# Framework for analysis of opportunistic schedulers: average sum rate vs. average fairness

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***Abstract*—Channel aware and opportunistic scheduling algorithms exploit the channel knowledge and fading to increase the average throughput. The maximum throughput scheduler (MTS) transmits only to the best user at a time. It is unfair to users at the cell edge. Obviously, there is a tradeoff between average throughput and fairness in the system. In this work, we study four representative schedulers, namely the MTS, the proportional fair scheduler (PFS), the (relative) opportunistic round robin scheduler (ORS), and the round robin scheduler (RRS). We show that the average sum rate performance and the average worst-case delay depend strongly on the user distribution within the cell. MTS gains from asymmetrically distributed users whereas the other three schedulers suffer. On the other hand, the average fairness of MTS and PFS decreases with asymmetrical user distribution. The key contribution of this paper is to put these tradeoffs and observations on a solid theoretical basis. The scaling laws of the average sum rate with the number of users as well as of the average worst-case delay are derived. Both the PFS and the ORS provide a reasonable performance in terms of throughput and fairness. However, PFS outperforms ORS for symmetrical user distributions, whereas ORS outperforms PFS for asymmetrical user distributions.**

## I. Introduction, recent results, and preliminaries

### A. Introduction and contributions

The optimal strategy for maximizing the sum capacity with perfect channel state information (CSI) of a cellular single-input single-output (SISO) multiuser channel is to allow only the user having the best channel conditions in terms of SNR to transmit at each time slot (TDMA). This result in [1] has induced the notion of multiuser diversity [2], i.e. the achievable capacity of the system increases with the number of users. The corresponding scheduling policy is called maximum throughput scheduler (MTS).

A major disadvantage of MTS is its unfairness against users at the cell edge. On the other hand, the most fair but channel unaware scheduler is the round robin scheduler (RRS) [3], that is, all transmissions take place in a strict cyclic order. In order to increase the fairness for users at the cell edge, the so called proportional fair scheduler can be applied. The proportional fair scheduler (PFS) weights the instantaneous transmission rates by their averages to find the best user[1] and achieves equal activity probability for all users [4]. Yet another scheduler, which is referred to as opportunistic round robin scheduling (ORS) was introduced in [5]. It is a combination of the RRS and MTS. The comparison of different schedulers with respect to different performance criteria is a highly viable research area. For example, in [6] the throughput guarantee violation probability is approximated and simulated for different schedulers in different channel models.

In order to quantitatively measure the impact of the scheduler on the fairness, different measures are proposed in the literature [7], [8], [9]. The Jain fairness index (JFI) defined in [7], also known as the Global Fairness Index (GFI) [10], provides a single number between zero and one that measures the fairness even for resource scheduling in finite windows. The average fairness defined in [8] is developed from an information-theoretic view. The worst-case delay as it is used in e.g. [9] measures the average number of transmissions needed until all users were active at least $m$ times.

Obviously, there exists a tradeoff between average throughput and average fairness. In this paper, we study this tradeoff for the four scheduling algorithms MTS, RRS, PFS, and ORS. The contributions of the paper are as follows: In Section I-E, closed-form expressions for the four schedulers for arbitrary user distributions are derived. The impact of the user distribution on the average sum rate is analyzed in Section II and it is shown that the average sum rate is increased with asymmetrical user distributions for MTS. For all other schedulers (RRS, PFS, and ORS) it decreases. Different fairness measures and their properties are discussed in Section III. Furthermore, we study the impact of the user distribution and its connection to the activity probabilities. The asymptotic performance for high SNR or large number of users is analyzed in Section IV. In section V, we illustrate the theoretical results with numerical single-cell multiuser simulations.

### B. System model

In the signal model, there are $K$ mobile users who are going to receive data from one base station. The single-antenna quasi-static block flat-fading channels $h_1, ..., h_K$ between the mobiles and the base are modeled as constant for a block of coherence length $T$ and from block to block as zero-mean independent complex Gaussian distributed with $\mathrm{CN}(0, c_k)$. The variance is $c_k = \mathbb{E}\left[|h_k|^2\right]$ for $1 \leq k \leq K$. The

[1] This is sometimes called normalized SNR scheduler.

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additive zero-mean white Gaussian noise $n_k(t)$, $1 \leq k \leq K$, at the receivers are independent identically distributed (iid) with variance $\sigma_n^2$ each. Furthermore, we assume that the sum transmit power is constrained to be $P$. The SNR is given by $\rho = \frac{P}{\sigma_n^2}$. The received signal at mobile $k$ at time $t$ is $y_k(t) = h_k \sum_{l=1}^{K} x_l(t) + n_k(t)$. In the following, we omit the time index for convenience. The statistics of the fading channel coefficients $h_k$ are completely characterized by their respective $c_k$. The transmit power directly corresponds to the variance of the transmit signals $p_k = \mathbb{E}\left[|x_k|^2\right]$ for $1 \leq k \leq K$. The $l_1$-norm of the power allocation vector $\mathbf{p} = [p_1, ..., p_K]$ is constrained to be $||\mathbf{p}|| = \sum_{k=1}^{K} p_k = P$. For $1 \leq k \leq K$ define $w_k$ by $||h_k||^2 = c_k w_k$, i.e. the $w_k$ are iid standard exponentially distributed random variables. We assume that the receivers have perfect CSI. Further on, we collect the channel states in a vector $\boldsymbol{h} = [h_1, ..., h_K]$.

In this work, we restrict our attention to a certain utility function namely the transmission rate. Another approach is to generalize to other utility functions of the type $U(x) = \frac{x^{1-\alpha}}{1-\alpha}$ where $x$ is the capacity share and $\alpha \geq 0$ is a free parameter. Our results correspond to the special case $\alpha = 0$.

### C. Measure of user distribution

The distance of the MS $k$ from the BS determines the average channel power $c_k$. In the following, we call the vector of average channel powers $\boldsymbol{c} = [c_1, ..., c_K]$, the user distribution. In order to guarantee a fair comparison between different user distributions, we constrain the sum variance to be equal to the number of users, i.e. $\sum_{k=1}^{K} c_k = K$. Without loss of generality, we order the users in a decreasing way according to their fading variances, i.e. $c_1 \geq c_2 \geq ... \geq c_K$. The constraint regarding the sum of the fading variances ensures that we compare scenarios in which the channel carries the same average sum power.

We need the following definitions [11]:

*Definition 1:* For two vectors $\boldsymbol{x}, \boldsymbol{y} \in R^n$ ordered in decreasing order one says that the vector $\mathbf{x}$ majorizes the vector $\mathbf{y}$ and writes $\mathbf{x} \succ \mathbf{y}$ if $\sum_{k=1}^{m} x_k \geq \sum_{k=1}^{m} y_k$ for $m = 1, ..., n-1$ and $\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k$. [2]

The next definition describes a function $\Phi$ which is applied to the vectors $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x} \succ \mathbf{y}$:

*Definition 2:* A real-valued function $\Phi$ defined on $\mathcal{A} \subset R^n$ is said to be *Schur-convex* on $\mathcal{A}$ if $\mathbf{x} \succ \mathbf{y}$ on $\mathcal{A}$ implies $\Phi(\mathbf{x}) \geq \Phi(\mathbf{y})$. Similarly, $\Phi$ is said to be *Schur-concave* on $\mathcal{A}$ if from $\mathbf{x} \succ \mathbf{y}$ on $\mathcal{A}$ follows $\Phi(\mathbf{x}) \leq \Phi(\mathbf{y})$.

Majorization is a useful tool to compare the impact of vectors which can be partially ordered. The common monotony properties of scalar functions corresponds to the Schur-convex property of vector functions. The reason why it is called Schur-convex and not Schur-monotone is that every symmetric and convex vector function is Schur-convex. Majorization is a large and active area of research in linear algebra, with entire books [11] devoted to its theory and application.

[2]Note that sometimes majorization is defined by the sum of the *smallest* $m$ components [12].

It is worth mentioning that majorization induces only a partial order on vectors with more than two components, i.e. not all possible vectors can be compared with each other. This is due to the fact that vectors with more than two components cannot be totally ordered. However, a sufficient number of vectors can be compared. Also, the extreme cases can be used for comparison with any other vector. For more information about this measure of user distribution and its application see [13, Section 4.2.1].

### D. High-SNR measures $\mathcal{S}_\infty$ and $\mathcal{L}_\infty$

The quantitative performance is analyzed using the high-SNR offset concept from [14]. Denote by $C(\rho)$ the average throughput as a function of the SNR. The two high-SNR measures are introduced as follows

$$\begin{aligned} \mathcal{S}_\infty &= \lim_{\rho\to\infty} \frac{C(\rho)}{\log(\rho)} \qquad \text{and} \\ \mathcal{L}_\infty &= \lim_{\rho\to\infty} \left(\log(\rho) - \frac{C(\rho)}{\mathcal{S}_\infty}\right). \end{aligned} \tag{1}$$

The measure $\mathcal{S}_\infty$ is called high-SNR slope and the measure $\mathcal{L}_\infty$ is called high-SNR power offset. At high SNR the average throughput behaves like $C(\rho) = \mathcal{S}_\infty \left(\frac{\rho[dB]}{3dB} - \mathcal{L}_\infty\right) + O(1)$. For convenience, these high-SNR measures are defined in 3-dB units. For further discussion, see [14, Section II]. These two high-SNR measures are useful if two systems are compared which differ either in their multiplexing gain, i.e. the slope of the average throughput curve at high SNR, or which have equal $\mathcal{S}_\infty$ but are shifted at high SNR.

### E. Types of (channel aware) scheduling

If perfect CSI is available at the base station, the sum rate is maximized by single-user transmission to the best user only [1], i.e. TDMA achieves the sum capacity. This result leads to the notion of multiuser diversity [2]. This scheduler is called MTS and the achievable average sum rate is given by

$$R_{sum}^{MT} = \mathbb{E}\left[\log\left(1 + \rho \max_{1\leq k\leq K} ||h_k||^2\right)\right]. \tag{2}$$

Note that the average sum rate of the MTS can be written in integral representation [15] as

$$R_{sum}^{MT} = \int_0^\infty \frac{\rho}{1+\rho t}\left[1 - \prod_{k=1}^{K}\left(1 - e^{-\frac{t}{c_k}}\right)\right] dt. \tag{3}$$

The case with symmetrically distributed users ($\boldsymbol{c} = \mathbf{1}$) has been derived in [16]. The MTS is unfair from a user-perspective because mobiles at the cell edge have less probability to be served.

The opposite type of scheduler is the round-robin scheduler (RRS). It is not channel aware. It minimizes the average worst-case delay, i.e. the average time until every user has been served at least once. Note that there exists also the weighted RRS that achieves max-min fairness in terms of throughput rather than in terms of transmission resource. The average sum rate of the unweighted RRS is given by

$$R_{sum}^{RR} = \mathbb{E}\left[\frac{1}{K}\sum_{k=1}^{K}\log\left(1+\rho||h_k||^2\right)\right]. \quad (4)$$

Note that (4) can be rewritten in closed form as

$$R_{sum}^{RR} = \frac{1}{K}\sum_{k=1}^{K}\mathrm{Ei}\left(1, \frac{1}{\rho c_k}\right)\exp\left(\frac{1}{\rho c_k}\right) \quad (5)$$

where the exponential integral is given by $\mathrm{Ei}(a,x) = \int_1^\infty \exp(-tx)t^{-a}dt$.

These two schedulers are the two most extreme cases. The MTS maximizes the average sum rate whereas the RRS minimizes the average worst-case delay. A compromise between the two is the proportional fair scheduler (PFS) [2]. For the analysis, we use the so called relative SNR scheduler. The user is served which has the highest ratio of the instantaneous SNR to average SNR. Hence, the achievable sum rate is given by

$$R_{sum}^{PF} = \mathbb{E}\left[\log\left(1+\rho||h_{k^*}||^2\right)\right] \text{ with} \quad (6)$$
$$k^* = \arg\max_{1\le k\le K}\frac{||h_k||^2}{c_k}.$$

Note that (6) can be rewritten as

$$R_{sum}^{PF} = \frac{1}{K}\sum_{k=1}^{K}\sum_{l=1}^{K}(-1)^{l-1}\binom{K}{l}\mathrm{Ei}\left(1, \frac{l}{\rho c_k}\right)e^{\frac{l}{\rho c_k}} \quad (7)$$

because the scheduling probability of all users is equal to $\frac{1}{K}$.

Another interesting channel aware scheduler is proposed in [5]. The one-round version [17] of the relative opportunistic round-robin scheduler (ORS) guarantees the same average worst case delay as the RRS but exploits a certain amount of multiuser diversity. It consists of $K$ rounds and initializes the set of available users $\mathcal{S}$ with $\mathcal{S} = \{1,...,K\}$. The relative best user $\max_{k\in\mathcal{S}}\frac{||h_k||^2}{c_k}$ out of the set of available users is picked and removed from the set within each step. After $K$ steps it is guaranteed that all users were active at least once. The sum rate performance is derived in [18, Eq. (8)]

$$\begin{aligned} R_{sum}^{OR} &= \frac{1}{K^2}\sum_{n=1}^{K}n\sum_{i=1}^{K}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j \\ &\cdot\frac{e^{\frac{1+j}{c_i}}}{1+j}\mathrm{Ei}\left(1,\frac{1+j}{c_i}\right). \end{aligned} \quad (8)$$

For our analysis, we prefer the representation in the following lemma.

*Lemma 3:* The average sum rate of the ORS (8) can be written as

$$\int_0^\infty\left[1-\frac{1}{K^2}\sum_{n=1}^{K}\sum_{i=1}^{K}\left(1-e^{-\frac{t}{c_i}}\right)^n\right]\frac{\rho}{1+\rho t}dt. \quad (9)$$

## II. Analysis of sum rate performance

The following result is proven in [19] and restated here for convenience. It states that a more asymmetrical user distribution increases the average sum rate with MTS.

*Theorem 4:* The average sum rate of the MTS is Schur-convex with respect to $\boldsymbol{c}$, i.e.

$$\boldsymbol{c}\succeq\boldsymbol{d}\Longrightarrow R_{sum}^{MT}(\boldsymbol{c})\ge R_{sum}^{MT}(\boldsymbol{d}). \quad (10)$$

The average sum rate of the RRS is Schur-concave with respect to the vector of average user powers $\boldsymbol{c}$, i.e.

$$\boldsymbol{c}\succeq\boldsymbol{d}\Longrightarrow R_{sum}^{RR}(\boldsymbol{c})\le R_{sum}^{RR}(\boldsymbol{d}). \quad (11)$$

The average sum rate of the PFS is Schur-concave with respect to the vector of average user powers $\boldsymbol{c}$, i.e.

$$\boldsymbol{c}\succeq\boldsymbol{d}\Longrightarrow R_{sum}^{PF}(\boldsymbol{c})\le R_{sum}^{PF}(\boldsymbol{d}). \quad (12)$$

The average sum rate of the ORS is Schur-concave with respect to the vector of average user power $\boldsymbol{c}$, i.e.

$$\boldsymbol{c}\succeq\boldsymbol{d}\Longrightarrow R_{sum}^{OR}(\boldsymbol{c})\le R_{sum}^{OR}(\boldsymbol{d}). \quad (13)$$

In conclusion, there is only one scheduler which improves for asymmetrically distributed users, namely the MTS. The average sum rates of the other schedulers, PFS, ORS, and RRS, decrease with more asymmetrically distributed user.

## III. Fairness analysis

### A. Analysis of average worst-case delay

In oder to capture the fairness of the different schedulers, the average worst-case delay is considered. The average worst-case delay $\mathbb{E}[D_{m,K}]$ measures the average number of transmissions that are needed until all $K$ users have been active at least $m$ times. We define $D_1 = \mathbb{E}[D_{1,K}]$.

The two most fair schedulers are the RRS and ORS. Both have an average worst-case delay of $mK$, because all users are guaranteed to be active within a block of $K$ transmissions. Especially, it takes $K$ transmissions until every user has transmitted exactly once, i.e.

$$D_1^{RRS} = D_1^{ORS} = K. \quad (14)$$

The PFS normalizes the users channels. Therefore, the probability of user $k$ being active is, independently of $k$, $1\le k\le K$, equal to $\frac{1}{K}$. Especially, it is independent of the user distribution $\boldsymbol{c}$. The result from [20] applies for $m=1$

$$D_1^{PFS} = K\int_0^\infty 1-(1-\exp(-x))^K dt \quad (15)$$

Note that (15) can be written as $D_1^{PFS} = K(\Psi(K+1)+\gamma)$ with the $\Psi$-function [21, 6.3] and Euler's constant $\gamma$ [21, 6.1.3].

The analysis of the MTS is more difficult. Rewrite the average worst-case delay [9, Section 3.3] without dropping probability as

$$D_1^{MTS} = n\int_0^\infty\left(1-\prod_{k=1}^{K}\left(1-\frac{\Gamma(m,d_kt)}{\Gamma(m)}\right)\right)dt. \quad (16)$$

For $m = 1$, the expression in (16) says how many packets are transmitted on average until every user has at least transmitted one. The coefficients $d_k$ in (16) are related to the probability that user $k$ is chosen $\pi_k = \frac{d_k}{K}$. For the MTS, we prove the following result.

*Theorem 5:* The average worst-case delay $\mathbb{E}[D_{1,K}]$ is Schur-convex with respect to $\boldsymbol{d}$, i.e.

$$\boldsymbol{d}_1 \succeq \boldsymbol{d}_2 \longrightarrow D_1^{MTS}(\boldsymbol{d}_1) \geq D_1^{MTS}(\boldsymbol{d}_2). \tag{17}$$

Theorem 5 formally states the intuitive fact that the average worst-case delay grows if some users are less frequently active on average. If the probability that user $k$ is active is equal to $\frac{1}{K}$, independently of $k$, then the expression in (16) is minimized. Note that a similar analysis has been performed in the different context of Birthday matching in [22].

### B. Jain's fairness index and dispersion

In [7], a quantitative measure of fairness is introduced. It is called Jain's fairness index (JFI) or Global Fairness Index (GFI) [10]. Define $x_k$ as the amount of a resource that is distributed to user $k$. Then, JFI is defined as [7, Eq. (2)] JFI $= \frac{\left(\frac{1}{K}\sum_{k=1}^{K} x_k\right)^2}{\frac{1}{K}\sum_{k=1}^{K} x_k^2}$. Let us specialize this general definition to the case in which one unit of resource is one transmission. The JFI is averaged over $L$ transmissions [18]

$$\text{JFI}(L) = \frac{\mathbb{E}_L\left(\frac{1}{K}\sum_{k=1}^{K} x_k\right)^2}{\mathbb{E}_L \frac{1}{K}\sum_{k=1}^{K} x_k^2}.$$

Denote by $\pi_k$ the probability that user $k$ is active within $L$ transmissions, then $x_k = \pi_k L$. Let $L \to \infty$ to obtain the long-term average JFI as JFI $= \frac{\left(\frac{1}{K}\sum_{k=1}^{K}\pi_k\right)^2}{\frac{1}{K}\sum_{k=1}^{K}\pi_k^2}$. Note that $\sum_{k=1}^{K}\pi_k = 1$ and this leads to the dispersion of $\boldsymbol{p}$,

$$\text{Dsp}(\boldsymbol{\pi}) = \frac{1}{\sum_{k=1}^{K}\pi_k^2} \tag{18}$$

Interestingly, this measure of fairness is closely related to Majorization theory.

*Theorem 6:* The dispersion is a Schur-concave function of the vector $\boldsymbol{\pi}$, i.e.

$$\boldsymbol{\pi}_1 \succeq \boldsymbol{\pi}_2 \Longrightarrow \text{Dsp}(\boldsymbol{\pi}_1) \leq \text{Dsp}(\boldsymbol{\pi}_2). \tag{19}$$

### C. Connection of user distribution, service probability, and delay

From the results in the last subsections follows that the impact of the user location on the different fairness measures depends on the resulting activity probability vector $\boldsymbol{p}$. Therefore, we have to map the user distribution vector $\boldsymbol{c}$ to the activity probability vector $\boldsymbol{\pi}$. The concrete mapping depends on the chosen scheduler. For PFS the activity probabilities of all users are equal to $\pi_k = \frac{1}{K}$ and independent of $\boldsymbol{c}$.

In order to apply Majorization theory to the analysis of the average worst-case delay as a function of the user distribution, we have to transfer the partial order for user distributions to the partial order for probability that a user $k$ is picked.

Define the vector of probabilities that user $k$ is picked $\boldsymbol{\pi} = [\pi_1, ..., \pi_K]$ as a function of the user distribution $\boldsymbol{c}$, i.e.

$$\pi_k(\boldsymbol{c}) = Pr\left[c_k w_k \geq \max_{l \neq k} c_l w_l\right] \tag{20}$$

Unfortunately, the next result is an impossibility result. It shows that it is not possible to say that if $\boldsymbol{c} \succeq \boldsymbol{d}$ then automatically $\boldsymbol{\pi}(\boldsymbol{c}) \succeq \boldsymbol{\pi}(\boldsymbol{d})$.

*Lemma 7:* The mapping from the vector of user distributions to the vector of service probabilities is not order preserving with respect to the partial order Majorization.

## IV. Asymptotic characterizations

In this section, we characterize the average sum rate of the different scheduling schemes for high SNR or for a large number of users. The scaling laws of the schemes are derived as a function of the user distribution.

### A. High SNR behavior

The high SNR slope $\mathcal{S}_\infty$ as defined in (1) for all four scheduling schemes is equal to one because

$$\mathcal{S}_\infty = \lim_{\rho\to\infty} \frac{\int_0^\infty \log(1+\rho x) pdf(x) dx}{\log(\rho)} = 1. \tag{21}$$

It is allowed to swap integration and limit by applying the Dominated Convergence Theorem. In general, any TDMA scheme could have at most a high SNR slope of one. The high SNR power offset is different for the four schedulers, which is shown in the following.

*Theorem 8:* For MTS, the maximum and minimum high SNR power offsets are given by

$$\begin{aligned} \max({\mathcal{L}_\infty}^{MT}) &= \gamma - \log(K) & (22)\\ \min({\mathcal{L}_\infty}^{MT}) &= \gamma + \sum_{k=1}^{K}(-1)^{k-1}\binom{K}{k}\log(k). \end{aligned}$$

For RRS, the high SNR power offset as a function of the user distribution is given by

$${\mathcal{L}_\infty}^{RR}(\boldsymbol{c}) = \frac{1}{K}\sum_{k=1}^{K}\gamma - \log(c_k). \tag{23}$$

For PFS, the high SNR power offset as a function of the user distribution is given by

$${\mathcal{L}_\infty}^{PF}(\boldsymbol{c}) = \gamma - \frac{1}{K}\sum_{k=1}^{K}\sum_{l=1}^{K}(-1)^{l-1}\binom{K}{l}\log\left(\frac{l}{c_k}\right). \tag{24}$$

For ORS, the high SNR power offsets as a function of the user distribution is given by

$$\begin{aligned} {\mathcal{L}_\infty}^{OR}(\boldsymbol{c}) &= \frac{1}{K^2}\sum_{n=1}^{K} n \sum_{k=1}^{K}\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{(-1)^j}{1+j}\cdot \\ &\quad \cdot\left(\gamma + \log\left(\frac{1+j}{c_k}\right)\right). \end{aligned} \tag{25}$$

The proof of Theorem 8 follows similar lines as in [23, Theorem 2] and is omitted. Note that the Schur-convexity of (23) can be directly observed and this proves the result in (11). However, in (24) and (25) the Schur-convexity cannot be directly observed because of the alternating sum.

The high-SNR power offsets obey the following inequality chain ${\mathcal{L}_\infty}^{MT} \leq \{{\mathcal{L}_\infty}^{PF}, {\mathcal{L}_\infty}^{OR}\} \leq {\mathcal{L}_\infty}^{RR}$. The order of PFS and ORS depends on the correlation scenario. Note that the average worst-case delay does not scale with the SNR.

### *B. Scaling with number of users*

First, consider the case in which the users are symmetrically distributed, i.e. $\boldsymbol{c} = \mathbf{1}$. The scaling behavior with $K \to \infty$ for fixed SNR $\rho$ can be easily shown by considering a simple upper and lower bound on the average sum rate. The average sum rate of RR does not scale with $K$ at all. The results can be compared to [24].

*Lemma 9:* For symmetrically distributed users $\boldsymbol{c} = \mathbf{1}$, the average sum rate of MTS, PFS, and ORS scale for large $K$ with $\log(\log(K))$, i.e.

$$\lim_{K\to\infty} \frac{R_{sum}^{MT}(K)}{\log\log(K)} = \lim_{K\to\infty} \frac{R_{sum}^{PF}(K)}{\log\log(K)} = \lim_{K\to\infty} \frac{R_{sum}^{OR}(K)}{\log\log(K)} = 1. \tag{26}$$

The case in which the users are not symmetrically distributed is discussed in the numerical results section.

The scaling of the average worst-case delay with the number of users is also of interest. Then next lemma gives the scaling for the case in which the users are symmetrically distributed. It follows directly from (14) and (**??**).

*Lemma 10:* For symmetrically distributed users, the average worst-case delay scales linearly with $K$ for RRS and ORS. For MTS and PFS, it scales as $K\log(K)$, i.e.

$$\lim_{K\to\infty} \frac{D_1^{RRS}(K)}{K} = \lim_{K\to\infty} \frac{D_1^{ORS}(K)}{K} = 1$$
$$\lim_{K\to\infty} \frac{D_1^{MTS}(K)}{K\log(K)} = \lim_{K\to\infty} \frac{D_1^{PFS}(K)}{K\log(K)} = 1. \tag{27}$$

## V. Illustrations

In this section, we present illustrations which validate and explain the theoretical results from the last sections. The performance for the case with symmetrically distributed users $\boldsymbol{c} = \mathbf{1}$ is compared to the case with asymmetrically distributed users. For the asymmetric user distribution, we choose an exponential decaying model $c_k = \exp(-tk)$ and normalize $\sum_{k=1}^{K} c_k = K$. Note that this does not model the path losses of the users. Each decay model corresponds to a SNR or user distribution scenario, e.g. the flat model $c_k = 1/K$ corresponds to the case where the SNR values are lined up equidistantly. In the numerical simulations we set $K = 20$ and $t = 0.2$, for each data point 100000 Monte Carlo runs are performed to compute the averages.

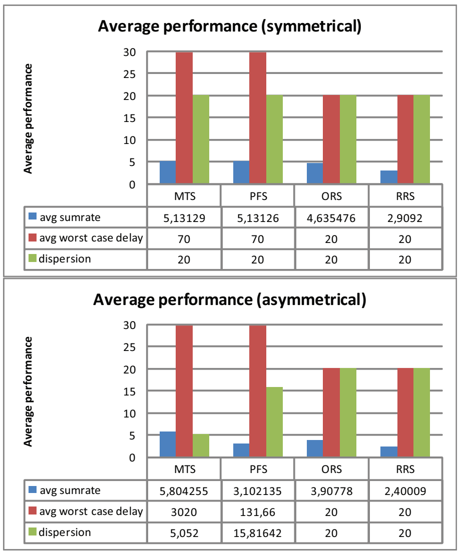


Fig. 1. Average sum rate, worst-case delay, and dispersion for $K = 20$ symmetrically and asymmetrically distributed users.

### *A. General results*

In Figure 1, the average sum rate, the average worst-case delay, and the dispersion are shown for the four studied schedulers. In the upper figure, the users are symmetrically distributed, i.e. $\boldsymbol{c} = \mathbf{1}$, whereas in the lower figure, the users are asymmetrically distributed with $t = 0.2$. The results in Figure 1 illustrate the following observations: The average sum rate of MTS increases with more asymmetrically distributed users (compare to equation (10)), while the average sum rate of all three other schedulers decreases (compare to equations (11), (12), and (13)). However, PFS outperforms ORS for the symmetrical scenario whereas it is the other way round for the assymetrical scenario. Another observation is that the average worst-case delay is more differentiated than the dispersion. This underlines that the average worst-case delay is better suited for fairness analysis than the JFI-based dispersion. Finally, the average worst-case delay for the asymmetrical scenario of the PFS and ORS tends to grow without bound. Therefore, taking the tradeoff between fairness and average sum rate into account, the PFS and ORS perform reasonably well. PFS is advantageous in symmetric scenarios whereas ORS performs better in asymmetric scenarios.

### *B. Scaling with number of users*

In figures (2) and (3), we show the average performance of the four scheduling algorithms for symmetrically distributed as well as asymmetrically distributed users. The derived scaling laws in equations (26) and (27) are confirmed. The interesting observation is that for the asymmetrical case, PFS outperforms OFS for a small number of users whereas it is the other way round for large number of users.

The average worst case delay for MTS and PFS increases with asymmetrical user distribution as predicted in Theorem 5. As soon as a single $c_k$ approaches zero, the average

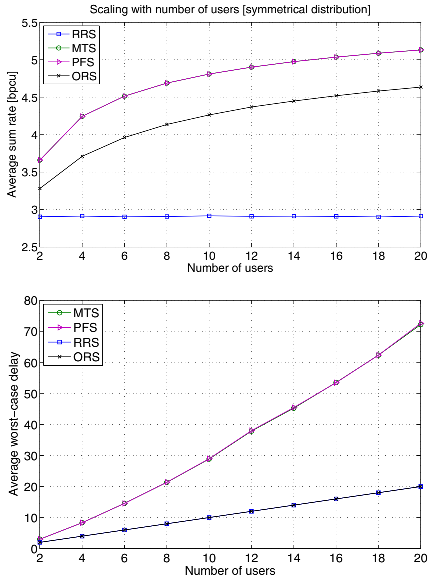


Fig. 2. Average sum rate and worst-case delay over number of users for symmetrically distributed users.

worst-case delay approaches infinity. The round-robin-based schedulers RRS and ORS are robust against the asymmetrical user distribution.

The main observation in this section is that for practical scenarios in which fairness is important as well as users are randomly distributed within the cell, ORS clearly outperforms PFS. Note that the results presented here hold for a static scenario in which we place the users only once inside the cell and simulate the small-scale fading. Mobility as well as traffic models are left for further research.

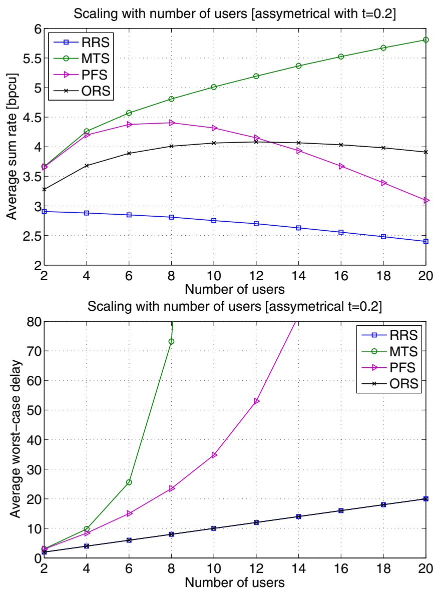


Fig. 3. Average sum rate and worst-case delay over number of users for asymmetrically distributed users.

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