On near-far gain in multiuser diversity systems

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***Abstract*—Multiuser diversity scheduling in a single-cell system is studied. The scheduler chooses a constant size subset of users with best short term fading gains and coordinates transmissions using superposition coding. Rate allocation is performed according to users' channel states assuming equal power allocation over users. Asymptotic analysis of system capacity is applied to study system behavior at low and high spectral efficiencies. It is found that if users have non-symmetric channel distributions, i.e. path loss is present, with large enough user population it becomes beneficial in the sum rate sense to schedule more than one user at a time.**

I. Introduction

We consider the uplink of a wireless communications system with K users. Each user k experiences a propagation channel gain d_k that is the product of two random variables: the short term fading f_k and the path loss s_k. Short term fading is assumed fixed over the signaling interval but i.i.d. across users and signaling intervals. Path loss is assumed i.i.d. across users and fixed in time for all users. Throughput optimal scheduling [1] in a system with path loss results often in a severely unfair result: only the user with the best channel quality, usually very close to the access point, is scheduled. As a result, alternative scheduling methods providing fairness of scheduling decisions have been proposed. One of these is the proportional fairness scheduler (PFS) [2], [3], [4]. In [5] it is proved that in the limit of infinite time window PFS gives the channel to whatever user has the best short term channel realization. Each user is then transmits with equal power, i.e. no power control optimization is applied [6] and as a result, the system does not provide any explicit rate guarantees to the users.

Based on the infinite time window interpretation of PFS, we devise a scheme where the K_A users with largest short term fading gains are scheduled simultaneously and transmit with optimal multiuser signaling. Rate allocation is performed assuming equal power allocation while the total transmitted power is normalized. We use tools from [7], [8] to analyze the behavior of system capacity as a function of total transmitted energy at low and high spectral efficiencies. Dependencies between the size of user population, the number of simultaneously scheduled users and channel statistics are found.

The paper is organized as follows. The scheduling principle and the utilized performance metrics are defined in Section II. In Sections III and IV the system is analyzed for vanishingly small and asymptotically high spectral efficiencies, respectively. Numerical examples are presented in Section V.

The elements of a vector or a sequence $\vec{x}$ are x_i. We denote by $x_{i:j}$ the ith smallest variable of j variables and the distribution of a random variable y by F_y. Expectation with respect to a random variable x is denoted by $\mathrm{E}_x[\cdot]$ and with respect to a sequence of variables by $\mathrm{E}_{\vec{x}}[\cdot]$. The base of logarithm is explicitly marked except for natural logarithms.

II. Group scheduling

PFS allows the user with the relatively best channel to transmit without considering the resulting rate provided. In effect, the fairness is restricted to the scheduling decisions, and does not consider fairness in transmitted rate. Generalizing this approach to allowing K_A users with best short term fading channel realizations transmitting simultaneously, we come to the following formulation.

Let $\mathsf{SNR} = 1/N_0$. At any given time t, the system capacity is expressed as a function of the channels of the set of scheduled users $\mathcal{A}$ as

$$\mathsf{C}(t) = \log\left(1 + \mathsf{SNR}\frac{\sum_{k\in\mathcal{A}} d_k}{K_A}\right). \tag{1}$$

If we now define $y = \sum_{k\in\mathcal{A}} d_k/K_A$, we can express the average system capacity and the corresponding system energy per bit, by

$$\mathsf{C}(\mathsf{SNR}) = \int_0^\infty \log\left(1 + y\mathsf{SNR}\right) dF(y) \tag{2}$$

$$\left(\frac{E_b}{N_0}\right)_{\text{sys}} = \frac{\mathsf{SNR}}{\mathsf{C}} \log(2) \tag{3}$$

Since the selection of the set of scheduled users $\mathcal{A}$ is performed so that the K_A users with the best relative channels are scheduled, the short term fading statistic follows the order statistic of K_A largest samples of the short term fading distribution. Note that the normalization of the transmitted power by K_A merely has the effect of holding the total transmitted power unchanged w.r.t. K_A.

Increasing the number of simultaneously scheduled users has the following effect. In terms of short term fading, users with relatively poorer channel conditions are scheduled causing the expected value of y decrease with increasing K_A. In terms of path loss, however, the distribution of y will concentrate nearer the expectation with increasing K_A.

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The former will bring a loss, and the latter, due to the concavity of logarithm, a gain in expected system capacity. The effective change in system capacity will thus depend on the distributions of short term fading and path loss, in addition to the SNR.

III. Low spectral efficiency behavior

Direct analysis of the behavior of capacity is difficult due to the received energy being the mean of ordered statistics weighted with a random path loss. We can still analyze the behavior of the system at high and low spectral efficiency regions as defined in [7], [8]. In the low SNR region the system behavior is characterized by the wideband slope $\mathcal{S}_0$ and the minimum system E_b/N_0, $(E_b/N_0)_{min}$, such that [7]

$$\left(\frac{E_b}{N_0}\right)_{\text{sys}}\Bigg|_{\text{dB}} = \left(\frac{E_b}{N_0}\right)_{\text{min}}\Bigg|_{\text{dB}} + \frac{\mathsf{C}}{\mathcal{S}_0} 10 \log_{10}(2) + o(\mathsf{c}), \tag{4}$$

where the capacity C is given in bits. By the results of [7], we have directly from (2) that

$$\begin{aligned} \left(\frac{E_b}{N_0}\right)_{\text{min}} &= \frac{\log(2)}{\mathsf{C}'(0)} && (5)\\ &= \frac{\log(2)}{\int_0^\infty \frac{y}{1+\mathsf{SNR}y} dF_y(y)\Big|_{\mathsf{SNR}=0}} && (6)\\ &= \frac{\log(2)}{\mathrm{E}[y]} && (7)\\ &= \frac{\log(2)}{\mathrm{E}[s]\frac{1}{K_A}\sum\limits_{i=K-K_A+1}^{K} \mathrm{E}[f_{i:K}]}. && (8) \end{aligned}$$

Note that due to

$$\mathrm{E}[f_{i:K}] > \mathrm{E}[f_{j:K}]$$

when $i > j$, the minimum system energy per transmitted bit increases as K_A grows while K remains fixed. Let us analyze the difference between scheduling one or K_A users. The ratio of the corresponding minimum system energies is then

$$\left(\frac{E_b}{N_0}\right)^{K_A}_{\text{min}} \Big/ \left(\frac{E_b}{N_0}\right)^{1}_{\text{min}} = \frac{\sum\limits_{i=K-K_A+1}^{K} \mathrm{E}[f_{i:K}]}{K_A \mathrm{E}[f_{K:K}]}. \tag{9}$$

In the special case of Rayleigh fading short term channel statistics with exponential received energy distribution, we have [9]

$$\mathrm{E}[f_{i:K}] = \sum_{j=K-i+1}^{K} j^{-1},$$

and (9) can be expressed as

$$\begin{aligned} \frac{(E_b/N_0)^{K_A}_{\text{min}}}{(E_b/N_0)^{1}_{\text{min}}} &= 1 - \frac{\sum\limits_{j=1}^{K_A-1} j^{-1}\left(1-\frac{j}{K_A}\right)}{\sum\limits_{j=1}^{K} j^{-1}} && (10)\\ &= 1 - \frac{\Psi(K_A)+\gamma-1+1/K_A}{\Psi(K)+\gamma} && (11) \end{aligned}$$

where $\Psi()$ is the digamma function and $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. The digamma function has the limit $\lim_{K\to\infty} \Psi(K) = \infty$. In case of finite K_A the second term of (11) vanishes as K grows to infinity. As a result, in terms of $(E_b/N_0)_{\text{min}}$, the loss of scheduling multiple users diminishes as the user population grows.

Also the wideband slope can be obtained by using the results of [7] and (2) as

$$\begin{aligned} \mathcal{S}_0 &= \frac{2\mathsf{C}'(\mathsf{SNR})^2}{-\mathsf{C}''(\mathsf{SNR})} && (12)\\ &= \frac{2\left(\int_0^\infty \frac{y}{1+\mathsf{SNR}y} dF_y(y)\Big|_{\mathsf{SNR}\to 0}\right)^2}{\int_0^\infty \left(\frac{y}{1+\mathsf{SNR}y}\right)^2 dF_y(y)\Big|_{\mathsf{SNR}\to 0}} && (13)\\ &= \frac{2\mathrm{E}[y]^2}{\mathrm{E}[y^2]} && (14)\\ &= 2\left(\sum_{i=K-K_A+1}^{K} \mathrm{E}[f_{i:K}]\right)^2 \Bigg/ \\ &\quad \sum_{i=K-K_A+1}^{K}\left(\frac{\mathrm{E}[s^2]}{\mathrm{E}[s]^2}\mathrm{E}[f^2_{i:K}] + \right.\\ &\quad \left.\sum_{j=K-K_A+1, j\neq i}^{K} \mathrm{E}[f_{i:K}f_{j:K}]\right). && (15) \end{aligned}$$

Note for the exponential path loss model detailed in Appendix A the ratio $\mathrm{E}[s^2]/\mathrm{E}[s]^2$ is an increasing function of the path loss exponent, which makes the behavior of the wideband slope depend on the first two moments of the path loss distribution. For Rayleigh fading, we have [9]

$$\mathrm{E}[f^2_{i:K}] = \mathrm{E}[f_{i:K}]^2 + \sum_{j=K-i+1}^{K} j^{-2},$$

and for any i, j,

$$\mathrm{E}[f_{i:K}f_{j:K}] = \mathrm{E}\left[f^2_{\min(i,j):K}\right] + \mathrm{E}[f_{i:K}]\,\mathrm{E}[f_{j:K}].$$

With the above, it is straightforward to demonstrate that in Rayleigh fading, the wideband slope increases with K_A.

It is also useful to analyze the behavior of the minimum energy and wideband slope when the number of scheduled users is asymptotically large. In such a case, we let $K_A =$

$\gamma K \to \infty$. Taking the value of (1) at the limit of $K_A \to \infty$ we get

$$\begin{aligned} C_\infty(\mathsf{SNR}) &= \lim_{K_A\to\infty} \log\left(1+\mathsf{SNR}\frac{1}{K_A}\sum_{i=1}^{K_A} d_i\right) \\ &= \log\left(1+\mathsf{SNR}\,\mathrm{E}[d]\right) \\ &= \log\left(1+\mathsf{SNR}\,\mathrm{E}[s]\,\mathrm{E}[f]\right), \end{aligned} \quad (16)$$

where the expectation over short term fading is taken over the scheduled users. At the limit of infinite K_A, the principle of scheduling K_A out of K users is identical to scheduling a constant fraction of the user population. This is effectively the same as choosing a short term fading threshold above which the users are scheduled.

Now, we can find the parameters of the asymptotic system as

$$\left(\frac{E_b}{N_0}\right)_{\min} = \frac{\log(2)}{\mathrm{E}[s]\mathrm{E}[f]} \quad (17)$$

$$\begin{aligned} \mathcal{S}_0 &= \frac{2\left(C'(0)\right)^2}{-C''(0)} \\ &= 2. \end{aligned} \quad (18)$$

IV. High spectral efficiency behavior

In the high spectral efficiency region, when $\mathsf{C} \to \infty$, the analysis is based on the spectral efficiency (high SNR) slope $\mathcal{S}_\infty$ and horizontal dB penalty $\mathcal{L}_\infty$ so that the energy-spectral efficiency function is approximated by [8]

$$\left.\left(\frac{E_b}{N_0}\right)_{\mathsf{sys}}\right|_{\mathsf{dB}} = \frac{\mathsf{C}}{\mathcal{S}_\infty}10\log_{10}(2) - 10\log_{10}(\mathsf{C}) + \mathcal{L}_\infty 10\log_{10}(2) + o(1), \quad (19)$$

where the capacity C is given in bits. The spectral efficiency slope is given by [8]

$$\begin{aligned} \mathcal{S}_\infty &= \lim_{\mathsf{SNR}\to\infty} \mathsf{SNR}\,\mathsf{C}'(\mathsf{SNR}) && (20) \\ &= \lim_{\mathsf{SNR}\to\infty} \mathsf{SNR}\int_0^\infty \frac{y}{1+\mathsf{SNR}y}dF_y(y) && (21) \\ &= 1, && (22) \end{aligned}$$

which is consistent with all schemes that uses all degrees of freedom of the system. The horizontal dB penalty is a more interesting measure of performance, and is given by

$$\begin{aligned} \mathcal{L}_\infty &= \lim_{\mathsf{SNR}\to\infty}\Bigg(\log_2 \mathsf{SNR} \\ &\quad - \int_0^\infty \log_2\left(1+y\mathsf{SNR}\right)dF_y(y)\Bigg). \\ &= -\mathrm{E}_y\left[\log_2(y)\right]. \end{aligned} \quad (23)$$

We can develop a simple upper bound on $\mathcal{L}_\infty$ to analyze the effects of multiuser scheduling on system capacity. We begin by noting that since $f_{i:K} < f_{j:K}$ for any $i < j$, we have

$$y \geq \frac{1}{K_A}\sum_{i=K-K_A+1}^{K} s_i f_{K-K_A+1:K},$$

with equality when $K_A = 1$. Thus, due to the fact that logarithm is a monotonically increasing function, we can bound $\mathcal{L}_\infty$ as

$$\begin{aligned} \mathcal{L}_\infty \leq &-\mathrm{E}_s\left[\log_2\left(\frac{1}{K_A}\sum_{i=1}^{K_A} s_i\right)\right] \\ &- \mathrm{E}_{f_{K-K_A+1:K}}\left[\log_2(f_{K-K_A+1:K})\right]. \end{aligned} \quad (24)$$

The horizontal dB penalty upper bound is thus the sum of two terms. The first term signifies the near-far gain and for a given path loss distribution a function of the number of scheduled users K_A. It is denoted as a gain since an increase in K_A will reduce the Jensen penalty of the logarithm. Note that due to the path loss being i.i.d. accross users we can change the indexing to that in (24).

The second term in the bound signifies the multiuser diversity loss due to the scheduling of users with relatively poorer instantaneous channel as K_A is increased. We can further analyze the this loss by analyzing the difference in this term between the cases of arbitrary K_A and $K_A = 1$. This will give us the upper bound of the multiuser diversity loss of scheduling more than one user as

$$\begin{aligned} &\mathrm{E}_{f_{K:K}}\left[\log_2(f_{K:K})\right] - \mathrm{E}_{f_{K-K_A+1:K}}\left[\log_2(f_{K-K_A+1:K})\right] \\ &= \mathrm{E}_{f_{K:K},f_{K-K_A+1:K}}\left[\log_2\left(\frac{f_{K:K}}{f_{K-K_A+1:K}}\right)\right], \end{aligned}$$

which for Rayleigh fading can be further developed as follows. Let $\{z_i,\ i \in N^+\}$ be exponentially distributed i.i.d. random variables with mean one. Then for Rayleigh fading

$$\begin{aligned} &\mathrm{E}_{f_{K:K},f_{K-K_A+1:K}}\left[\log_2\left(\frac{f_{K:K}}{f_{K-K_A+1:K}}\right)\right] \\ &= \mathrm{E}_{\vec{y}}\left[\log_2\left(1+\frac{\sum_{i=1}^{K_A-1} z_i/i}{\sum_{j=1}^{K-K_A+1} z_j/j}\right)\right] \\ &\leq \left(\sum_{i=1}^{K_A-1} z_i/i\right)\mathrm{E}_{\vec{y}}\left[\frac{1}{\sum_{j=1}^{K-K_A+1} z_j/j}\right] \\ &\leq \frac{\sum_{i=1}^{K_A-1} z_i/i}{\sum_{j=1}^{K-K_A+1} z_j/j} \\ &= \frac{\Psi(K_A)+\gamma}{\Psi(K-K_A+2)+\gamma}. \end{aligned}$$

This upper bound on the multiuser diversity loss vanishes asymptotically as $K \to \infty$ revealing that there exists a system

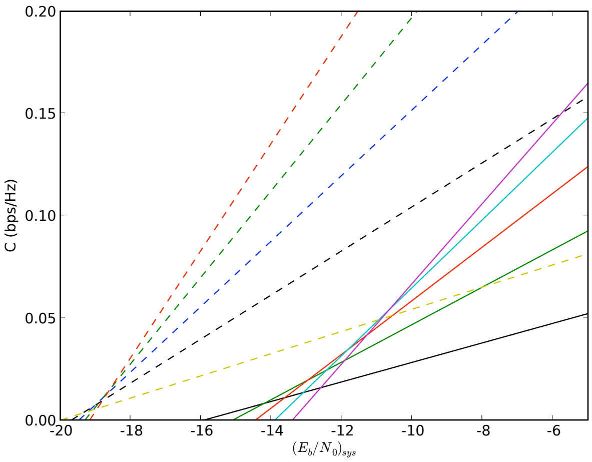

Fig. 1. Low spectral efficiency behavior when scheduling 1 to 5 users with system size 10 (solid) or 1000 (dashed).

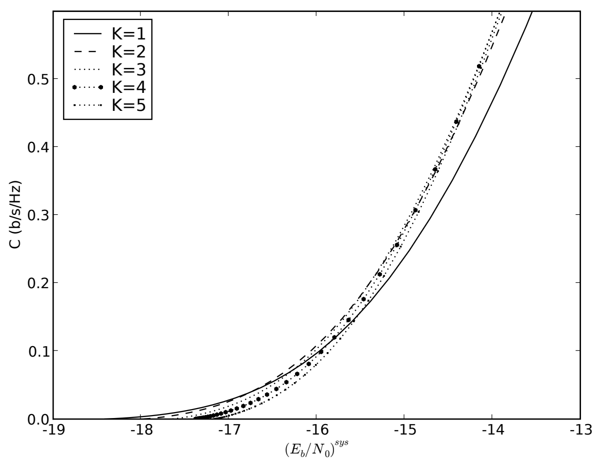

Fig. 2. System capacity behavior.

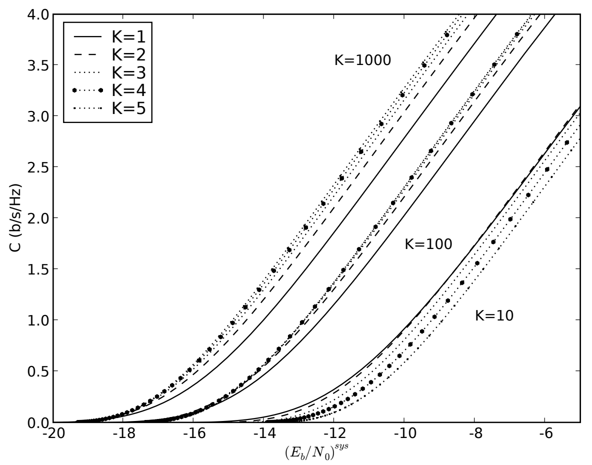

Fig. 3. System capacity behavior.

size K above which it is beneficial to schedule more than one user. It also shows the bound (24) is tighter as K increases. On the other hand, the near-far gain is bounded from above, since as $K_A \to \infty$

$$\mathrm{E}_{\vec{s}}\left[\log_2\left(\frac{1}{K_A}\sum_{i=1}^{K_A} s_i\right)\right] \to \log_2(\mathrm{E}\left[s\right]).$$

A crucial question is what are the resulting user rates of the devised scheme. A partial answer can be found examining the rate achieved by the user with the worst path loss (denote here as k^*). This user is scheduled whenever f_{k^*} is one of the K_A largest short term fading coefficients. The average rate of the user is then given as the expectation over fading realization and user ordering.

V. Numerical Results

The following numerical results correspond to a single-cell system where users are randomly placed on a unit disc with a forbidden region with radius $\delta = 0.01$ around the access point. Path loss is assumed exponential with loss exponent 2, while Rayleigh fading is assumed for short term fading. The details of the channel statistics can be found in the appendix A.

Figure 1 presents the low spectral efficiency linear approximation for system behavior as given by (4) and demonstrates the increase in wideband slope when scheduling more than one user simultaneously. Also the increase in required energy is readily visible. Changing the group size from 1 to 5 shows a significant increase in the wideband slope. While the behavior of the wideband slope across group sizes from 1 to 5 is similar between user populations of 10 and 1000, the loss in minimum $(E_b/N_0)_{\text{sys}}$ is visibly smaller with a larger user population.

Figure 2 gives the result of a Monte-Carlo integration of the exact system capacity behavior wih $K = 100$. While scheduling more than one user is an inferior strategy at very low spectral efficiencies, above a certain spectral efficiency (here approximately 0.15 b/s/Hz) it is beneficial to do so.

Figure 3 presents the system capacity characteristics of the scheme for three system sizes and demonstrates the effect of total system size K to the optimal group size. While it is almost never beneficial to schedule two or more users with system size $K = 10$, it is also almost never optimal to schedule only one user with system size $K = 1000$. One should note, furthermore, that the system capacity characteristic quickly settles to follow an asymptotic behavior. It is thus expected that beyond a certain spectral efficiency the relative ordering between scheduled group sizes K_A will remain fixed. Thus, examining the horizontal dB penalty for

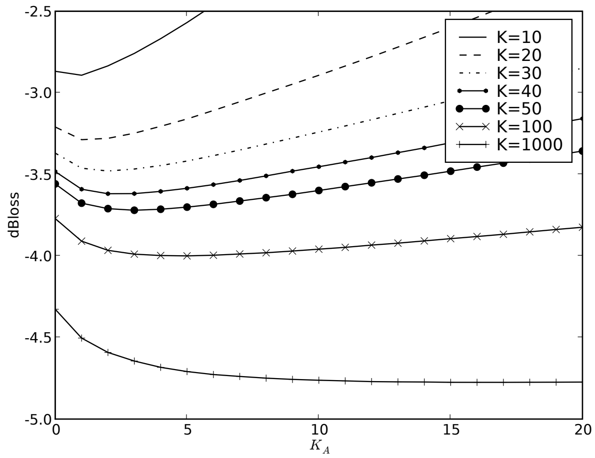

Fig. 4. Horizontal dB loss with varying K_A.

different group sizes will provide a meaningful optimization criterion for all but the smallest spectral efficiencies.

Figure 4 presents the horizontal dB loss of group sizes K_A up to 20 for different total system sizes K. Its shows an optimal group size can be found for all system sizes and that this optimal is larger for larger systems. Furthermore, the optimum is relatively flat for larger group sizes – one will lose very little by using an approximately optimal group size.

VI. Conclusions

An algorithm that schedules multiple users simultaneously with uniform power allocation was proposed. Asymptotic results revealed that the losses from scheduling multiple users instead or merely the user with the strongest short term fading vanish as the user population grows and the near-far gain from superposition coded scheduling becomes dominant. This is due to the fact that given a growing user population, consecutive ordered random variables become more and more indistinquishable. On the other hand, the gain from shrinking the Jensen-penalty of pure time-division scheduling (PFS) is a function of the number of scheduled users. The asymptotic horizontal dB penalty can be used to optimize the size of the scheduled group. For large user populations the optimum is very flat and a wide range of group sizes will perform close to optimal.

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Appendix

The path loss has the cumulative distribution function

$$F_s(x) = \begin{array}{cc} 0 & x < 1 \\ 1 - \frac{x^{-2/\alpha} - \delta^2}{1-\delta^2} & 1 \le x < \delta^{-\alpha} \\ 1 & x \ge \delta^{-\alpha} \end{array} \tag{25}$$

with the first two raw moments being

$$\mathrm{E}\left[s\right] = \begin{cases} \frac{2\left(\delta^{2-\alpha}-1\right)}{(1-\delta^2)(\alpha-2)} & \alpha \neq 2 \\ \frac{-2\ln\delta}{1-\delta^2} & \alpha = 2 \end{cases} \tag{26}$$

$$\mathrm{E}\left[s^2\right] = \frac{\delta^{2-2\alpha}}{(1-\delta^2)(\alpha-1)}. \tag{27}$$

The short term fading envelope is modeled as an exponential distribution with mean one. The channel distribution is given by

$$F_f(x) = 1 - e^{-x}, x \ge 0. \tag{28}$$