The metric dimensions of bridge graphs for some classes of graphs

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Abstract. There are many open problems in the metric dimension of a graph, espessially the bridge graph and the disconnected graph, that have not been resolved until now. This paper presents the metrics dimension of the bridge graph in several classes of graphs namely cycle, complete, and star graphs. We know that the metric dimensions of the complete, cycle or star graph have been obtained. The bridge graph $B\ (G_1, G_2, e)$ is a graph which is obtained from the operation of adding edge e to graphs G_1 and G_2. To obtain the metric dimension of the bridge graph from the graphs G_1 and G_2 on edge e, $pd(B\ (G_1, G_2, e))$, we used the graph structure of G_1, G_2 and the properties of the bridge graph based on the endpoint on the edge e. The results obtained the metric dimension of the bridge graph $B(G_1, G_2, e)$ for the cycle, complete, or star graphs.

Keywords: Metric Dimension, Bridge Graph, Cycle Graph, Complete Graph, Star Graph

1 Introduction

The metric dimension is a concept that uses a revolving set. The resolving set is a set of vertices as a marker component that distinguishes all vertices on the graph. The concepts have used in several fields namely chemical structures [1], robot navigation [2], and optimization of combinations [3]. However, the applications cannot be used optimally because all the metric dimensions of the graph cannot be obtained yet. Therefore, several researchers conducted a study of metric dimensions in various ways, including the use of graph operations. Some graph operations that have been published are sum-product, cross-product operations, and corona product. The metric dimension of the cross product of path order m with path order 2, $P_m \times P_2$ [1], the cartesian product [4], and the corona product [5]. In addition to the metric dimension, some researchers use another variant to solve the problem of the dimension of the graph, namely partition dimensions. The partition dimension appears in many published articles, namely the partition dimension of subdivision homogeneous caterpillar [6], homogeneous firecrackers [7], subdivision of a Complete graph [8] and a complete multipartite graph [9].
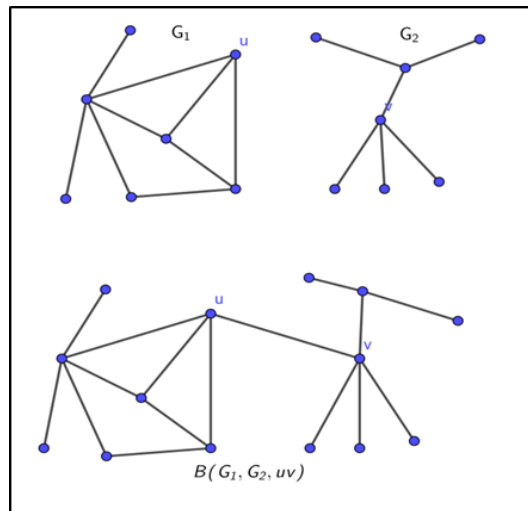
To find the metric dimension of the graphs, there is also another interesting operation of the graph to study in the metric dimension, namely the bridge of two graphs. Therefore this paper shows the metric dimensions on some bridge graphs. The bridge graph is the result of the operation of two graphs connected by two vertices of the two graphs in the form of edges. In this paper, it is devoted to discussing the metric dimensions of bridge graphs formed from cycle graphs C_n, star graphs $K_{1,n}$ and the complete graph K_n.

2 Preliminaries

Let $G = (V, E)$ be a connected graph.bridge graph and $v \in V(G)$. An ordered set of vertices $W = \{w_1, w_2, \cdots, w_k\} \subset V(G)$. The representation of v respect to W, $r(v|W)$, is a vector $(d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$. The set W is called the resolving set of G if any two distinct vertices x, y, then $r(W) \neq r(y|W)$. The representation $r(x|W)$ and $r(y|W)$ are called the distinct if there is a component of the vector which is distinct, in other words, there is a i-component which is distinct, $d(x, w_i) \neq d(y, w_i)$ for some $i \in \{1, 2, \cdots, k\}$. The basis of the bridge graph G is a resolving set with the minimum cardinality. The metric dimension G, $dim(G)$, is the cardinality of the basis of G. Let x, y be two distinct vertices of G. If $d(x, w_1) \neq d(y, w_1)$ then we say that the vertices x and y are distinguished by w_1.

2.1 The Bridge Graph

Let G_1 and G_2 be two connected graphs, $a \in V(G_1), b \in V(G_2)$. The bridge graph $B(G_1, G_2, ab)$ is a graph which is obtained from G_1 and G_2 with linking $a \in V(G_1)$ to $b \in V(G_2)$ [10] For example, look at Figure 1.

Fig. 1. The graphs G_1, G_2 and bridge graph $B(G_1, G_2, uv)$

2.2 Basic concepts

In section, we give some results of the metric dimension of the graph which is used in this research.

Theorem 1. [4]

Let G and P_n be a connected graph and a path graph of order n. The metric dimension $dim(G) = 1$ if and only if $G = P_n$.

Theorem 2. [4]

Let G and K_n be a connected graph and a complete graph of order n. The metric dimension $dim(G) = n - 1$ if and only if $G = K_n$.

Referring to the Theorems 1 and 2, it can be obtained that other graphs of order n only have the metric dimensions from 2 to $n - 2$.

Lemma 1.

If G is a connected graph, then $pd(G) \leq dim(G) + 1$.

Lemma 1 gives the properties interconnection between the metric dimension and the partition dimension of the graphs.

3 The Main Results

This section discusses the metric dimensions of bridge graphs from three class graphs namely a cycle graph, a star graph, and a complete graph. The first result, we show that the metric dimension of bridge graf from two-cycle graphs is two, as in the following Theorem 3. The cycle of order $n \geq 3$, C_n, is a regular graph of degree 2 and has n edges.

Theorem 3.

If C_n and C_m are two-cycle graphs with $n, m \geq 3$, $v_1 \in V(C_n); u_1 \in V(C_m)$, then $dim(B(C_n, C_m, v_1 u_1)) = 2$.

Proof

Let $G = B(C_n, C_m, v_1 u_1)$ and let $V(C_n) = \{v_1, v_2, v_3, \dots, v_n\}$ be a vertices set of C_n, and $V(C_m) = \{u_1, u_2, u_3, \dots, u_m\}$ be a vertices set of C_m. Since $B(C_n, C_m, v_1 u_1)$ is not a path, then by Teorema 1, we obtain

$$\dim(B(C_n, C_m, v_1 u_1)) \geq 2. \tag{1}$$

Suppose $W = \{v_n, u_m\} \subset V(G)$, We will prove that W is a basis of G.

Let x, y be two distinct vertices of $B(C_n, C_m, v_1 u_1)$. To prove that $dim(B(C_n, C_m, v_1 u_1)) \leq 2$, we consider the vertices x, y in two cases:

The first case. Suppose both $x, y \in C_n$.

Now we consider $d(x, v_1)$ and $d(y, v_1)$. If $d(x, v_1) = d(y, v_1)$, then clearly, $d(x, v_n) \neq d(y, v_n)$. So we obtain $r(x|W) \neq r(y|W)$. Next, If $d(x, v_1) \neq d(y, v_1)$, then we have $d(x, u_1) \neq d(y, u_1)$. As a consequence, we have $d(x, u_n) \neq d(y, u_n)$. So we obtain $r(x|W) \neq r(y|W)$. Similarly, apply the same argument for $x, y \in C_m$. We obtain $r(x|W) \neq r(y|W)$.

The second case. Suppose $x \in C_n$ and $y \in C_m$.

If $d(x, v_n) = d(y, v_n)$, then consider $d(x, u_m)$ and $d(y, u_m)$. Since $v_n, v_1, x \in V(C_n)$ and v_n is adjacent to v_1, then $d(x, u_m)$ is at least $d(x, v_n) + 1$. Whereas if $u_m, u_1, y \in V(C_n)$ and u_m is adjacent to u_1, then $d(y, u_m)$ is at most $d(x, v_n) - 1$. As a consequence, $d(y, u_m) < d(x, u_m)$. Therefore, we have $r(x|W) \neq r(y|W)$.

By these two cases, we have

$$dim(B(C_n, C_m, v_1u_1)) \leq 2. \tag{2}$$

As a consequence of these both two inequations (1) and (2), we obtain that the dimension of $B(C_n, C_m, v_1u_1)$ is 2.

If the bridge graph is obtained from the star graph and the cycle graph, then the metric dimension of the bridge graph depends on the bridge edge. Next, the metric dimension is discussed in the following Theorem 4. The star graph, $K_{1,n}$, is a tree with a vertex of degree n and n verices of degree 1. The vertex of degree 1 is called the *center* of $K_{1,n}$ and the others are the *leaves*.

Theorem 4.
Let $K_{1,n}$ be a star and C_m be a cycle graph.
If $a \in V(K_{1,n})$ and $b \in V(C_m)$, then
$dim\,dim\left(B\left(K_{1,n}, C_m, ab\right)\right) = \{n\ n-1\ if\ a\ is\ center\ of\ K_{\{1,n\}}\ others\ .$

Proof

Let $V\left(K_{1,n}\right) = \{u, u_1, u_2, u_3, \cdots, u_n\}$, $V(C_m) = \{v_1, v_2, v_3, \cdots, u_m\}$ and $G = B\left(K_{1,n}, C_m, ab\right)$
We consider the proof in two cases
Case 1. If $a = u$ and $b = v_1$, then we suppose $W = \{u_1, u_2, u_3, \cdots, u_{n-1}, v_n\}$.
We will show that W is the basis of G.
Let x, y be two distinct vertices of G. We consider two subcases
Subcase 1.1 the vertices $x, y \in V(C_m)$ or $x, y \in V(K_{1,n})$. If $x, y \in V(K_{1,n})$ and $x, y \notin W$ then $\{x, y\} = \{u, u_n\}$. So, the vertices x and y are distinguished by v_m. Next, we consider the vertices $x, y \in V(C_m)$. If $d(x, v_m) \neq d(y, v_m)$ then clearly $r(W) \neq r(y|W)$. Now, if If $d(x, v_m) = d(y, v_m)$ then we obtain $d(x, v_1) \neq d(y, v_1)$ because the vertex v_1 is adjacent to v_n in C_m. As a consequence, we have $d(x, u_1) \neq d(y, u_1)$. So, we can say that $r(W) \neq r(y|W)$.
Subcase 1.2 the vertex $x \in V(C_m)$ and $y \in V(K_{1,n})$. If $x, y \in V(K_{1,n})$ and $x, y \notin W$
If $d(x, v_m) \neq d(y, v_m)$, then the vertices x, y are distinguished by v_m. So, we have $r(x|W) \neq r(y|W)$. If $d(x, v_m) = d(y, v_m)$, then consider $y \in \{u, u_n\}$. If $y = u$ then $d(y, v\_m) = 2$, Therefore, since $d(y, v_m) = 2$ and $d(y, v_m) = d(x, v_m)$, then $x = v_2$ or $x = v_{m-2}$. Since $d(y, u_1) = 1$ and $d(x, u_1) \geq 3$, then we obtain $d(y, u_1) \neq d(x, u_1)$. If $y = u_n$ then $d(y, v\_m) = 2$, Therefore, $x = v_2$ or $x = v_{m-2}$. Since $d(y, u_1) = 1$ and $d(x, u_1) \geq 3$, then we obtain $d(y, u_1) \neq d(x, u_1)$. Therefore, we have $r(x|W) \neq r(y|W)$. As these consequences of two subcases, we obtain that W is a basis of G. So, we obtain
$dim(G) \leq n.$ ....(3)

Next, We show $dim(G) \geq n$. Suppose $dim(G) < n$.
Let W be a basis of G. Since $dim(G) < n$ then $|W| < n$. Let $W = \{a_1, a_2, \cdots, a_t\}$ with $t \leq n - 1$. If all vertices of W are in the vertices of $K_{1,n}$ subgraph of G then there are at least two vertices of C_m subgraph of G which have the same distance to all vertices of W namely v_2 and v_m. So we have $r(v_2|W) = r(v_m|W)$, a contradiction.

If there is at least one vertex of W in C_m subgraph of G. Suppose a_t in C_m the and $a_i = u_i$ for $i = 1,2,\cdots,t-1$. So, there are at least three vertices of $K_{1,n}$ subgraph of G which are not vertices of W. The three vertices of $K_{1,n}$ are u, u_n, and u_{n-1}. Consider the vertices u_1 and u_2, they have the same distance to all vertices of W. We have $r(v_2|W) = r(v_m|W)$, a contradiction. As the consequences, we obtain

$$dim(G) \geq n. \tag{4}$$

By inequations (3) and (4), we have $dim(G) = n$.

Case 2. If $a = u_n$ and $b = v_1$, then we suppose $W = \{u_1, u_2, u_3, \cdots, u_{n-2}, v_m\}$.
We will show that W is a basis of G
Let x, y be two distinct vertices of G. We consider in two subcases
Subcase 1.1 the vertices $x, y \in V(C_m)$ or $x, y \in V(K_{1,n})$. If $x, y \in V(K_{1,n})$ and $x, y \notin W$ then $\{x, y\} = \{u, u_{n-1}\}$. So, the vertices x and y are distinguished by v_m. Next, we consider the vertices $x, y \in V(C_m)$. If $d(x, v_m) \neq d(y, v_m)$ then clearly $r(W) \neq r(y|W)$. Now, if If $d(x, v_m) = d(y, v_m)$ then we obtain $d(x, v_1) \neq d(y, v_1)$ because the vertex v_1 is adjacent ot v_n in C_m. As a consequence, we have $d(x, u_1) \neq d(y, u_1)$. So, we can say that $r(W) \neq r(y|W)$.
Subcase 1.2 the vertex $x \in V(C_m)$ and $y \in V(K_{1,n})$. If $x, y \in V(K_{1,n})$ and $x, y \notin W$. If $d(x, v_m) \neq d(y, v_m)$, then the vertices x, y are distinguished by v_m. So, we have $r(x|W) \neq r(y|W)$. If $d(x, v_m) = d(y, v_m)$, then consider $y \in \{u, u_{n-1}\}$. If $y = u$ then $d(y, v_m) = 3$, Therefore, since $d(y, v_m) = 3$ and $d(y, v_m) = d(x, v_m)$, then $x = v_3$ or $x = v_{m-3}$. Since $d(y, u_1) = 1$ and $d(x, u_1) \geq 3$, then we obtain $d(y, u_1) \neq d(x, u_1)$, so, $r(W) \neq r(y|W)$. If $y = u_{n-1}$ then $d(y, v\_m) = 4$, Therefore, $x = v_4$ or $x = v_{m-4}$. Since $d(y, u_1) = 2$ and $d(x, u_1) \geq 4$, then we obtain $d(y, u_1) \neq d(x, u_1)$, so $r(W) \neq r(y|W)$.
Therefore, by two subcases, we have

$$dim(G) \leq n-1. \tag{5}$$

Next we will show $dim\ dim \geq n-1$.
Suppose $dim(G) \leq n-2$.
Let W be a basis of G. Since $dim(G) < n$ then $|W| < n$. Let $W = \{a_1, a_2, \cdots, a_t\}$ with $t \leq n-2$. If all vertices of W are in the vertices of $K_{1,n}$ subgraph of G then there are at least two vertices of C_m subgraph of G which have the same distance to all vertices of W namely v_2 and v_m. So we have $r(v_2|W) = r(v_m|W)$, a contradiction.
If there is at least one vertex of W in C_m subgraph of G. Suppose a_t in C_m the and $a_i = u_i$ for $i = 1,2,\cdots,t-1$. So, there are at least three vertices of $K_{1,n}$ subgraph of G which are not vertices of W. The three vertices of $K_{1,n}$ are u, u_n, and u_{n-1}. Consider the vertices u_1 and u_2, they have the same distance to all vertices of W. We have $r(v_2|W) = r(v_m|W)$, a contradiction. As the consequences, we obtain

$$dim(G) \geq n-1 \tag{6}$$

By inequations (5) and (6), we have $dim(G) = $ n-1.

In the next Theorem 5, we show the metric dimension of bridge graf from a complete graph and a cycle graph. The complete graph, K_n, is a graph formed by joining each pair of n vertices [10].

Theorem 5.
Let K_n be a complete graph and C_m be a cycle. If $a \in V(K_n)$ and $b \in V(C_m)$, then $dim\ dim\ \big(B(K_n, C_m, ab)\big) = n - 1$
Proof
Let $G = B(K_n, C_m, ab)$ and $V(K_n) = \{u_1, u_2, u_3, \cdots, u_n\}$, $V(C_m) = \{v_1, v_2, v_3, \cdots, u_m\}$
First, we will show $dim\ dim\ \big(B(K_n, C_m, ab)\big) \geq n - 1$.
Suppose $dim\ dim\ \big(B(K_n, C_m, ab)\big) \leq n - 2$. So there is a basis W of G with $|W|=n - 2$.
If all vertices of W are contained in $V(K_n)$, then there are two vertices of K_n which are not in W. We say that the vertices u_1 and u_2 are not in W. As a consequence, we have $d(u_1, z) = 1$ and $d(u_1, z) = 1$ for any $z \in W$. So we obtain $r(u_1|W) = r(u_2|W)$, a contradiction. If there is at least one vertex of W which is contained in $V(K_n)$, then there are at least three vertices of K_n which are not in W. So the vertices have the same representation, a contradiction. As a consequence we have

$$dim\ dim\ \big(B(K_n, C_m, ab)\big) \geq n - 1 \qquad (7)$$

Next, we will show that $dim\ dim\ \big(B(K_n, C_m, ab)\big) \leq n - 1$.
Let $V(K_n) = \{u_1, u_2, u_3, \cdots, u_n\}$, $V(C_m) = \{v_1, v_2, v_3, \cdots, u_m\}$, $a = v_1$ and $b = u_1$.
Suppose $W = \{v_m, u_3, u_4, \cdots, u_n\}$. We will show that W is the basis of G.
Let x, y be two distinct vertices of G which are not in W. If the vertices $x, y \in V(C_m)$ then we consider the distance between the vertices x, y to v_m. If $d(x, v_m) \neq d(y, v_m)$ then they are distinguished by v_m. If $d(x, v_m) = d(y, v_m)$ then $d(x, v_1) \neq d(y, v_1)$. Therefore, we obtain $d(x, v_3) \neq d(y, v_3)$. So the vertices x, y are distinguished by v_3. Then, we have $r(W) \neq r(y|W)$. If the vertices $x, y \in V(K_n)$ then $x = u_1$ and $y = u_2$. So, clearly, they are distinguished by v_m.
Next, we consider $x \in V(C_m)$ and $y \in V(K_n)$. We know $d(y, u_3) = 1$ and $d(x, u_3) \geq 2$. The consequence, the vertices x,y are distinguished by u_3. So we have $r(W) \neq r(y|W)$.
As the consequences, we obtain

$$dim(G) \leq n - 1. \qquad (8)$$

By inequations (7) and (8), we obtain $dim(G) = n - 1$

Theorem 6.
If $K_{1,n}$ is a star graph and K_m is a complete graph with $n, m \geq 3$, $v \in V(K_{1,n}); u \in V(K_m)$, then $dim(B(K_{1,n}, K_m, uv)) = \{(n + m) - 4\ if\ v\ is\ a\ leaf\ of\ K_{1,n}\ (n + m) - 3\ others$
Proof
Let $V(K_m) = \{u_1, u_2, \cdots, u_m\}$ and $V(K_{1,n}) = \{v_1, v_2, \cdots, v_n, c\}$ where c is a non-leaf of $K_{1,n}$
Let $G = B(K_{1,n}, K_m, uv)$. we consider the vertex u in two cases.
Case 1. The vertex v is a leaf of $K_{1,n}$
We will show that $dim(B(K_{1,n}, K_m, uv)) \leq (n + m) - 4$.

Since v is a leaf, so we suppose $u = u_1, v = v_1$ and $W = \{u_3, u_4, \cdots, u_m\} \cup \{v_3, v_4, \cdots, v_n\}$ where $|W| = n + m - 4$. We will show that W is the basis of G. Let x, y be two distinct vertices of G. Since there are only five vertices of G which are not in W namely c, v_1, v_2, u_1, u_2, then clearly, they have a different representation of W, $r(W) \neq r(y|W)$. As a consequence, we have

$$dim(B(K_{1,n}, K_m, uv)) \leq (n + m) - 4 \qquad (9)$$

Next, we will show that $dim(B(K_{1,n}, K_m, uv) \geq (n + m) - 4$. In another word, There is a $W \subset V(G)$ as a basis of G such that $|W| \geq m + n - 4$.

Suppose $dim(B(K_{1,n}, K_m, uv)) < (n + m) - 4$. So, we have a basis W such that $|W| = m + n - 5$. Since the order of G is $n + m + 1$, then there are at least six vertices of G that are not in W. we say the vertices $H = \{h_1, h_2, h_3, h_4, h_5, h_6\} \subset V(G)$ and every $h_i \notin W$ for $i \in \{1,2,\dots,6\}$.

We will show that there are at most two vertices $h_i \in V(K_n)$. Suppose there are three vertices $h_1, h_2, h_3 \in H$ and $h_i \in V(K_n)$. Let $h_1 = u$. So, we have $d(h_2, W) = d(h_3, W)$, a contradiction of W as a basis of G.

The second, Since there are at most two vertices $h_i \in V(K_n)$, we say h_1 and $h_2 \in V(K\_n)$, then there are four vertices h_3, h_4, h_5, h_6 of H which are in the $V(K_{1,n})$. Let $h_3 = c$ and $h_4 = v$. Now, we consider vertices h_5, h_6 which are a leaf of $K_{1,n}$. As a consequence, we have $d(h_5, W) = d(h_6, W)$, a contradiction of W as a basis of G.

These consequences, we obtain

$$dim(B(K_{1,n}, K_m, uv)) \geq (n + m) - 4. \qquad (10)$$

By inequations (9) and (10), we obtain $dim\ dim\left(B(K_{1,n}, K_m, uv)\right) = (n + m) - 4$.

Case 2. The vertex u is not a leaf of $K_{1,n}$. So we have $u = c$. We will show that $dim(B(K_{1,n}, K_m, uv)) \leq (n + m) - 3$. Since v is not a leaf, so we have $v = v_1$. Let $u = u_1$ and $W = \{u_3, u_4, \cdots, u_m\} \cup \{v_2, v_3, v_4, \cdots, v_n\}$ where $|W| = n + m - 3$. We will show that W is a basis of G. Let x, y be two distinct vertices of G. Since there are only four vertices of G which are not in W namely c, v_1, u_1, u_2, then clearly, they have a different representation of W, $r(W) \neq r(y|W)$.

As a consequence, we have

$$dim(B(K_{1,n}, K_m, uv)) \leq (n + m) - 3 \qquad (11)$$

Next, we will show that $dim(B(K_{1,n}, K_m, uv) \geq (n + m) - 3$. In other words, There is a $W \subset V(G)$ as a basis of G such that $|W| \geq m + n - 3$.

Suppose $dim(B(K_{1,n}, K_m, uv)) < (n + m) - 3$. So, we have a basis W such that $|W| = m + n - 4$. Since the order of G is $n + m + 1$, then there are at least five vertices of G that are not in W. we say the vertices $H = \{h_1, h_2, h_3, h_4, h_5\} \subset V(G)$ and every $h_i \notin W$ for $i \in \{1,2,\dots,5\}$.

We will show that there are at most two vertices $h_i \in V(K_n)$. Suppose there are three vertices $h_1, h_2, h_3 \in H$ and $h_i \in V(K_n)$. Let $h_1 = u$. So, we have $d(h_2, W) = d(h_3, W)$, a contradiction of W as a basis of G.

The second, Since there are at most two vertices $h_i \in V(K_n)$, we say h_1 and $h_2 \in V(K\_n)$, then there are four vertices h_3, h_4, h_5 of H which are in the $V(K_{1,n})$. Let $v = h_3 = c$. Now, we consider vertices h_4, h_5 which are a leaf of $K_{1,n}$. As a consequence, we have $d(h_4, W) = d(h_6, W)$, a contradiction of W as a basis of G.

These consequences, we obtain

$$dim(B(K_{1,n}, K_m, uv)) \geq (n+m) - 3. \tag{12}$$

By inequations (11) and (12), we obtain $dim\ dim\left(B(K_{1,n}, K_m, uv)\right) = (n+m) - 3$.

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