Investigating Bhattacharyya Parameters for Short and Long Polar Codes in AWGN and Rayleigh Fading Channels

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Abstract. Error correction coding is one of the most important things in wireless communication systems, where the fifth telecommunication generation (5G) New Radio (NR) uses Polar codes as channel coding scheme together with Low Density Parity Check (LDPC) codes. Polar codes are new efficient channel code having high performance with low computational complexity of encoding and decoding. Given the practical channel, the performances of Polar codes are depending on the channel quality. This paper investigates Bhattacharyya parameters as the parameter for selecting the frozen bits following the change of the channel quality. This paper considers Bhattacharyya parameters for short and long Polar codes based on Signal to Noise power Ratio (SNR) and compares Bit Error Rate (BER) performances of Polar codes under additive white Gaussian noise (AWGN) and Rayleigh fading channels. The result are useful to design Polar codes for different SNR and information block lengths.

Keywords: Polar codes, Bhattacharyya parameters, Polar Weight, AWGN channel, and Rayleigh fading channel.

1 Introduction

Technological evolutions are quickly evolve through the years. It conduce high data transmission demand on mobile communications. 5G technology were released to support that demand [1]. A lot of parameters considered to achieve the high data rate, such as bandwidth, energy consumption, latency, and complexity. Mobile communications have developed from first generation (1G) to fifth generation (5G). 1G focus on voice services, 2G fix voice services and text message services, 3G integrate voice services and internet mobile services, 4G support the high level of capacity on multimedia mobile services. In 2020 has been estimated that will be billion of devices are connected with internet, because of that 5G is sued for higher capacity with low latency.

Error correction coding technic get the high transformation, 2G use turbo codes for error correction coding, while 3G use low-density parity-check (LDPC) codes. Nevertheless, from 3G to 4G, there is no modification for error correction coding, but LDPC performance have been improved. The big transformation occurs in 4G to 5G, 5G use Polar codes and LDPC codes as

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error correction coding. It because of 5G have a new challenges, i.e. low energy consumption, high data demand, and the others of interface [2].

Polar codes proposed by Erdal Arikan with channel polarization method on binary-input memoryless channels, that obtain polarization rate, polarization scheme and iterative decoding [3]. On its development, the decoder is the popular research material of Polar codes. There are 3 types decoder of Polar codes, i.e. belief propagation (BP), successive cancellation decoding (SC) and list successive cancellation decoding (SCL). From [4], SCL with list L = 32 have a good performance, but need too much energy consumption. Polar is named Polar because

Polar codes be able to polarize. Polar determine channel capacity that use for encoder and decoder schemes. The determination of channel capacity are determined with calculat and sort the capacity from Bhattacharyya parameters [5].

From the result of [6], Polar codes have better performance than LDPC codes on mobile communications. Because of mobile communication system which only need low information bit (blocklength of information bits $k < 512$). Other than that, the good channel coding can resist with channel interference. Between LDPC and Polar codes, channel coding which can resist with channel interference is LDPC codes, but LDPC have high complexity and high energy consumption. On the other hands, Polar codes have a low complexity and low energy consumption. Bhattacharyya parameters is the one of method for calculate and compile channel capacity. Therefore, this paper focus on investigate the effect of Bhattacharyya parameters toward to channel capacity of Polar codes, so that obtain the efficient configuration of Bhattacharyya parameters in every single signal-to-noise power ratio (SNR).

2 Polar Codes Structure

In this section, we review encoder and decoder of Polar codes and Bhattacharyya parameters for channel capacity of Polar codes.

2.1 Polarization with Bhattacharyya Parameter

Polar codes were introduce by Erdal Arikan [3]. They can provably achieve the symmetric capacity of binary input discrete memoryless with low complexity decoder. Polar codes have two types of bits to transmit, there are information bits and frozen bits. Polar codes need to assign the position and the quantity among information bits and frozen bits. To assign the quantity, Polar codes use rate R to calculate the quantity of information bits and frozen bits depend on blocklength N which is the maximum quantity of information bits ($K = R\ N$) and frozen bits ($F = N - K$). To assign the position, Polar codes need to calculate the capacity channel, the good channel capacity used as information bits and the bad channel capacity used as frozen bits. Bhattcharyya parameter is a method to calculate channel capacity of channel during a polarization process. Bhattacharyya parameters have the fundamental formula as

$$Z(W) = \sum_x \sqrt{W(y|0)W(y|1)} \quad (1)$$

where W is Bhattacharyya parameters and $Z(W)$ is erasure probability for binary erasure channel (BEC). While on the Polar codes, Bhattacharyya parameters have own formulas for Polar codes [5] as:

$$Z(W^-) = 2Z(W) - Z(W)^2 \tag{2}$$
$$Z(W^+) = Z(W)^2 \tag{3}$$

So it can be conceived with Figure 1, variable W can be known as epsilon (ϵ) from binary erasure

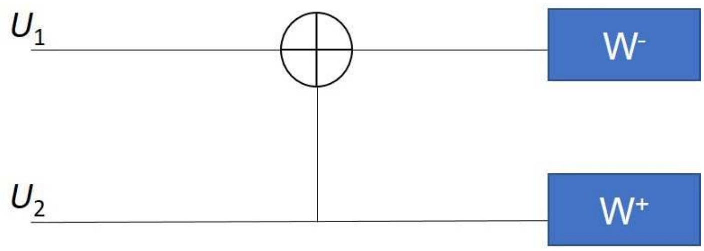

Figure 1. *Encoding* Polar *codes* with channel capacity.

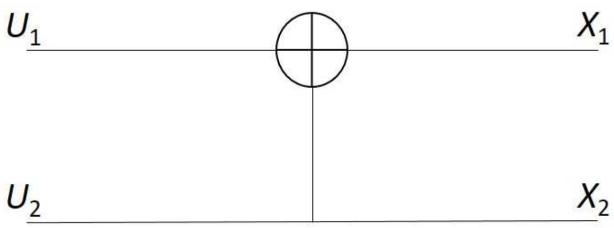

Figure 2. Construction Polar *codes* Encoder with blocklength $N = 2$ bits.

channel (BEC) and can be defined with

$$\epsilon = \exp\left(-R \cdot \frac{Eb}{N_0}\right)$$
$$\epsilon^- = 2\epsilon - \epsilon^2$$
$$\epsilon^+ = \epsilon^2. \tag{4}$$

2.2 Polar Codes Incoder

Encoding is coding technic to protect bits before their are transmitted. Theory of Polar encoder
[7] calculate by matrix operation between bit construction U and matrix generator depend on
N. Polar codes encoder can be defined with Figure 2, and formulas:

$$\begin{aligned} [X_1\ X_2\ ] &= [U]\,[G_m] \\ &= [U_1 U_2] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= [U_1 \oplus U_2 \quad U_2] \end{aligned} \tag{5}$$

where U is bit input consist of information bits and frozen bits. X is the result of encoding scheme. G_m is matrix generator which can be expand depend on Kronecker power with specific blocklength, the process can be defined with

$$\begin{aligned} G_N &= (G_2)^{\otimes u} \\ G_2 &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ G_4 &= \begin{bmatrix} G_2 & G_2 \\ G_2 & G_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{aligned} \tag{6}$$

2.3 Polar Codes Decoder

Decoding is coding technic to revert the signal after transmission become bits. There are various kinds of Polar decoder, such as successive cancellation decoding, iterative decoding, list decoding [8], and belief propagation decoding. This paper propose soft input successive cancellation decoding (SC), which is modification of successive cancellation decoding (SC) with soft decoding BPSK [9]. The input of decoding are log-likehood ratio (LLR), of which the value is defined as

$$L_N^{(i)}\left(y_1^N, u_1^{i-1}\right) = \frac{W_N^{(i)}\left(y_1^N, u_1^i - 1|0\right)}{W_N^{(i)}\left(y_1^N, u_1^i - 1|1\right)} \tag{7}$$

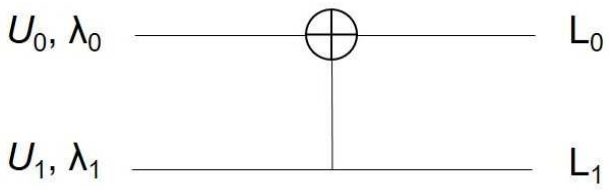

Figure 3. Fundamental decoding block Polar codes.

with the outputs are bits receiver (uˆi) with hard decision formula:

$$\hat{u} = \begin{cases} 0\,, if\ L_N^{(i)}\left(y_1^N, \hat{u}_1^{i-1}\right) \geq 1 \\ 1, otherwise \end{cases} \tag{8}$$

Decoding scheme can be illustrated by Figure 3, and defined with formula:

$$\begin{aligned} \lambda_0 &= f_0(L_0, L_1) \\ &= L_0 \boxplus L_1 \\ &= 2\tanh^{-1}(\tanh\tfrac{L_0}{2} - \tanh\tfrac{L_1}{2}) \\ &= sign(L_0)sign(L_1)\min\{|L_0|, |L_1|\}, \end{aligned} \tag{9}$$

$$\begin{aligned} \lambda_1 &= f_1(U_0, L_0, L_1) \\ &= (-1)^{U_0} * L_0 + L_1. \end{aligned} \tag{10}$$

3 Polar Codes Transmission Through The Channels

We consider Polar codes for white Gaussian noise (AWGN) channel and Rayleigh fading channel.

3.1 Model System

This paper design process transmitter and receiver on Polar codes with Bhattacharyya param- eters and illustrated on Figure 4, We assign U are bits input with blocklength N = (8, 16, 32, 64, 128) bits and R = 1 . On the transmitter side, there are Polar codes encoder C and modulation M. All of bits encode with Polar encoder and transform to symbol with binary phase shift keying (BPSK). On the receiver side, there are demodulation M −1 and Polar codes decoder C−1. A pack of symbols transform to LLR, then decode with soft input successive cancellation decoding and transform to output bits Uˆ. While variable h is channel to transmit signal from transmitter to receiver and n is noise from the antenna on receiver.

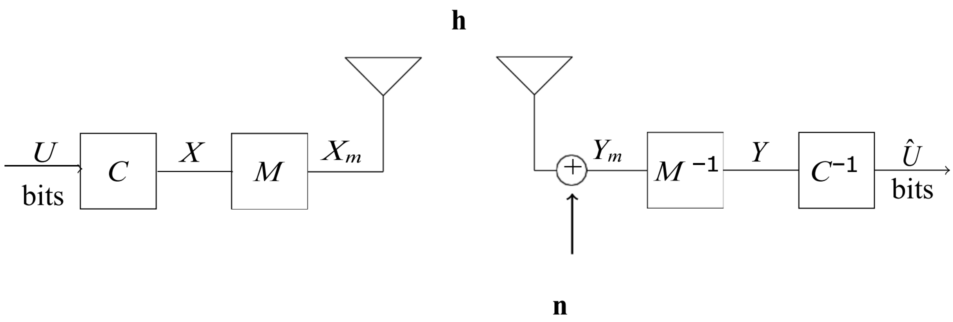

Figure 4. System model of transmission structure using Polar *codes*.

3.2 Additive White Gaussian Noise (AWGN) Channel

Gaussian known as Gaussian distribution or normal distribution. Whereas noise called with white because consists of all over frequencies with their white spectral [10]. White noise also known as WSS noise that has constant spectral density power. Additive mean the noise which merely add to signal and without multiplication operation and can be defined with

$$y = h \cdot x + n \tag{11}$$

where h = 1, c is bit transmit and n is noise. Noise power has uniform spectral density for every frequency, or called as white noise.

3.3 Rayleigh Fading

In communication system, signal transmit pass any path on the channel before it arrive on the receiver. For narrowband single carrier transmission, the signal experiences frequency-flat Rayleigh fading channels [11]. Rayleigh fading channel can be represented by:

$$y = h_f \cdot x + n \tag{12}$$

where hf is random variable distributed from Rayleigh PDF, with n is noise. The probability distribution function of Rayleigh fading is expressed as [11]

$$p(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{n^2}{\sigma^2}\right). \tag{13}$$

4 RESULT AND SIMULATIONS

In this section, we have simulated Bhattacharyya parameters and Polar codes scheme in a simulation environment.

4.1 Bhattacharyya Parameters Analysis

Bhattacharyya parameters analyze the effect of signal-to-noise ratio (SNR) to channel capa- city, so that effect to the position of frozen bits and information bits. This paper use SNR –15 until 50 dB with N = (8, 16, 32) bits and R = 1/2 . We have make a group of bits allocation depend on SNR and blocklength.

In Table I, we have 3 different patterns in N = 8. For SNR –15 until 25 dB, information bits located in index k = [4, 6, 7, 8]. For SNR 26 until 28 dB, information bits located in index k = [2, 3, 4, 5]. For SNR 29 until 50 dB, information bits located in index k = [1, 2, 3, 4].

Table 1. Bhattacharyya parameters with blocklengh 8.

SNR (dB)	Bit Index 1 2 3 4 5 6 7 8
-15	0 0 0 1 0 1 1 1
-10	0 0 0 1 0 1 1 1
-5	0 0 0 1 0 1 1 1
0	0 0 0 1 0 1 1 1
5	0 0 0 1 0 1 1 1
10	0 0 0 1 0 1 1 1
15	0 0 0 1 0 1 1 1
25	0 0 0 1 0 1 1 1
26	0 1 1 1 1 0 0 0
27	0 1 1 1 1 0 0 0
28	0 1 1 1 1 0 0 0
29	1 1 1 1 0 0 0 0
30	1 1 1 1 0 0 0 0
35	1 1 1 1 0 0 0 0
40	1 1 1 1 0 0 0 0
45	1 1 1 1 0 0 0 0
50	1 1 1 1 0 0 0 0

Table 2. Bhattacharyya parameters with blocklengh 16.

SNR (dB)	Bit Index
	1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6
-15	0 0 0 0 0 0 0 1 0 1 1 1 1 1 1 1
-10	0 0 0 0 0 0 0 1 0 1 1 1 1 1 1 1
-5	0 0 0 0 0 0 0 1 0 1 1 1 1 1 1 1
0	0 0 0 0 0 0 0 1 0 1 1 1 1 1 1 1
5	0 0 0 0 0 0 0 1 0 1 1 1 1 1 1 1
10	0 0 0 0 0 0 0 1 0 1 1 1 1 1 1 1
15	0 0 0 0 0 0 0 1 0 1 1 1 1 1 1 1
22	0 0 0 0 0 0 0 1 0 1 1 1 1 1 1 1
23	0 0 0 1 0 1 1 1 0 1 1 1 1 0 0 0
24	0 0 0 1 0 1 1 1 0 1 1 1 1 0 0 0
25	0 0 0 1 0 1 1 1 0 1 1 1 1 0 0 0
26	0 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0
27	0 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0
28	0 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0
29	1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0
40	1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0
50	1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0

Table 3. Bhattacharyya parameters with blocklengh 32.

SNR (dB)	Bit Index
	1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 8 9 0 1 2
-15	0 0 0 0 0 0 0 1 0 0 0 1 0 1 1 1 0 0 0 1 0 1 1 1 0 1 1 1 1 1 1 1
-10	0 0 0 0 0 0 0 1 0 0 0 1 0 1 1 1 0 0 0 1 0 1 1 1 0 1 1 1 1 1 1 1
-7	0 0 0 0 0 0 0 1 0 0 0 1 0 1 1 1 0 0 0 1 0 1 1 1 0 1 1 1 1 1 1 1
-6	0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 0 0 1 0 1 1 1 1 1 1 1 1 1 1 1
0	0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 0 0 1 0 1 1 1 1 1 1 1 1 1 1 1
2	0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 0 0 1 0 1 1 1 1 1 1 1 1 1 1 1
3	0 0 0 0 0 0 0 1 0 0 0 1 0 1 1 1 0 0 0 1 0 1 1 1 0 1 1 1 1 1 1 1
10	0 0 0 0 0 0 0 1 0 0 0 1 0 1 1 1 0 0 0 1 0 1 1 1 0 1 1 1 1 1 1 1
22	0 0 0 0 0 0 0 1 0 0 0 1 0 1 1 1 0 0 0 1 0 1 1 1 0 1 1 1 1 1 1 1
23	0 0 0 1 0 1 1 1 0 1 1 1 1 1 1 1 0 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0
24	0 0 0 1 0 1 1 1 0 1 1 1 1 1 1 1 0 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0
25	0 0 0 1 0 1 1 1 0 1 1 1 1 1 1 1 0 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0
26	0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
27	0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
28	0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
29	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
40	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
50	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

In Table II, we have 4 different patterns in N = 16. For SNR –15 until 22 dB, information bits located in index k = [8, 9, 10, 11, 12, 13, 14, 15, 16]. For SNR 23 until 25 dB, information bits located in index k = [4, 5, 6, 7, 8, 9, 10, 11]. For SNR 26 until 28 dB, information bits located in index k = [2, 3, 4, 5, 6, 7, 8, 9]. For SNR 29 until 50 dB, information bits located in index k = [1, 2, 3, 4, 5, 6, 7, 8].

In Table III, we have 5 different patterns in N = 32. For SNR –15 until –7 dB and 3 until 22 dB, information bits located in index k = [8, 12, 14, 15, 16, 20, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32]. For SNR –6 until 2 dB, information bits located in index k = [12, 14, 15, 16, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. For SNR 23 until 25 dB, information bits located in index k = [4, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22]. For SNR 26 until 28 dB, information bits located in index k = [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. For SNR 29 until 50 dB, information bits located in index k = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

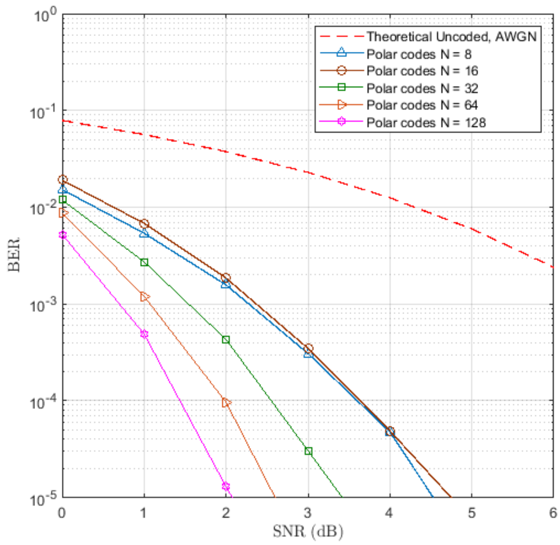

Figure 5. BER Performance

Figure 5. BER Performance Polar codes with Bhattacharyya Bhattacharyya with N = (8, 16, 32, 64, 128) bits in AWGN channel.

4.2 BER Performance Polar Codes in AWGN

BER performance evaluate with perfect channel and coefficient channel value h is 1 (maximum). BER performance in AWGN channel illustrated in Figure 5. From the Figure 5, we can show that the performances are great. To achieve BER 10^{-4}, N = 8 require SNR 3.6 dB, N = 16 require SNR 3.63 dB, N = 32 require 2.55 dB, N = 64 require 2 dB, and N = 128 require

1.36 dB. We know that the longer blocklength have the better performance, except in N = 16. BER performances with N = 16 are worse than N = 8, it because the pattern of capacity channel on N = 16 look similar with N = 8 for SNR -15 until 22 dB.

4.3 BER Performance Polar Codes in Rayleigh fading

BER performance calculate with coefficient channel values are random complex number. In Rayleigh fading be examined for SNR 0 untill 40 dB. BER Performance Polar codes through Rayleigh fading in Figure 6. But from this Figure, we get bad BER performances, so

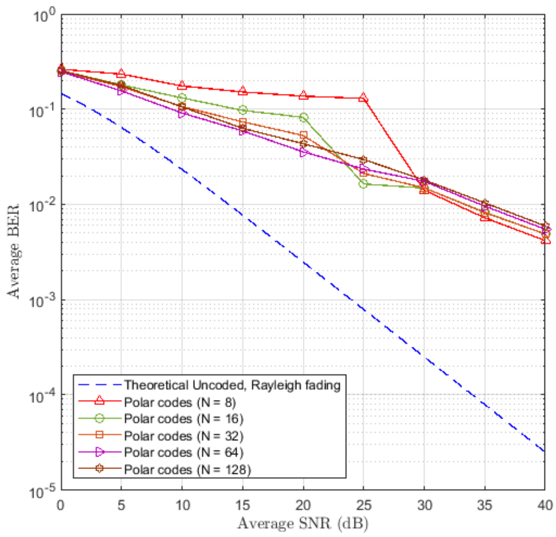

Figure 6. BER Performance Polar

Figure 6. BER Performance Polar codes with Bhattacharyya Bhattacharyya with N = (8, 16, 32, 64, 128) bits in Rayleigh fading channel.

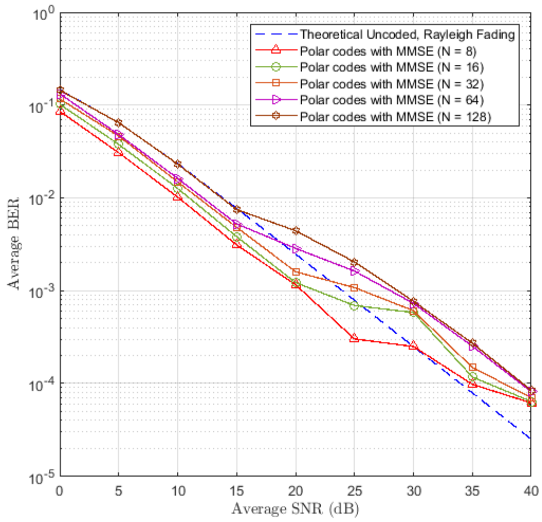

Figure 7. BER Performance Polar codes with Bhattacharyya Bhattacharyya with N = (8, 16, 32, 64, 128) bits in Rayleigh fading channel with MMSE.

We need to improve with add equalizer minimum mean square error (MMSE) on the Polar codes scheme. The performances in Figure 7, shows the performances are better than fading Polar codes without MMSE. But, for SNR 20, blocklength 64 and 128 have bad performance because their performances are over than BER theory threshold. The performances are also bad on the other blocklengths. So, we need to improve with feedback method. Because of Rayleigh fading channel is randomly distributed channel untill we can get randomly SNR and changes the patterns of capacity channel. With feedback, we need transmit twice. Firstly, system transmit to get information about SNR in receiver, then the system transmit again and calculate the capacity channel based on feedback SNR. The result, the performance of Polar codes in Rayleigh fading channel are great on Figure 8. The performances achieve BER = 10−4 for SNR of 30 to 35 dB. Different from the BER performances on AWGN Channel, in Rayleigh fading, the longer blocklength have worse performance. It happens because we do not use multipath, so the increment of blocklength also affect number of bits are interfered.

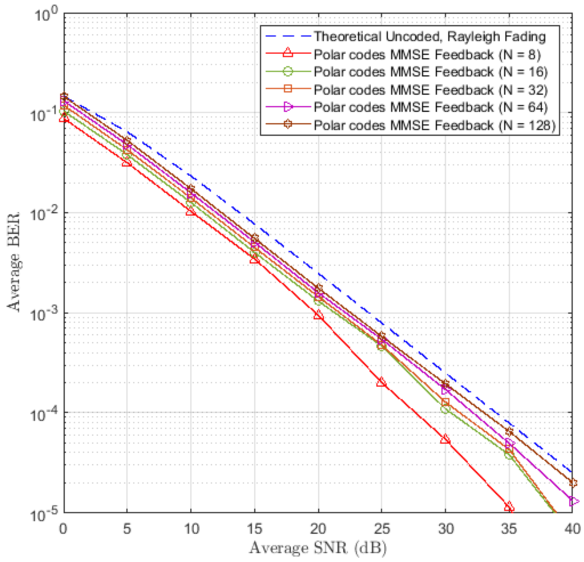

Figure 8. BER Performance

Figure 8. BER Performance Polar codes with Bhattacharyya Bhattacharyya with N = (8, 16, 32, 64, 128) bits in Rayleigh fading channel with MMSE and Feedback.

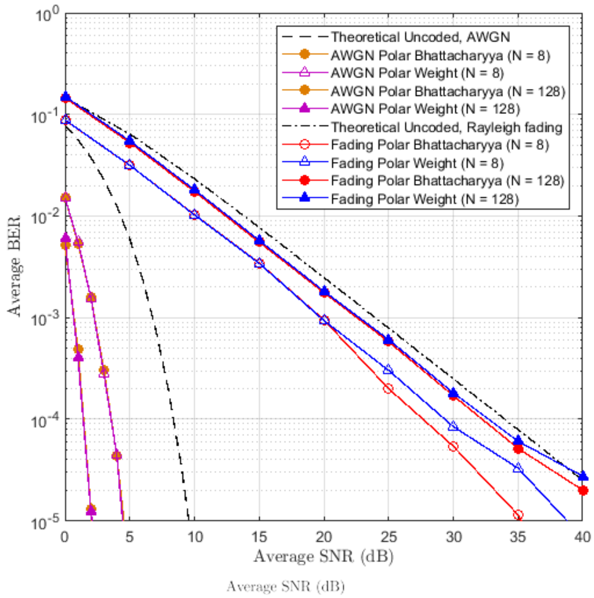

Figure 9. Comparison BER

Figure 9. Comparison BER Performance Polar codes between Bhattacharyya Bhattacharyya MMSE-Feedback and Polar Weight.

4.4 Bhattacharyya Parameters vs Polar Weight

In this paper, we also compare the performance of Polar codes with Bhattacharyya parameters and Polar codes with Polar Weight [12]. In Figure 9, we show that the BER performances between Bhattacharyya parameters and Polar Wieght literally same. Nevertheless, BER performance of Bhattacharyya parameters in high SNR are better than Polar Weight. On the other side, the energy consumption of Bhattacharyya parameters are higher than Polar Weight in Rayleigh fading channel. Because of that, Polar Weight dont't need feedback method, so they aren't transmit twice.

5 Conclusion

This paper has evaluated the effect of Bhattacharyya parameters on Polar codes. SNR ef- fect the calculation of channel capacity with Bhattacharyya parameters and the allocation of information bits and frozen bits. The increament of blocklengths effect more pattern of that allocations. BER Performance of Polar codes with Bhattacharyya parameters in AWGN channel are great and achieve 10−5 dB for SNR 2 to 5 dB. BER Performance of Polar codes with Bhattacharyya parameters in Rayleigh fading channel need MMSE and feedback methods to achieve great performance (under Rayleigh fading theory). BER performance of Polar codes with Bhattacharyya parameters and Polar Weight are literally same. Nevertheless, BER performance of Bhattacharyya parameters in high SNR are better than Polar Weight. On the other side, Bhattacharyya transmission are complicated than Polar Weight transmission, because Bhattacharyya need a feedback to achieve good performance in Rayleigh fading channel.

References

[1] P. Allison, “A Case Study Of E-Learning initiatives in new Zeland’s Secondary schools. Doctoral Dissertation,” *J. Chem. Inf. Model.*, vol. 53, no. 9, p. 261, 2011, doi: 10.1017/CBO9781107415324.004.

[2] T. Specification, “TS 138 212 - V15.3.0 - 5G; NR; Multiplexing and channel coding (3GPP TS 38.212 version 15.3.0 Release 15),” vol. 0, 2018.

[3] E. Arıkan, “Channel Polarization: A Method for Constructing Capacity-Achieving Codes for Symmetric Binary-Input Memoryless Channels,” *Proc. - IEEE Int. Symp. Circuits Syst.*, vol. 55, no. 7, 2009, doi: 10.1109/TIT.2009.2021379.

[4] M. Hu, J. Li, and Y. Lv, “A comparative study of polar code decoding algorithms,” *Proc. 2017 IEEE 3rd Inf. Technol. Mechatronics Eng. Conf. ITOEC 2017*, vol. 2017-January, pp. 1221–1225, 2017, doi: 10.1109/ITOEC.2017.8122551.

[5] S. Zhao, P. Shi, and B. Wang, “Designs of Bhattacharyya Parameter in the construction of Polar codes,” *7th Int. Conf. Wirel. Commun. Netw. Mob. Comput. WiCOM 2011*, pp. 2–5, 2011, doi: 10.1109/wicom.2011.6040179.

[6] O. Işcan, D. Lentner, and W. Xu, “A comparison of channel coding schemes for 5G short message transmission,” *2016 IEEE Globecom Work. GC Wkshps 2016 - Proc.*, 2016, doi: 10.1109/GLOCOMW.2016.7848804.

[7] S. B. Korada, E. Şaşŏglu, and R. Urbanke, “Polar codes: Characterization of exponent, bounds, and constructions,” *IEEE Int. Symp. Inf. Theory - Proc.*, pp. 1483–1487, 2009, doi: 10.1109/ISIT.2009.5205865.

[8] I. Tal and A. Vardy, “List Decoding of Polar Codes,” *IEEE Trans. Inf. Theory*, vol. 61, no. 5, pp. 2213–2226, 2015, doi: 10.1109/TIT.2015.2410251.

[9] H. Kim, J. Shin, and J. Ahn, “Performance analysis of polar codes with soft input successive cancellation,” *Int. Conf. Adv. Technol. Commun.*, vol. 2015-February, pp. 563–566, 2015, doi: 10.1109/ATC.2014.7043452.

[10] E. Abbe and A. Barron, “Polar coding schemes for the AWGN channel,” *IEEE Int. Symp. Inf. Theory - Proc.*, pp. 194–198, 2011, doi: 10.1109/ISIT.2011.6033892.

[11] A. Bravo-Santos, “Polar codes for the rayleigh fading channel,” *IEEE Commun. Lett.*, vol. 17, no. 12, pp. 2352–2355, 2013, doi: 10.1109/LCOMM.2013.111113.132103.

[12] D. P. Zhu, “Polar code for 5G NR,” *Huawei Fellow, IEEE Fellow,* 2018.