Comparison Between Convex and Quasiconvex Functions on ℝ With Development

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Abstract. In this paper, we will introduce some basic properties about convex function and quasiconvex fucnction on ℝ. Some comparison between properties of convex function and quasiconvex function will be presented also. Our results are quasiconvex function can be represented as monotone function either or at most as two combination of monotone functions, as consequences it is discontinuous mostly at countable points, and it is not differentiable at zero measure set only. However the results are still fundamental, but we will learn as clear as we can.

Keywords: Convex Function, Quasiconvex Function, Convex Function Space, Quasiconvex, Function Space

1 Introduction

Functional analysis is a branch of analysis in which among others are discussed about functions, function spaces, sequences, sequence spaces, series, convergence, metrics spaces, normed spaces, Hilbert spaces, operators, and general topology.

In relation to the function space, researchers have drawn a lot of attention to study further, including the continuous function space $C[a,b]$ and integrable function space $L_p[a,b]$ are complete classical spaces, discussed by [1-2]. Other findings about the row space and function space put forward [3-7].

Specifically [4],[5] conducted a study related to the convex function in the Euclid space ℝ whose formation was motivated by a domain in the form of intervals at ℝ**,** then extended in sub-sets in convex and linear convex sub norms in ℝ. Furthermore, it will be examined more deeply about the properties that apply to the convex function space and quasi-convex space.

1.1 Research Methodology

In study of convex and quasiconvex functions, we want to see the shape of convex and quasiconvex function as clear as we want. By surveying at locally points such as extreme points, we will get some unique properties. Some supporting theories related to locally points is presented below.

Definition 1. *Let function f on* ℝ , $x, y \in \mathbb{R}$ *and* $x < y$. *We say that f is increasing if it satisfies the*

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inequalities $f(x) \leq f(y)$ *and f is descreasing if it satisfies the inequalities* $f(x) \geq f(y)$.

Definition 2. *Let* $A \subseteq \mathbb{R}$ *and function* $f: A \to \mathbb{R}$. *Point* $x \in A$ *is said to be a local minimum point if there is a Neighborhood* $N_\epsilon(x) = (x - \epsilon, x + \epsilon)$ *such that*

$$f(x) \leq f(y)$$

for every $y \in N_\epsilon(x) \cap A$.

Theorem 3 (*Nested Interval*s). *If* $I_n = [a_n, b_n], n \in \mathbb{N}$ *is a nested sequence of closed bounded intervals, then there exist a number* $s \in \mathbb{R}$ *such that* $s \in I_n$ *for all* $n \in \mathbb{N}$.

Theorem 4*(Mean Value Theorem)***.** *If function f continuouss on* $[a, b]$ *and differentiable on* (a, b), *than exis c in* (a, b) *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 5 *(Taylor). Let* $n \in \mathbb{N}, I = [a, b], and\ f: I \to \mathbb{R}$ *such that* $f, f', f'', \dots, f^n$ *continuous on I and* f^{n+1} *exist on* (a, b). *If* $x_0 \in I$ *than any* $x \in I$ *there is c between x and* x_0 *such that*

$$f(x) = f(x_0) + f^{;}(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x - x_0)^n + \frac{f^{n+1}(x_0)}{(n+1)!}(x - x_0)^{n+1}$$

2 Result and Discussion

2.1 Convex Function

Definition 1. (Convex function [1]). Function $f : I \to \mathbb{R}$ is convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

For any $x, y \in I$, $\boldsymbol{\lambda} \in (0,1)$.

Examples :

1. $f(x) = |x|$ on $(-\infty, \infty)$ is convex.
2. $g(x) = \sin x$ on $[-\pi, 0]$, is strictly convex.

Definition 2. Let $f : I \to \mathbb{R}$. Epigraf of function f is defined as

$$epi(f) = \{(x, y): x \epsilon I, y \geq f(x)\}$$

Theorem 3. Function f is convex if only if $epi(f)$ is convex set

Proof ($\Leftarrow$) Let $(x_1, y_1), (x_2, y_2) \in epi(f)$. Since $epi(f)$ is convex then $\alpha(x_1, y_1) + (1 - \alpha)(x_2, y_2) = (\alpha x_1 + (1 - \alpha)x_2, \alpha y_1 + (1 - \alpha)y_2) \in epi(f)$ for every $\alpha \in [0,1]$.

Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. The choice of y_1 and y_2 is valid since $(x_1, f(x_1)), (x_2, f(x_2)) \in epi(f)$. Then, $\alpha f(x_1) + (1-\alpha)f(x_2) \geq f(\alpha(x_1) + (1-\alpha)(x_2))$.
($\Rightarrow$) Let f is a convex function, then $f(\alpha x + (1-\alpha)y \leq \alpha f(x)(1-\alpha)f(y)$ for each $0 < \alpha < 1$.
Let $(x_1, y_1), (x_2, y_2) \in epi(f)$, then $f(x_1) \leq y_1$ and $f(x_2) \leq y_2$, therefore for every $\alpha \in [0,1]$ we have

$$\alpha f(x_1) \leq \alpha y_1 \text{ and } (1-\alpha)f(x_2) \leq (1-\alpha)y_2$$

According to f convergence, then $f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha y_1 + (1-\alpha)y_2$. Therefore, $(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) \in epi(f)$.

Theorem 4. If function f is convex on open interval I then f is continuous on I.
Proof. Let $x \in I$. We can choose $N_\epsilon(x) = (x-\epsilon, x+\epsilon)$ such that $N_\epsilon(x)$ is proper subset of I. Since f is convex then

$$g_x(y) = \frac{|f(y) - f(x)|}{|y - x|}$$

is bounded on $N_{\frac{\epsilon}{2}}(x)$. Let M_ϵ be the bound. So that $|f(y) - f(x)| \leq M_\epsilon |y - x|$ for every $y \in N_{\frac{\epsilon}{2}}(x)$. By taking limit $y \to x$, we have $\lim_{y \to x} |f(y) - f(x)| = 0$.

Theorem 5. Every local minimum of convex function is a global minimum.
Proof. Suppose that interval I is the domain of convex function f. Let x be local minimum. Then, there exist a neighborhood $N_\epsilon(x) = (x-\epsilon, x+\epsilon)$ such that

$$f(x) \leq f(y)$$

for every $y \in N_\epsilon(x) \cap I$. Let $z \in I$ be any points outside of $N_\epsilon(x)$. Then there exist $w \in N_\epsilon(x) \cap I$ such that w is between z and x, thus $w = \alpha x + (1-\alpha)z$ for some $\alpha \in (0,1)$. Since f is convex, then

$$f(x) \leq f(w) \leq \alpha f(x) + (1-\alpha)f(z).$$

Thus we have $f(x) \leq f(z)$.

Theorem 6. Convex function $f : I \to \mathbb{R}$ has one-side partial derivative on $int(I)$.
Proof. Since function f is convex, then for each $x \in int(I)$ with $y < z$ we have

$$\frac{f(z) - f(x)}{z - x} \geq \frac{f(y) - f(x)}{y - x}$$

This shows that the function $g_x(w)$ defined as

$$g_x(w) = \frac{f(w) - f(x)}{w - x}$$

is monotone on $int(I)$. Since monotone functions always have one side limit, then we have $\lim_{w \to x^-} g_x(w)$ and $\lim_{w \to x^+} g_x(w)$ exists.

Theorem 7. Let $f \in C^2$, then f convex on $\mathbb{R}$ if only if $f'' \geq 0$.
Proof. ($\Rightarrow$) Since f is convex and $f \in C^2$ then f' is increasing, therefore f'' is non-negative.
($\Leftarrow$) Let x_1, x_2 be interior points of the domain of f, $x_1 \neq x_2$. Suppose that $t \in (0,1)$ and $x_0 = (1-t)x_1 + tx_2$. By Taylor Theorem, there exists c_1 between x_0 and x_1 such that

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \frac{f''(c_1)}{2!}(x_1 - x_0)^2,$$

and there exists c_2 between x_0 and x_2 such that
$$f(x_2) = f(x_0) + f'(x_0)(x_2 - x_0) + \frac{f''(c_2)}{2!}(x_2 - x_0)^2$$
Since f'' is non-negative, then
$$R = \frac{1}{2}(1-t)f''(c_1)(x_1 - x_0)^2 + \frac{1}{2}tf''(c_2)\,(x_2 - x_0)^2$$
is non-negative. Therefore,
$$\begin{aligned}(1-t)f(x_1) + tf(x_2) &\geq f(x_0) + f'(x_0)\big((1-t)(x_1 - x_0) + t(x_2 - x_0)\big)\\ &\geq f(x_0) + f'(x_0)\big((1-t)x_1 + tx_2 - x_0\big)\\ &= f(x_0)\\ &= f\big((1-t)x_1 + tx_2\big)\end{aligned}$$

Theorem 8. Suppose that function f is continuous and differntiable on $\mathbb{R}$, then f is convex on $\mathbb{R}$ if only if
$$f(y) - f(x) \geq f'(x)(y - x)$$
For every $x, y \in \mathbb{R}$.
Proof. ($\Rightarrow$) Since f is convex, we have $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$. By mean value theorem, there exists c between x and y such that
$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$
Without loss of generality, let $x < c < y$. Since f is convex, we have $f'(c) \geq f'(x)$, therefore

$$\frac{f(y) - f(x)}{y - x} \geq f'(x) \text{ or } f(y) - f(x) \geq f'(x)(y - x).$$
($\Leftarrow$) Suppose that $f(y) - f(x) \geq f'(x)(y - x)$. For every z between x and y, we have
$$f(x) - f(z) \geq f'(z)(x - z)$$
and
$$f(y) - f(z) \geq f'(z)(y - z).$$
Therefore,
$$\alpha f(x) - \alpha f(z) \geq \alpha f'(z)(x - z)$$
and
$$(1-\alpha)f(y) - (1-\alpha)f(z) \geq (1-\alpha)f'(z)(y - z).$$
As consequences, we have
$$\alpha f(x) + (1-\alpha)f(y) - f(z) \geq f'(z)(\alpha(x - z) + (1-\alpha)(y - z))$$
or
$$\alpha f(x) + (1-\alpha)f(y) \geq f(z) + f'(z)(\alpha x + (1-\alpha)y - z)$$

Choose $z = \alpha x + (1-\alpha)y$ with $0 < \alpha < 1$, then we have

$$\alpha f(x) + (1-\alpha)f(y) \geq f(z)$$
$$\alpha f(x) + (1-\alpha)f(y) \geq f(\alpha x + (1-\alpha)y).$$
Therefore, f is a convex function

2.2 Quasi-convex function

Definition 9. Real-valued function f on interval I is said to be quasi-convex if
$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}$$

For every $x, y \in I$ and $\alpha \in [0,1]$.

Theorem 10. Every local minimum of quasiconvex function f is a global minimum or f is constant on a neighborhood

Proof: Assume that x be a local minimum but it is not constant on a neighborhood. Then there exists neighborhood $N_\epsilon(x) = (x - \epsilon, x + \epsilon)$ such that
$$f(x) < f(y)$$
for every $y \in N_\epsilon(x)$. Otherwise, if there exists $y \neq x$ such that $f(x) = f(y)$, then by quasiconvexity of f we have $f(z) = f(x)$ for every z between x and y, contradiction with assumption.
Let w is any point outside of $N_\epsilon(x)$, then we should have
$$f(x) < f(w).$$
Otherwise, we have $y \in N_\epsilon(x)$ such that $f(x) < f(y)$ and $f(y) > f(w)$, contradiction with f is quasiconvex.

Theorem 11. Let f quasi-convex on I, then $x \in I^0$ is a global minimum point if only if f is decreasing on $(-\infty, x] \cap I$ and increasing on $I \cap [x, \infty)$.

Proof: ($\Longrightarrow$) Let $x_1, x_2 \in (-\infty, x] \cap I$ with $x_1 < x_2$. Then, $x_2 = \alpha x_1 + (1 - \alpha)x$ which $\alpha = \frac{x - x_2}{x - x_1}$.

Therefore,
$$f(x_2) \leq \max\{f(x_1), f(x)\} = f(x_1).$$
Similarly for $x_1, x_2 \in I \cap [x, \infty)$.

($\Longleftarrow$)Since $f(y) \geq f(x)$ for $y \leq x$ and $y \geq x$, then x is global minimum.

Corollary 12. Let f quasi-convex on I then $x \in I^0$ is local minimum if only if there is a neighborhood $V(x)$ such that f is decreasing on $(-\infty, x] \cap V(x)$ and increasing on $V(x) \cap [x, \infty)$.

Theorem 13. If f quasi-convex on interval I, then f is monotone either or there is $t \in int(I)$ such that f is decreasing on $(-\infty, t] \cap I$ and increasing on $I \cap [t, \infty)$.

Proof. Assume that function f is not monotone. Define

$$m = inf\{\alpha: |L_\alpha| > 0\}$$
with $|L_\alpha|$ is lebesgue measure of L_α. First, in case of f is bounded below, then there is M such that $f(x) > M$ for every $x \in I$. Since $L_M = \emptyset$, thus $|m| < \infty$. According to infimum properties, there is a sequence (a_n) such that $a_n \to m$. By monotonic properties of L_α, we have
$$H = \bigcap_{\alpha > m} \overline{L_\alpha} = \bigcap_{n=1}^{\infty} \overline{L_{a_n}}.$$
According to Nested Interval Theorem, we have $H \neq \emptyset$.
For cases of f is not bounded below, then $L_\alpha \neq \emptyset$ for each α. Then

$$H = \bigcap_{\alpha \in \mathbb{R}} \overline{L_\alpha} = \bigcap_{n=1}^{\infty} \overline{L_{-n}}.$$

Since quasiconvex function f is not monotone, then there are x_1, x_2, x_3 with $x_1 < x_2 < x_3$ such that $f(x_1) > f(x_2)$ and $(x_3) > f(x_2)$. Consequently, $H \subseteq L_{f(x_2)} \subset (x_1, x_3) \subseteq int(I)$. Thus H is a bounded set. Therefore, $\overline{L_\alpha}$ or $\overline{L_{-n}}$ are closed bounded interval for n large enough. According to Nested Interval Theorem, $H \neq \emptyset$.

Let $t \in H$. Since $H \subset int(I)$, then $I \cap (-\infty, t)$ and $I \cap (t, \infty)$ are not empty. Take any $x, y \in I$ with $x < y < t$. According to definition of H, we have $t \in \bar{L}_{f(x)}$. Since $L_{f(x)}$ is an interval, thus $\bar{L}_{f(x)}$ is also an interval. Therefore, $y \in (x, t) \subseteq int\left(L_{f(x)}\right)$. This shows that f is decreasing on I ∩(−∞,t). Similarly, f is increasing on $I \cap (t, \infty)$.

Next, by quasiconvexity of f, we have f is decreasing on $I \cap (-\infty, t]$ and increasing on $I \cap [t, \infty)$. Otherwise, there exists $x_1 \in I \cap (-\infty, t]$ and $x_2 \in I \cap [t, \infty)$ such that $f(t) > f(x_1)$ and $f(t) > f(x_2)$, contradiction.

Theorem 5 stated that convex functions can be decomposed at most into two monotonous functions. According to Lebesgue theorem, the function is discontinuous only on countable set and not differentiable only on zero measure set.

Corollary 14. If f quasi-convex then f only discontinuous on countable set and not differentiable on zero measure set.

Corollary 15. Suppose that f is quasiconvex function, then f has one sided derivatives for every points.

Proof. By Theorem 13, we have f is at most can be decomposed as two monotone functions. Since monotone function always have one sided derivatives, as well as f.

3 Conclusion

According to definition of convex and quasi-convex functions on ℝ, several theorems have been acquired that can be seen in the Table 1.

Table 1. Theorems of convex and quasi-convex function on R.

Convex	**Quasi-convex**
Every local minimum is a global minimum	*Every local minimum is a global minimum or f is constant in an neighborhood*
f is convex if and only if $\boldsymbol{epi(f)}$ *is convex set*	*f is quasiconvex if and only if level set* L_α *is convex set for every* α
f countinuous everywhere on interior of its domain	*f continuous almost everywhere. Specifically, f is not continuous at countably many points only.*

Has one sided derivatives at any points	*Has one sided derivatives at any points*
If function f is continuous and differentiated in $\mathbb{R}$*, f is convex in* $\mathbb{R}$ *if and only if* $\boldsymbol{f(x) - f(y) \geq f'(x)(y - x)}$	*f is quasi-convex on interval I if only if f is monotone or there is* $t \in I$ *such that f is decreasing on* $(-\infty, x] \cap I$ *and increasing on* $I \cap [x, \infty)$.
Assume $\boldsymbol{f} \in \boldsymbol{C^2}$*, therefore f is convex if and only if* $\boldsymbol{f''} \geq \boldsymbol{0}$	*Lef* $f \in C^2$*, if* $f'' \geq 0$ *then f is quasiconvex*
Differentiable almost everywhere	*Differentiable almost everywhere*

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