A Class Of Mean Field Interaction Models for Computer and Communication Systems

Michel Benaïm   Jean-Yves Le Boudec

*Abstract***— In this presentation we review a generic mean field interaction model where N objects are evolving according to an object's individual finite state machine and the state of a global resource. We show that, in order to obtain mean field convergence for large N to an Ordinary Differential Equation (ODE), it is sufficient to assume that (1) the intensity, i.e. the number of transitions per object per time slot, vanishes and (2) the coefficient of variation of the total number of objects that do a transition in one time slot remains bounded. No independence assumption is needed anywhere. We find convergence in mean square and in probability on any finite horizon, and derive from there that, in the stationary regime, the support of the occupancy measure tends to be supported by the Birkhoff center of the ODE. We use these results to develop a critique of the fixed point method sometimes used in the analysis of communication protocols.**

Consider a system of N objects; object n has a state $X_n^N(t) \in \{1, ..., I\}$. Time t is discrete. Objects interact with each other and with a resource; the state of the resource is $R^N(t) \in \{1, ..., J\}$. The global system $Y^N(t) = \left(X_1^N(t), ..., X_N^N(t), R^N(t)\right)$ is assumed to be a homogeneous Markov chain. We say that this system is a *Mean Field Interaction Model* if objects are observable only through their state. Formally, this means that the transition matrix of Y^N is invariant under permutations of the set of object labels $\{1, ..., N\}$. Note that we do not assume any form of independence; we allow multiple object transitions to depend on each other, and to depend on transitions of the resource.

We call *Occupancy Measure* $M^N(t)$ the vector whose ith component is the proportion of objects in state i at time t. We are interested in the scaling limit where $N \to \infty$ whereas I and J remain fixed. A number of papers find that, in the scaling limit, the occupancy measure can be approximated by a deterministic, usually non linear, dynamical system, called the mean field limit. The mean field limit is in discrete or continuous time, depending on how the model scales with the number of objects. More precisely, if the expected number of transitions per object per time slot remains constant when N grows, the limit is in discrete time [1]; if, in contrast, it tends to 0, the mean field limit is in continuous time; this is the case considered in this presentation.

Our goal is twofold: (1) find results that are widely applicable in practice, i.e. the model should be as little constrained as possible and (2) the technical assumptions should be reasonably simple to verify. In the presentation, we review a number of existing models. The model of Benaïm and Weibull [2] comes close to these goals, but its applicability is limited in some cases, as it does not allow a resource nor, for example, pairwise meetings of objects. The model of Bordenave, McDonald and Proutière [3] offers more expression power; it supports a resource, and this was used in [3] to provide the first mean field analysis of the 802.11 MAC protocol.

Still, the model in [3] has some limitations, which can be overcome by the results discussed in this presentation:

1) The assumptions required in [3] are complex, perhaps because [3] allows an infinite, enumerable state space for one object and for the resource. In contrast, we consider only the finite case.
2) Convergence to the mean field is established in [3] with compactness arguments typical of weak convergence over infinite horizons, which does not allow to make statements for the stationary regime other than in the case where the mean field limit has a unique global attractor. We use a method of proof inspired by the large body of results for stochastic approximation algorithms [4], [5], [6], [7] and find results generally valid for the stationary regime.
3) The model in [3] assumes that objects independently decide whether they will attempt to make a transition. This limits the applicability of their model; for example, in a pairwise interaction model, exactly two objects do a transition at every time slot; given that two objects have decided to do a transition, all other objects cannot. We find that this restriction is unnecessary; it appears that convergence to the mean field derives from exchangeability arguments and not from independence.

In this presentation we propose a generic mean field interaction model for N interacting objects and a resource. We show that, in addition to mild regularity assumptions that are trivial to verify, the only hypotheses we need are:

1) the intensity, i.e. the number of transitions per object per time slot vanishes,
2) the coefficient of variation of the total number of objects that do a transition in one time slot remains bounded.

We give several examples to illustrate the assumptions.

We show that there is convergence to a deterministic system, solution of an Ordinary Differential Equation (ODE). We also explain on one example how to derive the ODE in a straighforward manner. The convergence is in mean square and in probability over any finite horizon. For large N, the stationary distribution of the occupancy measure of all objects tends to be supported by the Birkhoff center of the ODE (introduced in the presentation).

We review the link between convergence to mean field, the mean field approximation and the decoupling assumption. If

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the ODE has a unique global attractor, we recover that the stationary distribution of the occupancy measure is concentrated at this attractor. However, we point to the well known fact that uniqueness of a stationary point of the ODE does not imply convergence to this stationary point, and develop from there a critique of the so-called fixed point method, sometimes used to analyze systems of interacting objects in stationary regime.

We call "mean field approximation" the independence assumption that is asymptotically true when N is large. This should not be confused with the approximation that consists in replacing a non-mean field interaction model with a mean field interaction model. This is also sometimes called the mean field approximation, as e.g. in [8].

The full text of this presentation can be found in [9].

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