Hölder's inequality in Discrete Morrey spaces

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Abstract. Morrey spaces are a generalization of Lebesgue's spaces. There are two categories of Morrey spaces, i.e. continuous Morrey spaces and discrete Morrey spaces. Many authors have discussed about continuous Morrey spaces. In this paper, first we review definitions of these types, and then we present sufficient condition for Hölder's inequality in discrete Morrey spaces and in weak discrete Morrey spaces. One of the keys to prove our results is to use the Hölder's sequence of the balls in $\mathbb{Z}$.

Keywords: Hölder's Inequality, Sufficient Condition for Hölder's Inequality, Discrete Morrey Spaces, Weak Discrete Morrey Spaces

1 Introduction

Morrey spaces discussions and researches were firstly initiated by C. B. Morrey in [1]. We can consider Morrey spaces as Banach spaces and generalized version of Lebesgue spaces. We can distinguish Morrey spaces into two categories. The first type is 'continuous' Morrey spaces denoted by $\mathcal{M}_q^p$ and the second one is discrete Morrey spaces which can be denoted by ℓ_q^p for $1 \leq p \leq q < \infty$.

Let us firstly review the definition of 'continuous' Morrey spaces and discrete Morrey spaces. In this article, we define the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ as the set of all integrable functions f on $\mathbb{R}^n$ such that, for $1 \leq p \leq q < \infty$, we have the expression is finite

$$\| f \|_{\mathcal{M}_q^p} := \sup_{a \in \mathbb{R}^d, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

By $B(a,r)$, we mean an open ball centered at $a \in \mathbb{R}^n$ and equipped with radius $r > 0$. Meanwhile, $|B(a,r)|$ represents its Lebesgue measure. Next, observe that $\|\cdot\|_{\mathcal{M}_q^p}$ stands for a norm on $\mathcal{M}_q^p(\mathbb{R}^n)$ and $\mathcal{M}_q^p(\mathbb{R}^n)$ is Banach spaces equipped with the norm $\|\cdot\|_{\mathcal{M}_q^p}$. In addition, note that if $q = p$, then we have Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ becomes Lebesgue space $L^p(\mathbb{R}^n)$. In the nest paragraph, we will explicate the discrete version with similar fashion.

Now, we will define the discrete Morrey spaces. First, write $S_{m,N} := \{m - N, \dots m, \dots, m + N\}$ in which $m \in \mathbb{Z}$ and $\mathrm{N} \in \{0,1,2,3, \dots\}$. Then, we can define discrete

MSCEIS 2019, October 12, Bandung, Indonesia

DOI 10.4108/eai.12-10-2019.2296398

Morrey spaces $\ell_q^p(\mathbb{Z})$ as the set of all sequences $X := (x_n)_{n\in\mathbb{Z}}$ such that the following condition holds true

$$\|x\|_{\ell_q^p(\mathbb{Z})} = \sup_{m\in\mathbb{Z},N\in\mathbb{N}_0} \left| S_{m,N} \right|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{n\in S_{m,N}} |x_n|^p\right)^{\frac{1}{p}} = \sup_{m\in\mathbb{Z},N\in\mathbb{N}_0} \left| S_{m,N} \right|^{\frac{1}{q}-\frac{1}{p}} \|X\|_{\ell_p(S_{m,N})} < \infty.$$

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Analog with 'continuous' Morrey spaces, if we shift our standpoint by taking $q = p$, then we can consider a Morrey space $\ell_q^p(\mathbb{Z}) = \ell_p(\mathbb{Z})$ as a p-summable sequence space. This is indeed not a surprising fact, but we argue that it is worthy to mention here as we will use this important idea later in our main result.

Next, we will define both generalized Morrey spaces and generalized weak Morrey spaces that we will use for the purpose of this article. For $1 \le p \le q < \infty$, the weak Morrey space $w\mathcal{M}_q^p(\mathbb{R}^n)$ is constituted of all measurable functions f on $\mathbb{R}^n$ in which the inequality $\| f \|_{w\mathcal{M}_q^p} < \infty$ is true and its norm is defined by

$$\| f \|_{w\mathcal{M}_q^p} := \sup_{a\in\mathbb{R}^d, r,\gamma>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \gamma |\{x \in B(a,r) : |f(x)| > \gamma\}|^{\frac{1}{p}}.$$

Note that $\|\cdot\|_{w\mathcal{M}_q^p}$ defines a quasi-norm on $w\mathcal{M}_q^p(\mathbb{R}^n)$. If $q = p$, then $w\mathcal{M}_q^p(\mathbb{R}^n) = wL^p(\mathbb{R}^n)$. With that being said, we can perceive $w\mathcal{M}_q^p(\mathbb{R}^n)$ as a way of generalizing the weak Lebesgue space $wL^p(\mathbb{R}^n)$.

Again, write $S_{m,N} := \{m-N, \dots m, \dots, m+N\}$ for $m \in \mathbb{Z}$ and $N \in \{0,1,2,3,\dots\}$. The discrete weak Morrey spaces $w\ell_q^p(\mathbb{Z})$ is the set of all sequences $X := (x_n)_{n\in\mathbb{Z}}$ such that

$$\|x\|_{w\ell_q^p(\mathbb{Z})} = \sup_{m\in\mathbb{Z},N\in\mathbb{N}_{0,\gamma>0}} \gamma |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left|\{n \in S_{m,N} : |x_n| > \gamma\}\right|^{\frac{1}{p}} < \infty.$$

There are growing and intense discussions on Morrey spaces and its discrete version (see [2-7], etc.). For example, in 2015, Gunawan *et al.* [9] obtained sufficient and necessary conditions for the inclusion properties of Morrey spaces. Later, in 2019, Gunawan *et al.* also established the sufficient and necessary conditions for the inclusion properties of discrete Morrey spaces. On the other hand, the following two theorems tells us the sufficient and necessary conditions for generalized Hölder's inequality within 'continuous' Morrey spaces which Ifronika *et al.* have obtained in 2018.

Theorem 1. *Let $1 \le p \le q < \infty$, $1 \le p_1 \le q_1 < \infty$, and $1 \le p_2 \le q_2 < \infty$. Then, these two statements are equivalent:*

(1) $\frac{1}{p_1} + \frac{1}{p_2} \le \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$.

(2) $\| fg \|_{\mathcal{M}_q^p} \le \| f \|_{\mathcal{M}_{q_1}^{p_1}} \| g \|_{\mathcal{M}_{q_2}^{p_2}}$ for every $f \in \mathcal{M}_{q_1}^{p_1}(\mathbb{R}^n)$ and $g \in \mathcal{M}_{q_2}^{p_2}(\mathbb{R}^n)$.

Theorem 2. *Let $1 \le p \le q < \infty$, $1 \le p_1 \le q_1 < \infty$, and $1 \le p_2 \le q_2 < \infty$. Then, these two statements are equivalent:*

(1) $\frac{1}{p_1} + \frac{1}{p_2} \le \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$.

(2) $\| fg \|_{w\mathcal{M}_q^p} \le 2 \| f \|_{w\mathcal{M}_{q_1}^{p_1}} \| g \|_{w\mathcal{M}_{q_2}^{p_2}}$ for every $f \in w\mathcal{M}_{q_1}^{p_1}(\mathbb{R}^n)$ and $g \in w\mathcal{M}_{q_2}^{p_2}(\mathbb{R}^n)$.

In order to continue the discussion and fill in a research gap, we would like to carefully investigate the generalized Hölder's inequality in discrete Morrey spaces and in their weak types. We sought for ways to scrutiny the conditions in which the generalized Hölder's inequality holds true.

To obtain our result, we will use generalized Hö$l$der's inequality in p-summable sequence spaces. We contend that from our results, parameters play significant roles in the Hölder's inequality within the discrete Morrey spaces and their weak type.

2 H$\ddot{o}$lder's inequality in discrete Morrey spaces

Our main findings are summarized in the following theorems. The following theorem describes the sufficient and necessary condition for Hö$l$der's inequality in discrete Morrey spaces and we follow it up with its proof by leveraging the p-summable sequence spaces.

Theorem 3. *Let* $1 \le p \le q < \infty$, $1 \le p_1 \le q_1 < \infty$, *and* $1 \le p_2 \le q_2 < \infty$. *If* $\frac{1}{p_1} + \frac{1}{p_2} \le \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, then we have

$$\| XY \|_{\ell_q^p} \le \| X \|_{\ell_{q_1}^{p_1}} \| Y \|_{\ell_{q_2}^{p_2}}$$

for every $X := (x_n) \in \ell_{q_1}^{p_1}(\mathbb{Z})$ $and := (y_n) \in \ell_{q_2}^{p_2}(\mathbb{Z})$.

Proof.

Let $\sum_{i=1}^{m} \frac{1}{p_i} \le \frac{1}{p}$ and $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$ hold. Put $\frac{1}{p^*} := \sum_{i=1}^{m} \frac{1}{p_i}$. Clearly $p^* \ge p$. Now take an arbitrary $S_{m,N} \subseteq \mathbb{N}$, $X := (x_n) \in \ell_{q_1}^{p_1}(\mathbb{Z})$ *and* $Y := (y_n) \in \ell_{q_2}^{p_2}(\mathbb{Z})$. By the Hölder's inequality in p-summable sequence spaces [2], we have

$$\begin{aligned} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \|X\|_{\ell_p(S_{m,N})} &\le |S_{m,N}|^{\frac{1}{q}-\frac{1}{p^*}} \|X\|_{\ell_{p^*}(S_{m,N})} \\ &\le \left(|S_{m,N}|^{\frac{1}{q_1}-\frac{1}{p_1}} \|X\|_{\ell_{p_1}(S_{m,N})} \right) \left(|S_{m,N}|^{\frac{1}{q_1}-\frac{1}{p_1}} \|X\|_{\ell_{p_1}(S_{m,N})} \right). \end{aligned}$$

Taking the supremum over we obtain $m \in \mathbb{Z}$ and $N \in \mathbb{N}_0$ we have $\| XY \|_{\ell_q^p} \le \| X \|_{\ell_{q_1}^{p_1}} \| Y \|_{\ell_{q_2}^{p_2}}$ for every $X := (x_n) \in \ell_{q_1}^{p_1}(\mathbb{Z})$ $and := (y_n) \in \ell_{q_2}^{p_2}(\mathbb{Z})$.

3 H$\ddot{o}$lder's inequality in weak discrete Morrey spaces

Now, we come to suffcient conditions for H$\ddot{o}$lder's inequalty in Morrey spaces which is presented in the Theorem 4. In the following proof, we use a similar technique which we used in proving Theorem 3.

Theorem 4. *Let* $1 \le p \le q < \infty$, $1 \le p_1 \le q_1 < \infty$, *and* $1 \le p_2 \le q_2 < \infty$. *If* $\frac{1}{p_1} + \frac{1}{p_2} \le \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, then we have

$$\| XY \|_{w\ell_q^p} \le 2 \| X \|_{\mathrm{w}\ell_{q_1}^{p_1}} \| Y \|_{\mathrm{w}\ell_{q_2}^{p_2}}$$

for every $X := (x_n) \in \mathrm{w}\ell_{q_1}^{p_1}(\mathbb{Z})$ $and := (y_n) \in w\ell_{q_2}^{p_2}(\mathbb{Z})$.

Proof.

Let $\sum_{i=1}^{m}\frac{1}{p_i}\leq\frac{1}{p}$ and $\sum_{i=1}^{m}\frac{1}{q_i}=\frac{1}{q}$ hold. Put $\frac{1}{p^*}:=\sum_{i=1}^{m}\frac{1}{p_i}$. Clearly $p^*\geq p$. Now take an arbitrary $S_{m,N}\subseteq\mathbb{N}$, $X:=(x_n)\in \mathrm{w}\ell_{q_1}^{p_1}(\mathbb{Z})$ and $Y:=\left(y_n\right)\in \mathrm{w}\ell_{q_2}^{p_2}(\mathbb{Z})$. By the Hölder's inequality in weak p-summbale sequence spaces [2], we have

$$|S_{m,N}|^{\frac{1}{q}-\frac{1}{p}}\|X\|_{w\ell_p(S_{m,N})}\leq|S_{m,N}|^{\frac{1}{q}-\frac{1}{p*}}\|X\|_{w\ell_{p*}(S_{m,N})}$$
$$\leq 2\left(|S_{m,N}|^{\frac{1}{q_1}-\frac{1}{p_1}}\|X\|_{w\ell_{p_1}(S_{m,N})}\right)\left(|S_{m,N}|^{\frac{1}{q_1}-\frac{1}{p_1}}\|X\|_{w\ell_{p_1}(S_{m,N})}\right).$$

Taking the supremum over we obtain $m\in\mathbb{Z}$ and $N\in\mathbb{N}_0$ we have

$$\parallel XY\parallel_{\mathrm{w}\ell_q^p}\leq 2\parallel X\parallel_{w\ell_{q_1}^{p_1}}\parallel Y\parallel_{w\ell_{q_2}^{p_2}}$$

for every $X:=(x_n)\in w\ell_{q_1}^{p_1}(\mathbb{Z}^n)$ $and:=(y_n)\in \mathrm{w}\ell_{q_2}^{p_2}(\mathbb{Z}^n)$.■

4 Conclusion

In this short paper, we have shown the sufficient condition for H$o$lder's inequality in discrete Morrey spaces and in their weak type. More specifically, from Theorems 3 and 4, we can state that the condition $\frac{1}{p_1}+\frac{1}{p_2}\leq\frac{1}{p}$ and $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$ are sufficient conditions for H$o$lder's inequality in discrete Morrey spaces and in their weak type. We would like to call for more mathematical explorations in the related Holder's inequality and Morey spaces research in which we believe, there are still more results to be worthily expanded and deepened.

Acknowledgement. This research was made possible by the following grants supports: Hibah Penguatan Kompetensi UPI and Hibah Afirmasi dan Pembinaan Dosen UPI 2019.

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