i = 1,2,3, \dots, m$, by using Lemma 1, we have Equation (17)

$$(2N+1)^{1/p} \le \prod_{i=1}^{m}\left\|\xi_k^{m,N}\right\|_{w\ell_{p_i}(\mathbb{Z})} \le C^m (2N+1)^{\sum_{i=1}^{m}\frac{1}{p_i}} \tag{17}$$

or $(2N+1)^{\frac{1}{p}-\left(\sum_{i=1}^{m}\frac{1}{p_i}\right)} \leq C^m$ for every $N \in \{0, 1, 2, 3, \ldots\}$. Hence, we can conclude that $\frac{1}{p} \leq \sum_{i=1}^{m}\frac{1}{p_i}$.

4 Conclusion

We have shown the sufficient and necessary conditions for generalized Hölder's inequality in $\mathrm{w}\ell_{\mathrm{p}}(\mathbb{Z})$ space, we can state that the condition $\frac{1}{p} \leq \sum_{i=1}^{m}\frac{1}{p_i}$ is a necessary condition for generalized Hölder's inequality in $\mathrm{w}\ell_{\mathrm{p}}(\mathbb{Z})$ space.

Acknowledgment. This research is supported by Hibah Afirmasi dan Pembinaan Dosen UPI 2019.

References

[1] Carothers, N. L.: A Short Course on Banach Space Theory, Department of Mathematics and Statistics, Bowling Green State University (2000)
[2] Castillo, R. E., Rafeiro, H.: An Introductory Course in Lebesgue Spaces, CMS books in Mathematics, Springer: Canadian Mathematical Society (2016)
[3] Gunawan, H., Kikianty, E., Schwanke, C.: Discrete Morrey spaces and their inclusion properties Math. Nachr. pp. 1 – 14 (2017)
[4] Li, Y., & Gu, X. M.: The weighted am-gm inequality is equivalent to the Hölder's inequality research report (2017)
[5] Masta, A. A., Gunawan, H., & Setya-Budhi, W.: An inclusion property of Orlicz spaces Proc.The 5th Annual Basic Science International Conference 2015. Vol. 5 ISSN: 2338-0128. pp. 274 – 276 (2015)
[6] Masta, Al. A., Gunawan, H., Setya-Budhi, W.: An inclusion property of Orlicz and weak Orlicz spaces J. Math. Fund. Sci. Vol 48-3. pp. 193-203 (2017)
[7] Masta, A. A., Gunawan, H., & Setya-Budhi W.: An inclusion property of Orlicz-Morrey spaces J. Phys.: Conf. Ser., Vol. 893. pp. 012015 (2017)
[8] Masta, A. A., Gunawan, H., & Setya-Budhi W.: Oninclusion properties of two versions of Orlicz-Morrey spaces Mediterr. J. Math., Vol. 14(6). pp. 228 – 239 (2017)
[9] Masta, A. A., Sumiaty, E., Taqiyuddin, M., & Pradipta, I.: The sufficient condition for inclusion properties of discrete weighted Lebesgue spaces, J. Phys.: Conf. Ser. Vol. 1013. pp. 012152 (2018)
[10] Nekvinda, A.: Embeddings between discrete weighted Lebesgue spaces with variable exponents Math. Ineq. Appl. Vol. 10-1. pp. 165 – 172 (2007)
[11] Osançliol, A.: Inclusion between weighted Orlicz spaces J. Inequal. Appl. pp. 390 – 397 (2014)
[12] Taqiyuddin, M., Masta, A. A.: Inclusion properties of Orlicz spaces and weak Orlicz spaces generated by concave function IOP Conf. Ser.: Mater. Sci. Eng., Vol. 288. pp. 012103 (2018)
[13] Ifronika., Idris, M., Masta, A. A., Gunawan, H.: Generalized Holder's Inequality on Morrey Spaces, to appear in Matematicki Vesnik (2018)
[14] Fatimah, S., Masta, A. A., Arsisari, A., Putra, Y. Y., & Apriani, F.: Sufficient and necessary conditions for generalized Hölder's inequality in p-summable sequence spaces, to appear in J. Phys.: Conf. Ser. Vol. 1280. pp. 022046 (2019)
[15] Gunawan, H., Hakim, D. I., & Idris, M.: A Note on Inclusions of Discrete Morrey Spaces research report. Vol. 1905. pp. 10021 (2019)