On Cyclically Ordered Groups and Theirs Direct Product

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Abstract. For any groups A and B, we can construct the direct product $A \times B$ in a natural way. In many cases, the product $A \times B$ together with suitable maps are very useful as a tool to observe and identify many important properties related to both groups A and B. In this paper we review the definition, examples, and main properties of cyclically ordered groups. For two cyclically ordered groups A and B, we consider the product $A \times B$ whether could be viewed as a new cyclically ordered group. We give some conditions for the direct product $A \times B$ to be a cyclically ordered group.

Keywords: cyclic order, cyclically ordered group, direct product

1 Introduction

The concept of cyclically ordered group has been studied by many researchers. It has closed relation to the concept of (cyclically) ordered sets. For basic definitions and related results, one may refers to [1,2,3,4]. Especially, the paper due to Swierczkowski gives many basic results about cyclically ordered group. Meanwhile, Cernak [5] reviewed and presented the concept of completion of cyclically ordered groups.

For two groups A and B, we can produce the direct product $A \times B$ in a natural way. If $(a_1, b_1), (a_2, b_2)$ are elements in $A \times B$, we can define $(a_1, b_1)\,(a_2, b_2) \coloneqq (a_1 a_2, b_1 b_2)$. Through this operation, we view $A \times B$ as a new group obtained from the previous ones. Moreover, this direct product can be a tool to develop the related concept.

If both of A and B are cyclically ordered groups, we are interested whether the product $A \times B$ can be viewed as a cyclically ordered group. Precisely, what are the necessary and sufficient conditions to make $A \times B$ as cyclically ordered group. In the case that both of A and B are linearly ordered groups, the researchers have shown that the direct product $A \times B$ can be viewed as a cyclically order group. Actually, this result is one implication of the fact that a linear order induces a cyclic order.

Now, we present basic definitions and some important facts related to cyclically ordered groups.

Definition 1. Let $\boldsymbol{G}$ be a group (not necessarily abelian) with operation +. A ternary relation $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ on $\boldsymbol{G}$ is called a cyclic order if for each $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{G}$, all of the following conditions are satisfied:

i. $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}] \Rightarrow \boldsymbol{x} \neq \boldsymbol{y} \neq \boldsymbol{z} \neq \boldsymbol{x}$ and if $\boldsymbol{x} \neq \boldsymbol{y} \neq \boldsymbol{z} \neq \boldsymbol{x}$ it must be $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ or $[\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x}]$.

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ii. $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}] \Rightarrow [\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{x}]$.
iii. $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ and $[\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{z}]$ implies $[\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{z}]$.
iv. $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ implies $[\boldsymbol{a}+\boldsymbol{x}+\boldsymbol{b}, \boldsymbol{a}+\boldsymbol{y}+\boldsymbol{b}, \boldsymbol{a}+\boldsymbol{z}+\boldsymbol{b}]$.

The group G equipped with this relation is called a cyclically ordered group. Note that for more general expression of property iv, we can write $[x, y, z] \Rightarrow [axb, ayb, azb]$.

One may realize that a linear order has a closed relation to a cyclic order. The following observation confirms the conjecture.

Example 1. (Example 2.1 [2]). Let $\boldsymbol{G}$ be a linearly ordered group. For elements $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{G}$ which are distinct, we set $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ if one of these conditions holds:

$$x < y < z, \text{ or } y < z < x, \text{ or } z < x < y. \tag{1}$$

Then we have fact that $[x, y, z]$ is a cyclic order in G. Therefore, we conclude that every linear order induces a cyclic order, and we can view a linearly ordered group as a cyclically ordered group.

Example 2. (Example 2.2 [2]). If $\boldsymbol{L} = [\boldsymbol{0}, \boldsymbol{1})$ and + is defined on $\boldsymbol{L}$ as addition mod 1, then $\boldsymbol{L}$ is a cyclically orderd group.

Note that the torus $T = \{x \in C : |x| = 1\} = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ can be equipped with a cyclic order, as well as the group Z_n. The reader may refer [6,7] for detail explanation about the construction of the order.

The following Lemma due to Jakobik gives us more information about a linear order and its induced cyclic order.

Lemma 3. (Lemma 3.1 [2]). Let $(\mathbf{G}, \leq)$ be a linearly ordered group and let $[.,.,.]$ be the corresponding induced cyclic order on $\mathbf{G}$. Then the linear order on $\mathbf{G}$ can be uniquely reconstructed from the cyclically ordered group.

Proof. Use the fact: $x > 0 \Leftrightarrow [-x, 0, x]$.

One most important fact in the proof of Lemma 3, that we can use the positive cone $\{x : x > 0\}$ to define the corresponding cyclic order in the set. Roughly speaking, if we can define a positive cone in a set, with certain conditions, we can construct a cyclic order in that set. So, in many cases the identification of the existence of positive cone is an important step to obtaine a cyclic order.

If we have a cyclically odered group G, what is a sufficient condition for G to be a linearly ordered group? The following Lemma due to Jakubik [2] gives the such condition.

Lemma 4. (Lemma 3.3 [2]). Let G be a cyclically ordered group. Then the following conditions are equivalent:

i. G is linearly ordered
ii. Each nonzero subgroup of G is infinite and for each $\boldsymbol{g} \in \boldsymbol{G}$ and each positive integer **n** the relation $[-\boldsymbol{g}, \boldsymbol{0}, \boldsymbol{g}] \Rightarrow [-\boldsymbol{g}, \boldsymbol{0}, \boldsymbol{ng}]$ is valid.

Related to Lemma 4, one may consider the existence of a linear ordered subgroup in a cyclically ordered group. For the observation and discussion, the reader may consult [7]. Some examples are presented and are analyzed there.

One important question bears in mind, if we have two cyclically ordered groups G and H, how we can produce an induced cyclic order in their direct product $G \times H$. The related problem to solved is what is the necessary condition for $G \times H$ to be a cyclically ordered group.

The following example due to Swierczkowski [1] gives us a basic fact about cyclic order on a direct product of groups.

Example 5. (Example 2 [2]). Let $\boldsymbol{L}$ be a linearly ordered group and $\boldsymbol{K} = [\mathbf{0}, \mathbf{1})$, and consider the direct product $\boldsymbol{K} \times \boldsymbol{L}$. For distinct elements $\boldsymbol{u} = (\boldsymbol{a}, \boldsymbol{x}), \boldsymbol{v} = (\boldsymbol{b}, \boldsymbol{y}), \boldsymbol{w} = (\boldsymbol{c}, \boldsymbol{z})$, we set $[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$ if some of the following conditions is satisfied:

i. $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]$;
ii. $\boldsymbol{a} = \boldsymbol{b} \neq \boldsymbol{c}$ and $\boldsymbol{x} < \boldsymbol{y}$;
iii. $\boldsymbol{b} = \boldsymbol{c} \neq \boldsymbol{a}$ and $\boldsymbol{y} < \boldsymbol{z}$;
iv. $\boldsymbol{c} = \boldsymbol{a} \neq \boldsymbol{b}$ and $\boldsymbol{z} < \boldsymbol{x}$;
v. $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{c}$ and $[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$.

It can be verified that $K \times L$ is a cyclically ordered group.

We see that the linear order in L is used in a natural way to obtain the cyclic order. The similar argument can be used in other cases.

2 Method

We use and analyze the previous results to obtain some information about a cyclic order in a direct product of groups. For example, the fact $x > 0 \iff [-x, 0, x]$ is essential in the proof of Lemma 4. As long as the condition $[-x, 0, x]$ holds, we can define the positive cone $\{x : x > 0\}$. Moreover, this later condition is used in example 2.

3 Result

It has been stated above that a linear order can induce a cyclic order. Moreover, we also can use this fact to induce a cyclic order in a direct product as shown by example above. By virtue of this observation, we have the following.

Proposition 3. Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be two cyclically ordered groups with their cyclic order $[.,.,.]_{\boldsymbol{G}}, [.,.,.]_{\boldsymbol{H}}$, respectively. If $\boldsymbol{H}$ has a positive cone $\boldsymbol{H}^{+} = \{\boldsymbol{x} \in \boldsymbol{H} | \ \boldsymbol{x} > \mathbf{0}\}$ and for any distinct elements $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{H}$, the conditions

$$[\boldsymbol{x}\boldsymbol{y}^{-1}, \mathbf{0}, \boldsymbol{y}\boldsymbol{x}^{-1}]_{\boldsymbol{H}}, [\boldsymbol{y}\boldsymbol{z}^{-1}, \mathbf{0}, \boldsymbol{z}\boldsymbol{y}^{-1}]_{\boldsymbol{H}}, [\boldsymbol{z}\boldsymbol{x}^{-1}, \mathbf{0}, \boldsymbol{x}\boldsymbol{z}^{-1}]_{\boldsymbol{H}} \quad (1)$$

hold, then the direct product $\boldsymbol{G} \times \boldsymbol{H}$ is a cyclically ordered group.

Proof. Use the fact that $[yx^{-1}, 0, xy^{-1}]_H$ is equivalent to $xy^{-1} > 0 \Leftrightarrow x > y$, and then we use conditions as shown in the example 5.

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