Complex Synchronizations in Markovian Models: a Tensor-based Proof of Product Form

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ABSTRACT

We consider complex synchronizations in a generalized network of queues with signals (Gnetwork) or Stochastic Automata Networks without functions. Both models allow to describe their continuous time Markov chain as a summation of tensor (or Kronecker) products and sums of local description of queues (or automata). We give a purely algebraic proof of the product form results based on properties of the tensor products. These results generalizes many well-known results in queueing theory but also on all the models which allow a tensor based representation such as Stochastic Petri Nets or Stochastic Process Algebra.

Categories and Subject Descriptors

C.4 [**Performance of Systems**]: Modeling techniques; Performance attributes; G.3 [**Probability and Statistics**]: Markov processes

Keywords

Product form, Markovian Networks of Queues, Tensor Algebra

1. INTRODUCTION

In the last decade, the component based approach in performance modelling has received considerable attention. Many High Level formalisms have been developed and the translation into multidimensional Continuous Time Markov Chain (CTMC in the following) was in general proved to analyse quantitative properties of systems while qualitative ones are studied neglecting the timing information in the model (see for instance the various Stochastic Process Algebras which have been proposed). Performance analysis tools have been developed and they generally allow to perform simulations and numerical analysis.

Most of these High Level formalisms translation into DTMC are based on tensor product and sums (see a formal definition in the next section). For instance Plateau [18] has

proved that the transition rate matrix of a CTMC associated to a Stochastic Automata Network (SAN) is the summation of the tensor products and sums of small matrices which describe the automata:

$$Q = \oplus_{l=1}^{n} \mathbf{L_l} + \sum_{r=1}^{s} (\otimes_{i=1}^{n} \mathbf{M_i^r} - \mathbf{N^r}), \qquad (1)$$

n is the number of automata, s is the number of synchronizations, $\mathbf{L_l}$, $\mathbf{M_i^r}$ and $\mathbf{N^r}$ are description matrices of the automata and $\otimes$ and $\oplus$ denote the tensor product and tensor sum. This tensor representation proved for SAN has been generalized to Stochastic Petri Nets [6] and Stochastic Process Algebra [16]. We can also describe Markovian Queueing Networks with such a tensor representation and many models in the literature are described by tensors and matrices. However this representation is only used to present the models and the analysis does not in general take advantage of the algebraic properties of these operators.

Recently, the tensor representation was also associated to analytical closed-form solutions such as product form. The main idea is the simple fact that a product form distribution is the tensor product of marginal distributions. Therefore as the solution and the transition rate matrix are both described by tensors products and sums, it may be possible to verify the balance equations using properties of tensor algebra. This approach has been used in [10, 11, 8] to find a very simple algebraic sufficient condition for product form of steady-state solution of continuous-time or discrete-time SAN without synchronizations where interactions are limited to functional rates. This condition relies on the existence of a common vector in the kernel of all the matrices obtained when the functional rates change. The results obtained by this approach generalize Boucherie's theory of Markov chains in competition [1] and some of Robertazzi's theorems on Petri nets [19].

Furthermore, we have proved in [4] that we can generalize many results on the product form of open queueing networks using their tensor representation and algebraic arguments. In classical Markovian queueing network theory, the synchronization between the components is due to the customer movement. This is denoted as Master/Slave synchronization. The Master queue sends a customer to the Slave queue. The approach in [4] allows to prove algebraically well-known results such as Jackson network or queues with catastrophes and many recent theorems on generalized network of queues with negative customers [14] or resets [13]. Generalized networks of queues (Gnetworks for short) have been introduced in [14] and [2]. In a Gnetwork, customers can become a sig-

nal when they move from one queue to another one. The signals visit queue, they do not require service but they have an immediate action upon the queue. A signal may disappear if some conditions are fulfilled at the queue it is visiting or it may continue its visit to another queue. Therefore Gnetwork with signals are associated to much more complex synchronizations between queues. In [9] we have shown that the same algebraic techniques we have developed for simple synchronization such as Master/Slave also allows to prove that networks with a complex synchronization denoted as a Domino has a product form steady-state distribution. The domino synchronization is described in section 5.

Here we consider a much more complex set of synchronizations between an arbitrary number of automaton (or equivalently a signal propagating among an arbitrary number of queues with an arbitrary effect on each of them). The propagation of the signal may depend on the state of the queue it is visiting. Using the same kind of algebraic arguments we prove a sufficient condition for the Markovian model to have a product form steady-state distribution. We also show that this result generalizes many known results in queueing theory. We present the models using Stochastic Automata Network of Markovian networks of queues with signals. Both formalisms are equivalent. The real property we use is the representation of the transition rate matrix using tensor product (i.e. refer to Eq. 1).

The rest of the paper is as follows. First in section 2 we give a brief description of tensor (or Kronecker) operators on matrices and we give a first result on networks with jump over blocking to introduce the technique and show that algebraic techniques provide a more general sufficient condition than the one which were previously known. Then in section 3 we present the notation and concepts about synchronizations. A synchronization is defined as an ordered list of automaton (or queue visited by the signal). Section 4 is devoted to the main theorem on the synchronizations with product form. We give two proofs of the result. The first one (the simplest) assume that there is no duplication in the list while the second proof does not require such an assumption. This proof is postponed into an appendix for the sake of readability. Finally in section 5 we present many examples to illustrate the approach.

2. KRONECKER ALGEBRA AND PRODUCT FORM

In this paper, we restrict ourself to continuous-time SAN without functions. The generator is based on the tensor sum and product of local components. Recall that with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

the tensor product $A \otimes B$ is:

$$\begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{13} \\ a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} \\ a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} & a_{12}b_{31} & a_{12}b_{32} & a_{12}b_{33} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{13} \\ a_{21}b_{21} & a_{21}b_{22} & a_{21}b_{23} & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} \\ a_{21}b_{31} & a_{21}b_{32} & a_{21}b_{33} & a_{22}b_{31} & a_{22}b_{32} & a_{22}b_{33} \end{pmatrix},$$

and the tensor sum $A \oplus B$ is given by: $A \otimes I_B + I_A \otimes B$ where I_A and I_B are Identity matrices whose size is respectively equal to the size of A and B. The tensor product and sums

have many algebraic properties (see [7] or [17] for proofs). We only give some of them in the following for the sake of completeness as they are used in the proofs. Note that these properties have also been used in [10, 11, 8, 4] to prove product form steady-state distributions for other types of SAN. The key property is the fact that a product form solution of n distributions $(\pi_l)_{l=1..n}$ is a tensor product: i.e. it is equal to $C\pi_1 \otimes \pi_2 \otimes \ldots \otimes \pi_n$ where C is a normalization constant.

PROPERTY 2.1 (BASIC PROPERTIES OF TENSOR). *Let A, B, C, and D be arbitrary matrices, the following properties hold:*

- *Associativity: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.*

- *Distributivity over Addition:*

$$(A + C) \otimes (B + D) = A \otimes B + A \otimes D + C \otimes B + C \otimes D.$$

- *Compatibility with matrix multiplication: For all vectors π_A and π_B whose sizes are consistent we have:*

$$(\pi_A \otimes \pi_B)(A \otimes B) = (\pi_A A) \otimes (\pi_B B).$$

We assume that the state space is the Cartesian product of the states of the components (i.e. all the states are reachable). The reachable state space is in general only a subset of this Cartesian product.

Let us now give some notation and definitions on the matrices used to design the example of models.

Id_n the identity matrix of size n,

Up_n the matrix of size n full of 0 except the main upper diagonal which is 1, this matrix is used to represent arrival of customers,

Low_n the matrix of size n full of 0 except the main lower diagonal which is 1, this matrix is used to represent departure of customers,

I_n size n, all elements equal to 0 except $I_n(n,n) = 1$,

I_n^0 size n, all elements equal to 0 except $I_n^0(1,1) = 1$.

Note that we accept the case where $n = \infty$. In that case, the index n will be omitted.

DEFINITION 2.2. *Let M be a square matrix. The matrix $\sigma(M)$ is a diagonal matrix with the size of M such that for all index i, $\sigma(M)[i,i] = \sum_j M[i,j]$.*

Let us now prove that the basic properties of tensor are sufficient to prove that the steady-state solution of a model described by Kronecker products has product form.

2.1 Three queues with jump over blocking

Before giving a general and complicated definition of the synchronization with multiple steps in section 3, let us begin with a simpler example to illustrate: the jump over blocking queue.

Consider a network consisting of 3 queues: $Q1,Q2$, and $Q3$. Queue $Q2$ has a finite buffer with size K. Arrivals to queue $Q1$ follow independent Poisson processes. The other queues do not receive customers from the the outside. Services have exponential durations and they are independent. Customers leaving $Q3$ depart from the network. The arrival

rate from the outside to queue $Q1$ is λ_1. The service rate at queue i is μ_i. At the completion of its service in queue $Q2$, a customer jumps to queue $Q3$. When its service at queue $Q1$ is completed, a customer jumps to queue $Q2$ if there is a place available in the buffer; otherwise it jumps to queue $Q3$. Such a network is known to have a product form solution [2, 5].

To give a completely algebraic proof, we do the following. We do not take into account the exact representation of an arrival or a service but we describe them as transition rate matrices. More precisely, arrivals in queue $Q1$ and service in $Q3$ are local transitions and they are respectively represented by transition rate matrices L_1 and L_3. A customer movement between queue $Q2$ and queue $Q3$ is described by matrix D_2 for queue $Q2$ and transition probability matrix S_1 for $Q3$. D_2 is a transition rate matrix without diagonalisation (i.e. $\sigma(D_2) \neq 0$). When a customer succeeds to move from Q_1 to Q_2, its behaviour is modelled by matrix D_1 (with $\sigma(D_1) \neq 0$) for queue $Q1$ and transition probability matrix S_2 for queue $Q2$. When the movement is refused by queue $Q2$ and the customer moves from $Q1$ to $Q3$, the model is represented by D_1 on queue $Q1$, S_4 for $Q2$ and S_3 for queue $Q3$. Thus the transition rate matrix of the CTMC is:

$$
\begin{aligned}
Q \quad = \quad & L_1 \oplus L_3 \\
+ \quad & Id \otimes D_2 \otimes S_1 - Id \otimes \sigma(D_2) \otimes \sigma(S_1) \\
+ \quad & D_1 \otimes S_2 \otimes Id - \sigma(D_1) \otimes \sigma(S_2) \otimes Id \\
+ \quad & D_1 \otimes S_4 \otimes S_3 - \sigma(D_1) \otimes \sigma(S_4) \otimes \sigma(S_3).
\end{aligned}
$$

In the "language" of synchronization, in this example, a synchronization r may have 2 or 3 steps. At each step, an automaton will be effected by a matrix (an automaton does not appear in the synchronization associated to the Identity matrix). Let us give the detail in the following tabular

$r = (Q2, Q3)$		Q2	Q3
M		D_2	S_1
$r = (Q1, Q2)$	Q1	Q2	
M	D_1	S_2	
$r = (Q1, Q2, Q3)$	Q1	Q2	Q3
M	D_1	S_4	S_3

To illustrate, the explicit representing of the matrices will be given after the algebraic proof.

THEOREM 2.3. *Assume that Q is ergodic and that we have $S_4 = \sigma(S_4)$ and $\sigma(S_2) + \sigma(S_4) = I$. Let X_i the size of queue indexed by i. Assume that the following conditions hold:*

$$
\begin{cases}
g_1 D_1 = \Gamma^{12} g_1, \\
g_2(D_2 + \Gamma^{12} S_4) = \Gamma^{23} g_2, \\
g_1 H_1 = 0, \quad g_2 H_2 = 0, \quad g_3 H_3 = 0, \\
g_3 S_1 = g_3 S_3,
\end{cases}
$$

where

$$
\begin{cases}
H_1 = L_1 + D_1 - \sigma(D_1), \\
H_2 = D_2 - \sigma(D_2) + \Gamma^{12}(S_2 - \sigma(S_2)), \\
H_3 = L_3 + \Gamma^{23}(S_1 - \sigma(S_1)),
\end{cases}
$$

then the steady-state solution has product form:

$$
Pr(X_1, X_2, X_3) = C \prod_{i=1}^{3} g_i(X_i).
$$

We just give here the proof to highlight the tensor properties we need. Assume that the steady-state solution has

product form and consider $Pr(X_1, X_2, X_3)Q$. We will verify that the product is equal to zero. Because of the distributivity and the compatibility with ordinary product we have:

$$
\begin{aligned}
(g_1 \otimes g_2 \otimes g_3)Q \quad = \quad & g_1 L_1 \otimes g_2 \otimes g_3 \\
+ \quad & g_1 \otimes g_2 \otimes g_3 L_3 \\
+ \quad & g_1 \otimes g_2 D_2 \otimes g_3 S_1 \\
- \quad & g_1 \otimes g_2 \sigma(D_2) \otimes g_3 \sigma(S_1) \\
+ \quad & g_1 D_1 \otimes g_2 S_2 \otimes g_3 \\
- \quad & g_1 \sigma(D_1) \otimes g_2 \sigma(S_2) \otimes g_3 \\
+ \quad & g_1 D_1 \otimes g_2 S_4 \otimes g_3 S_3 \\
- \quad & g_1 \sigma(D_1) \otimes g_2 \sigma(S_4) \otimes g_3 \sigma(S_3).
\end{aligned}
$$

Remember that $g_1 D_1 = \Gamma^{12} g_1$ and that $g_3 S_1 = g_3 S_3$. Note that $\sigma(S_1) = \sigma(S_3) = I$. Substitute into the equation to obtain:

$$
\begin{aligned}
(g_1 \otimes g_2 \otimes g_3)Q \quad = \quad & g_1 L_1 \otimes g_2 \otimes g_3 \\
+ \quad & g_1 \otimes g_2 \otimes g_3 L_3 \\
+ \quad & g_1 \otimes g_2 D_2 \otimes g_3 S_1 \\
- \quad & g_1 \otimes g_2 \sigma(D_2) \otimes g_3 \\
+ \quad & g_1 \otimes g_2 \Gamma^{12} S_2 \otimes g_3 \\
- \quad & g_1 \sigma(D_1) \otimes g_2 \sigma(S_2) \otimes g_3 \\
+ \quad & g_1 \otimes g_2 \Gamma^{12} S_4 \otimes g_3 S_1 \\
- \quad & g_1 \sigma(D_1) \otimes g_2 \sigma(S_4) \otimes g_3.
\end{aligned}
$$

Factorize the 6th with the 8th terms of right hand side, do the same for the 3rd and the 7th terms. Remember that $\sigma(S_2) + \sigma(S_4) = I$ to get:

$$
\begin{aligned}
(g_1 \otimes g_2 \otimes g_3)Q \quad = \quad & g_1 L_1 \otimes g_2 \otimes g_3 \\
+ \quad & g_1 \otimes g_2 \otimes g_3 L_3 \\
+ \quad & g_1 \otimes g_2(D_2 + \Gamma^{12} S_4) \otimes g_3 S_1 \\
- \quad & g_1 \otimes g_2 \sigma(D_2) \otimes g_3 \\
+ \quad & g_1 \otimes g_2 \Gamma^{12} S_2 \otimes g_3 \\
- \quad & g_1 \sigma(D_1) \otimes g_2 \otimes g_3.
\end{aligned}
$$

Now we take into account that $g_2(D_2 + \Gamma^{12} S_4) = g_2 \Gamma^{23}$. We substitute in the equation and we factorize the second and the third terms, and the first and the last terms:

$$
\begin{aligned}
(g_1 \otimes g_2 \otimes g_3)Q \quad = \quad & g_1(L_1 - \sigma(D_1)) \otimes g_2 \otimes g_3 \\
+ \quad & g_1 \otimes g_2 \otimes g_3(L_3 + \Gamma^{23} S_1) \\
- \quad & g_1 \otimes g_2 \sigma(D_2) \otimes g_3 \\
+ \quad & g_1 \otimes g_2 \Gamma^{12} S_2 \otimes g_3
\end{aligned}
$$

We add and we subtract $g_1 D_1 \otimes g_2 \otimes g_3$ and we note that $g_1 D_1 \otimes g_2 = g_1 \Gamma^{12} \otimes g_2 = g_1 \otimes g_2 \Gamma^{12}$. Similarly we add and we subtract $g_1 \otimes g_2(D_2 + \Gamma^{12} S_4) \otimes g_3$ and we take into account that $g_2(D_2 + \Gamma^{12} S_4) \otimes g_3 = g_2 \otimes \Gamma^{23} g_3$. After substitution and factorization we get:

$$
\begin{aligned}
(g_1 \otimes g_2 \otimes g_3)Q \quad = \quad & g_1(L_1 + D_1 - \sigma(D_1)) \otimes g_2 \otimes g_3 \\
+ \quad & g_1 \otimes g_2 \otimes g_3(L_3 + \Gamma^{23}(S_1 - I)) \\
+ \quad & g_1 \otimes g_2 G \otimes g_3
\end{aligned}
$$

where $G = (D_2 - \sigma(D_2) + \Gamma^{12}(S_2 + S_4 - I)$. Finally some terms cancel because $I = \sigma(S_2) + \sigma(S_4)$ and $S_4 = \sigma(S_4)$. We recognize matrices M_1, M_2 and M_3. Thus,

$$
\begin{aligned}
(g_1 \otimes g_2 \otimes g_3)Q \quad = \quad & g_1 H_1 \otimes g_2 \otimes g_3 \\
+ \quad & g_1 \otimes g_2 \otimes g_3 H_3 \\
+ \quad & g_1 \otimes g_2 H_2 \otimes g_3
\end{aligned}
$$

Therefore $(g_1 \otimes g_2 \otimes g_3)Q = 0$. The solution has product form.

Let us discuss about the above assumptions. Matrices H_1, H_2 and H_3 may be seen as the total transition rate matrix

associated to $Q1$, $Q2$ and $Q3$ in isolation. These generators take into account the local transitions (matrices L_i), the emission of signals (matrices D_i) and the effect of signals (matrices S_i). The eigenvalues Γ^{12} and Γ^{23} can be consider as the total arrival rate to queues $Q2$ and $Q3$

Now let us turn back to the network of queue and give the representation of the matrices used to describe the ordinary queueing model:

$$L_1 = \lambda_1 Up - \sigma(\lambda_1 Up), \ \ L_2 = 0, \ \ L_3 = \mu_3 Low - \sigma(\mu_3 Low),$$
$$D_1 = \mu_1 Low, \quad D_2 = \mu_2 Low_K, \quad S_1 = Up,$$
$$S_2 = Up_K, \quad S_3 = Up, \quad S_4 = I_K.$$

Clearly the assumptions of the theorem are satisfied. Solving the system, one obtains that g_i has geometric distribution with rate ρ_i given by

$$\rho_1 = \frac{\lambda_1}{\mu_1}, \qquad \rho_2 = \frac{\Gamma^{12}}{\mu_2}, \qquad \rho_3 = \frac{\Gamma^{23}}{\mu_3},$$

where $\Gamma^{12} = \mu_1 \rho_1 (= \lambda_1)$ and $\Gamma^{23} = \mu_2 \rho_2 (= \lambda_1)$.

Note however that the theorem and the proof is more general than the well known result of queues with jump over blocking. We only require that $g_3 S_1 = g_3 S_3$ while the classical results require that the effect of a customer who jumps over $Q2$ is identical to the effect of a customer completing its service at $Q2$. One can restate this last condition as $S_1 = S_3$. Clearly the algebraic proof can deal with more abstracted behaviour of customers and signals. It also allows to obtain weaker sufficient conditions.

3. MULTISTEP SYNCHRONIZATION

We now consider synchronizations with multiple steps. A synchronization takes place according to the order of a list of automata or queues. For instance we can consider networks with finite buffers or networks with signals which trigger customers.

The generator is given by:

$$Q = \oplus_{i=1}^{N} F_i + \sum_{r \in \mathcal{S}} \otimes_{i=1}^{N} M_i^r, \tag{2}$$

where $\mathcal{S}$ is a set of all synchronizations. F_i denotes the local transition in automaton i and does not contain normalization. Here we slightly change the description of the model to simplify the proofs. We have $L_i = F_i - \sigma(F_i)$. Without loss of generality we now do not suppose that matrices F_i and Q have the sum of each line equal to 0 (we do not take into account the normalization term in the matrix). To prove that the Markov chain modelling the queueing network or the SAN has a steady-state distribution π, it is equivalent to have $\pi Q = \pi \sigma(Q)$.

Let us now detail all the possible synchronizations. Synchronization r consists a list of pairs of an automaton and a type:

$$\text{List}(r) = \{(l_1, t_1), (l_2, t_2), \cdots, (l_{n(r)}, t_{n(r)})\}, \tag{3}$$

where l_i is an automaton and t_i is a type in set $\mathcal{T}$ and suppose that $|\text{List}(r)| \geq 2$. Here, for a multistep synchronization r, there are $n(r)$ consecutive steps (or $n(r)$ automatons (queues) will be effected). At step i, the synchronization r is at automaton l_i and the "action" type to l_i is t_i (in the example at the end of this section, t_i can be an "arrival" or a "departure" or a "signal"). For synchronization r associated

to a list given in (3), let us define:

$$\text{List}^1(r) = \{l_1, l_2, \cdots, l_n\}.$$

Furthermore, define $\text{Order}(i, r)$ the sequence in order of automaton i's appearance in $\text{List}(r)$ ($\text{List}^1(r)$), or

$$\text{Order}(i, r) = \{\alpha_1, \alpha_2, \cdots, \alpha_n \mid i_j < i_{j+1}\}$$

satisfies $\text{List}^1(r)(k) = (i)$ iff $k \in \text{Order}(i, r)$.

Matrices M_i^r are determined as follows:

1. If $\text{Order}(i, r) = \emptyset$ ($i \notin \text{List}^1(r)$, or the automaton i does not appear in the synchronization r)then $M_i^r = Id$.

2. If $\text{Order}(i, r) = \{\alpha_1, \cdots, \alpha_n\}$ (the automaton i appears n times in the synchronization r) then

$$M_i^r = M_i^r(1) M_i^r(2) \cdots M_i^r(n), \tag{4}$$

where $M_i^r(k)$ is defined by:

$$\begin{cases} E^{(l_{\alpha_k}, t_{\alpha_k}), (l_{\alpha_k+1}, t_{\alpha_k+1})}, & 1 < k < n; \\ E^{(l_{\alpha_1}, t_{\alpha_1}), (l_{\alpha_1+1}, t_{\alpha_1+1})}, & k = 1, i \neq l_1; \\ E^{(l_{\alpha_n}, t_{\alpha_n}), (l_{\alpha_n+1}, t_{\alpha_k+1})}, & k = n, i \neq l_{n(r)}; \\ E_0^{(l_{n(r)}, t_{n(r)})}, & k = n, i = l_{n(r)}; \\ E_0^{(l_1, t_1), (l_2, t_2)}, & k = 1, i = l_1; \end{cases}$$

All matrices E are positive. Figure 1 illustrates a synchronization with $n(r)$ steps and the corresponding matrices.
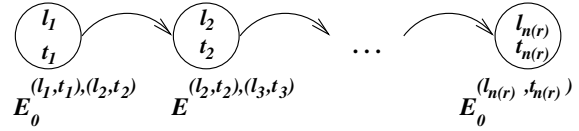

Figure 1: A synchronization r with $n(r)$ steps

Let us define:

$$\text{First} = \{t \in \mathcal{T} \mid E_0^{(j,t)} = E^{(j,t),(i,u)} = 0, \forall j, i, u\} \tag{5}$$

the set of all types which can be present only at the first step of a synchronization. Assume that the set $\mathcal{T} \setminus \text{First}$ can be decomposed in T disjoint subsets

$$\mathcal{T} \setminus \text{First} = \cup_{a=1}^{T} \mathcal{T}^a$$

which satisfies for all $v \in \mathcal{T}^a$:

$$E^{(i,u),(j,v)} = E^{(i,u),(j,\mathcal{T}^a)}, \ \ E_0^{(i,u),(j,v)} = E_0^{(i,u),(j,\mathcal{T}^a)}. \tag{6}$$

Equation (6) states that the step before a step of all type in $\mathcal{T}^a$ have the same matrix. In general, $|\mathcal{T}^a| = 1$. However, in some special cases, for example the jump over blocking network previously studied, $|\mathcal{T}^a| > 1$.

Let us define now $E^{(j,v)} = E_0^{(j,v)} + \sum_{k,a} E^{(j,v),(k,\mathcal{T}^a)}$. This is the "total action" applied on automaton j with type v: it is the sum of a synchronization's last step and all intermediate steps from automaton j with type v to another automaton k with a type in $\mathcal{T}^a$. Define also

$$E^{(j,\mathcal{T}^a)} = \sum_{v \in \mathcal{T}^a} E^{(j,v)}$$

the "total action" upon automaton j with type in $\mathcal{T}^a$. Suppose that

$$\sigma(E^{(j,\mathcal{T}^a)}) = Id. \qquad (7)$$

Define

$$
\begin{aligned}
E^{(i,u),j} &= \sum_v \mathbb{1}_{E^{(i,u),(j,v)} \neq 0} E^{(j,v)}, \\
E_0^{(i,u),j} &= \sum_v \mathbb{1}_{E_0^{(i,u),(j,v)} \neq 0} E^{(j,v)},
\end{aligned}
$$

the "total action" upon automaton j from automaton i with type u at an intermediate step and at a first step.

Assume that the following conditions hold:

$$
\begin{aligned}
\sigma(E^{(i,u),j})(\sigma(E^{(i,u),j}) - Id) &= 0, \\
\sigma(E_0^{(i,u),j})(\sigma(E_0^{(i,u),j}) - Id) &= 0. \qquad (8)
\end{aligned}
$$

Conditions (8) mean that at one step (say (i,u)), the summation of the matrices for all the possible synchronizations after this step to another automaton (say j) is a stochastic matrix ($\sigma(E^{(i,u),j}) = Id$, $\sigma(E_0^{(i,u),j}) = Id$). It implies that

$$\sigma(ME^{(i,u),j}) = \sigma(M), \quad \sigma(ME_0^{(i,u),j}) = \sigma(M), \quad (9)$$

for all matrix M with convenient size. This is the generalization of condition ($\sigma(S_4) + \sigma(S_2) = I$) in section 2.

The hypothesis (7) is interpreted as: if $\sigma(E^{(i,u),j}) = Id$ (or $\sigma(E_0^{(i,u),j}) = Id$) then $E^{(i,u),j}$ (or $E_0^{(i,u),j}$) is the sum taken over all v in some subset $\mathcal{T}^a$.

For a set $\mathcal{T}^a$ and a type v in $\mathcal{T}^a$, define

$$\text{Prev}(v) = \{u \in \mathcal{T} \mid \exists i,j \text{ s.t } E^{(i,u),(j,v)} + E_0^{(i,u),(j,v)} \neq 0\} \qquad (10)$$

the set of all types can be presented before the type v in some synchronization.

To illustrate these concepts, let us first consider some examples.

EXAMPLE 3.1. *Three automata studied in section 2.1.*
Set $\mathcal{T}$ is $\mathcal{T} = \{d, a, f\}$, where "d", "a" and "f" stand for "departure", "arrival" and "full". Set First is given by First $= \{d\}$, and $T = 1$, $\mathcal{T}^1 = \{a, f\}$.
Set Prev is Prev$(a) = \{d, f\}$, Prev$(f) = \{d\}$.
The matrices which define the synchronizations are given as in section 2.1. The condition (7) (and (8)) are satisfied, as $\sigma(Up_K + I_K) = Id$.

EXAMPLE 3.2. *We now consider the general jump over blocking network with more than three queues. As in the previous case, sets $\mathcal{T}$, First and $\mathcal{T}^a$ are given by*

$$\mathcal{T} = \{d, a, f\}, \qquad First = \{d\}, \qquad \mathcal{T}^1 = \{a, f\}.$$

However, in general, set Prev is

$$Prev(a) = Prev(f) = \{d, f\}.$$

The matrices used to define the synchronizations will be given later in the section 5.1.

EXAMPLE 3.3. *A signal triggers customer departure as in a Gnetwork with triggers. Set $\mathcal{T}$ is*

$$\mathcal{T} = \{d, s, a\}$$

where "d", "a" and "s" stand for "departure", "arrival" and "signal". Set First is given by First $= \{d\}$, and $T = 2$, $\mathcal{T}^1 = \{a\}$, $\mathcal{T}^2 = \{s\}$.
Set Prev is given by $Prev(a) = Prev(s) = \{d, s\}$. For the definition of the matrices associated with synchronizations, see section 5.3 .

4. PRODUCT FORM SOLUTION

4.1 Main theorem

We now state the main result of this paper.

THEOREM 4.1. *Consider a SAN with N automata or a Gnetwork with signals. Assume that the CTMC obtained from this high level specification is ergodic. Let $\mathcal{S}$ be the set of synchronizations with multiple steps. Assume that g_i is an eigenvector of all matrices $(E_{Prev(\mathcal{T}^a),\mathcal{T}^a}^{ij})_{a \leq T}$, where*

$$
\begin{aligned}
E_{Prev(\mathcal{T}^a)\mathcal{T}^a}^{ij} &= \sum_{u \in Prev(\mathcal{T}^a)} \big\{ E_0^{(i,u),(j,\mathcal{T}^a)} \qquad (11) \\
&\quad + \sum_{\substack{k \neq i \\ u \in \mathcal{T}^b}} \Gamma_{Prev(\mathcal{T}^b)\mathcal{T}^b}^{ki} E^{(i,u),(j,\mathcal{T}^a)} \big\},
\end{aligned}
$$

where $\Gamma_{Prev(\mathcal{T}^b)\mathcal{T}^b}^{ki}$ is the eigenvalue of the matrix $E_{Prev(\mathcal{T}^b)\mathcal{T}^b}^{ki}$. If g_i is in the kernel of matrix

$$
\begin{aligned}
H_i &= F_i - \sigma(F_i) \\
&\quad + \sum_{\substack{j \neq i \\ a \leq T}} (E_{Prev(\mathcal{T}^a)\mathcal{T}^a}^{ij} - \sigma(E_{Prev(\mathcal{T}^a)\mathcal{T}^a}^{ij})) \\
&\quad + \sum_{\substack{j \neq i \\ a \leq T}} \Gamma_{Prev(\mathcal{T}^a)\mathcal{T}^a}^{ji} (E_0^{(i,\mathcal{T}^a)} - \sigma(E_0^{(i,\mathcal{T}^a)})),
\end{aligned}
$$

where $E_0^{(i,\mathcal{T}^a)} = \sum_{u \in \mathcal{T}^a} (E_0^{(i,u)})$, then the steady-state distribution has product form solution:

$$Pr(X_1, X_2, \cdots, X^N) = C \prod_{i=1}^N g_i(X_i), \qquad (12)$$

and C is a normalization constant.

In (11), matrix $E_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij}$ can be viewed as the "total action matrix" applied to automaton j with a type in $\mathcal{T}^a$ from automaton i. The value $\Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ki}$ can be consider as the "total arrival rate" to automaton j with a type in $\mathcal{T}^a$ from automaton i.

4.2 A simple case

The complete proof of theorem (4.1) will be given in Appendix. In this section, we consider a simpler case: each automaton can appear at most once in List[1] of one synchronization and set $\mathcal{T}^a$ has only one element $|\mathcal{T}^a| = 1$. In this case, one has

$$|\text{Order}(i, r)| \leq 1,$$

hence, the form of M_i^r given in (4) is the product of at most one factor.

For simplicity, from now on, all the written tensor product will be automatically rewritten in the natural order. It simply means that $g_{i_1} P_{i_1} \otimes \cdots \otimes g_{i_N} P_{i_N}$ is the product $g_1 P_1 \otimes \cdots \otimes g_N P_N$.

Now, we will prove Theorem 4.1 in this simplified case. As mentioned before, we will prove that $\pi Q = \pi\sigma(Q)$. Define

$$\mathcal{S}(n) = \{r \in \mathcal{S} \mid |\text{List}(r)| = n\}$$

the set of all synchronizations with length n. In the following definitions, for synchronization r in $\mathcal{S}(n)$,

$$\text{List}(r) = \{(l_1, t_1), \cdots, (l_n, t_n)\}.$$

Let us define $Q(n)$ by:

$$Q(n) = \sum_{r \in \mathcal{S}(n)} M_1^r \otimes \cdots \otimes M_N^r.$$

In this case, $Q(n)$ is the sum over all $\mathcal{S}(n)$ of the tensor product $E_0^{(l_1,t_1),(l_2,t_2)} \otimes E^{(l_2,t_2),(l_3,t_3)} \otimes \cdots \otimes E_0^{(l_n,t_n)}$ in a convenient order. Furthermore we define $\widehat{Q}(n)$ and $\widetilde{Q}(n)$ by:

$$\widehat{Q}(n) = \sum_{r \in \mathcal{S}(n)} M_1^r \otimes \cdots \otimes \widehat{M}_{l_1}^r \cdots \otimes M_N^r,$$

$$\widetilde{Q}(n) = \sum_{r \in \mathcal{S}(n)} M_1^r \otimes \cdots \otimes \widetilde{M}_{l_n}^r \cdots \otimes M_N^r,$$

where

$$\widehat{M}_{l_1}^r = \sum_{k \notin \text{List}^1(r)} \Gamma_{\text{Prev}(t_1)\{t_1\}}^{kl_1} E^{(l_1,t_1),(l_2,t_2)},$$

$$\widetilde{M}_{l_n}^r = \sum_{k \notin \text{List}^1(r)} \sum_{t \in \mathcal{T} \backslash \text{First}} E^{(l_n,t_n),(k,t)}.$$

For the sake of simplification, for $\mathcal{T}^a = \{u\}$, let us denote by E_u^{ij} and Γ_u^{ij} the terms $E_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij}$ and $\Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij}$, respectively. Equations $g_i E_u^{ij} = \Gamma_u^{ij} g_i$ and (11) imply:

$$\begin{aligned}
\pi Q(2) &= \sum_{(i,u),(j,v)} g_i E_0^{(i,u)(j,v)} \otimes g_j E_0^{(j,v)} \\
&= \sum_{i,j,v} \Big[g_i E_v^{ij} \otimes g_j E_0^{(j,v)} \\
&\quad - \sum_{\substack{u \in \text{Prev}(v) \\ k \neq i}} g_i \Gamma_u^{ki} E^{(i,u),(j,v)} \otimes g_j E_0^{(j,v)} \Big] \\
&= \sum_{i,j,v} g_i \Gamma_v^{ij} \otimes g_j E_0^{(j,v)} - \pi\widehat{Q}(2). \quad (13)
\end{aligned}$$

For $n \geq 2$, substitution of $E_{t_1}^{kl_1}$ by the sum of two terms $\sum_{u \in \text{Prev}(t_1)} E_0^{(k,u),(l_1,t_1)}$ and $\sum_{j \neq k, u \in \text{Prev}(t_1)} \Gamma_u^{jk} E^{(k,u),(l_1,t_1)}$, one obtains:

$$\begin{aligned}
\pi\widehat{Q}(n) &= \sum_{r \in \mathcal{S}(n)} g_{l_1} \sum_{k \notin \text{List}^1(r)} \Gamma_{t_1}^{kl_1} E^{(l_1,t_1)(l_2,t_2)} \\
&\quad \otimes_{\alpha=2}^{n-1} g_\alpha E^{(l_\alpha,t_\alpha),(l_{\alpha+1},t_{\alpha+1})} \otimes g_{l_n} E_0^{(l_n,t_n)} \\
&= \sum_{\substack{r \in \mathcal{S}(n) \\ k \notin \text{List}^1(r)}} g_k E_{t_1}^{kl_1} \otimes g_{l_1} E^{(l_1,t_1)(l_2,t_2)} \\
&\quad \otimes_{\alpha=2}^{n-1} g_\alpha E^{(l_\alpha,t_\alpha),(l_{\alpha+1},t_{\alpha+1})} \otimes g_{l_n} E_0^{(l_n,t_n)} \\
&= \pi\widehat{Q}(n+1) + \pi Q(n+1), \quad (14)
\end{aligned}$$

where $\text{List}(r) = \{(l_1, t_1), \cdots, (l_n, t_n)\}$. Equations (13) and (14) yield that

$$\pi Q = \sum_i g_i F^i + \sum_{\substack{i,j \\ v \in \mathcal{T} \backslash \text{First}}} g_i \Gamma_v^{ij} \otimes g_j E_0^{(j,v)}. \quad (15)$$

Using equations (7) and (6) to evaluate the second component of the right-hand side of (15) we get:

$$\begin{aligned}
R &= \sum_{i,j,v} g_i \Gamma_v^{ij} \otimes g_j E_0^{(j,v)} \\
&= \sum_{i,j,v} g_i \Gamma_v^{ij} \otimes g_j (E_0^{(j,v)} - \sigma(E_0^{(j,\mathcal{T}^a)})) \\
&\quad + \sum_{i,j,v} g_i(E_v^{ij} - \sigma(E_v^{ij})) + \sum_{i,j,v} g_j \Gamma_v^{ij} \sigma(E_0^{(j,v)}) \\
&\quad + \sum_{i,j,v} g_i \sigma(E_v^{ij}) - \sum_{i,j,v} g_j \Gamma_v^{ij} \\
&= \sum_i g_i(H_i - F_i + \sigma(F_i)) + \sum_{i,j,v} g_j \Gamma_v^{ij} \sigma(E_0^{(j,v)}) \\
&\quad + \sum_{i,j,v} g_i \sigma\Big(\sum_{u \in \text{Prev}(v)} E_0^{(i,u),(j,v)} \\
&\quad + \sum_{k \neq i} \sum_{u \in \text{Prev}(v)} \Gamma_u^{ki} E^{(i,u),(j,v)} \Big) - \sum_{i,j,v} g_j \Gamma_v^{ij} \\
&= \sum_i g_i(-F_i + \sigma(F_i)) - \sum_{i,j,v} g_j \Gamma_v^{ij} \\
&\quad \times \Big[Id - \sigma\big(E_0^{(j,v)} + \sum_{\substack{k \neq j \\ t}} E^{(j,v)(k,t)}\big) \Big] \\
&\quad + \sum_{\substack{i,j,v \\ u \in \text{Prev}(v)}} g_i \sigma(E_0^{(i,u),(j,v)}) \otimes g_j \sigma(E^{(j,v)}) \\
&= \sum_i g_i(-F_i + \sigma(F_i)) + \pi\sigma(Q(2)) + \pi\sigma(\widetilde{Q}(2)). \quad (16)
\end{aligned}$$

For $n \geq 2$, using Equations (8), (9) and the definition of $E^{(l_n,t_n),k}$, one obtains:

$$\begin{aligned}
\pi\sigma(\widetilde{Q}(n)) &= \sum_{r \in \mathcal{S}(n)} g_1 E_0^{(l_1,t_1),(l_2,t_2)} \\
&\quad \otimes_{\alpha=2}^{n-1} g_\alpha E^{(l_\alpha,t_\alpha),(l_{\alpha+1},t_{\alpha+1})} \\
&\quad \otimes g_{l_n} \sigma\big(\sum_{k \notin \text{List}^1(r)} \sum_t E^{(l_n,t_n)(k,t)}\big) \\
&= \pi\sigma(Q(n+1)) + \pi\sigma(\widetilde{Q}(n+1)). \quad (17)
\end{aligned}$$

where $\text{List}(r) = \{(l_1, t_1), \cdots, (l_n, t_n)\}$. Finally Equations (15),(16) and (17) imply that:

$$\pi Q = \pi\sigma(Q).$$

This completes the proof.

4.3 The General Result

The proof of theorem (4.1) without any restriction on the list is in the Appendix. Define

$$\text{Last} = \{t \in \mathcal{T} \mid E_0^{(j,t),(i,u)} = E^{(j,t),(i,u)} = 0, \forall j, i, u\}$$

the set of all types can be present only at the last step.

In theorem (4.1), all the types in Last only appear in the hypothesis $H_i g_i = 0$, with the term

$$\Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ji} (E_0^{(i,\mathcal{T}^a)} - \sigma(E_0^{(i,\mathcal{T}^a)})).$$

Hence, one may consider a more general model (with more possibility of matrices): for a type u in Last, $\{u\} = \mathcal{T}^a$, one can have several matrices $E_0^{(i,u)}$ depending on synchronization r : $E_0^{(i,u,r)}$ for each possible synchronization r. The

assumptions we now need are: if $E_0^{(i,u,r)} \neq 0$ (which means synchronization r at its last step at automaton i and with type u), then:

$$\sigma(E_0^{(i,u,r)}) = \sigma(E^{i,u}) = Id,$$
$$g_i E_0^{(i,u,r)} = g_i E_0^{(i,u)}.$$

With the above modification, the network still has product form solution.

5. EXAMPLES

Let us now consider some examples to show the effectiveness of the approach.

5.1 A finite buffer network with jump over blocking, exponential services, Poisson arrivals

Consider a network of finite buffer queues. At queue i, the capacity is K_i, the arrival rate is λ_i, the service rate is μ_i. A customer after completion of its service at queue i arrives to queue j with probability p_{ij} or leaves the queue i immediately with probability d_i. When a customer arrives at queue i and finds K_i customers present, it jumps over the queue as a customer completing service and it routes to the next queue with the same probability.

Remember that: $\mathcal{T} = \{a, d, f\}$, $\mathcal{T}^1 = \{a, f\}$, and $\mathrm{Prev}(\mathcal{T}^1) = \{d, f\}$. Matrices F_i which describe the local transitions are:

$$F_i = \lambda_i Up_{K_i} + \lambda_i d_i I_{K_i} + \mu_i d_i Low_{K_i}.$$

To define the synchronizations we use the following matrices:

$$E_0^{(i,d),(j,\mathcal{T}^1)} = E_0^{(i,d),(j,a)} = E_0^{(i,d),(j,f)} = \mu_i p_{ij} Low_{K_i},$$
$$E_0^{(i,f),(j,\mathcal{T}^1)} = E_0^{(i,f),(j,a)} = E_0^{(i,f),(j,f)} = \lambda_i p_{ij} I_{K_i},$$
$$E^{(i,f),(j,\mathcal{T}^1)} = E^{(i,f),(j,f)} = E^{(i,f),(j,a)} = p_{ij} I_{K_i},$$
$$E_0^{i,a} = Up_{K_i}, \quad E_0^{i,f} = d_i I_{K_i},$$

and all the other matrices are null. Matrix $E^{j,\mathcal{T}^1}$ is given by:

$$E^{j,\mathcal{T}^1} = Up_{K_i} + d_i I_{K_i} + \sum_j p_{ji} I_{K_i} = Up_{K_i} + I_{K_i}.$$

We can rewrite Assumption (11) as g_i being an eigenvector of:

$$\begin{aligned}
E_{\{d,f\}\{a,f\}}^{ij} &= \mu_i p_{ij} Low_{K_i} + \lambda_i p_{ij} I_{K_i} \\
&\quad + \sum_k \Gamma_{\{d,f\}\{a,f\}}^{k,i} p_{ij} I_{K_i}.
\end{aligned}$$

After substitution in the system, one obtains that g_i has a geometric distribution with rate ρ_i such that:

$$\rho_i = \frac{\lambda_i + \sum_j \Gamma_{\{d,f\}\{a,f\}}^{ji}}{\mu_i},$$

where $\Gamma_{\{d,f\}\{a,f\}}^{ji} = \mu_j \rho_j p_{ji}$. This is the typical flow equation in such a network of queues.

5.2 Network with jump over blocking with a state dependent service rate

Consider a network similar to the network in section 5.1, except that the service rate in queue i is $\mu_i(n)$ when n customers are present. Define matrix D^i of size K_i as:

$$D^i = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \mu_i(1) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \mu_i(K_i) & 0 \end{pmatrix}.$$

One can deduce that g_i is an eigenvector of $E_{\{d,f\}\{a,f\}}^{ij}$:

$$\frac{g_i(n)\mu_i(n)p_{ij}}{g_i(n-1)} = \Gamma_{\{d,f\}\{a,f\}}^{ij}.$$

Solving the system after substitution, one can find that g_i is given by:

$$g_i(n) = g_i(0) \prod_{i=1}^n \frac{\Gamma^i}{\mu_i(n)},$$

where Γ_i satisfies the flow equations: $\Gamma_i = \lambda_i + \sum_j \Gamma_j p_{ji}$ and $\Gamma_i p_{ij} = \Gamma_{\{d,f\}\{a,f\}}^{ij}$.

5.3 Network with negative signals

Consider an open network of N queue with exponential services and Poisson arrivals from the outside. The service rate in queue i is μ_i. Customers arrive to queue i from the outside with rate λ_i. A customer departing from queue i arrives at queue j as a customer with probability p_{ij}, as a signal with probability p_{ij-}, or leaves the network with probability d_i. A signal arrives to queue i from the outside with rate λ_i^-. A signal arrives to queue i will trigger a customer from queue i to queue j, if there is, as a customer with probability p_{i-j}, as a signal with probability p_{i-j-}. Therefore the first signal emitted from the queue can have an instantaneous effect on several queues because of the transformation of customers into signals. This is much more general than the Gnetworks with triggers where only one customer can be triggered. Remember that:

$$\mathcal{T} = \{a, d, s\}, \qquad \mathcal{T}^1 = \{a\}, \qquad \mathcal{T}^2 = \{s\},$$
$$\mathrm{Prev}(\mathcal{T}^1) = \mathrm{Prev}(\mathcal{T}^2) = \{d, s\}.$$

Matrices F_i are used to describe the local transitions:

$$F_i = \lambda_i Up + \mu_i d_i Low,$$

while we model the synchronizations with the following matrices:

$$E_0^{(i,d),(j,a)} = \mu_i p_{ij} Low, \ E_0^{(i,d),(j,s)} = \mu_i p_{ij-} Low,$$
$$E_0^{(i,s),(j,a)} = \lambda_i^- p_{i-j} Low, \ E_0^{(i,s),(j,s)} = \lambda_i^- p_{i-j-} Low,$$
$$E^{(i,s),(j,a)} = p_{i-j} Low, \ E^{(i,s),(j,s)} = p_{i-j-} Low,$$
$$E_0^{i,a} = Up, \ E_0^{i,s} = I^0.$$

Furthermore $E^{j,a} = Up$ and $E^{j,s} = I^0 + Low$. One can rewrite Assumption (11) as g_i being an eigenvector of both matrices:

$$\begin{aligned}
E_{\{d,s\}\{a\}}^{ij} &= (\mu_i p_{ij} + \lambda_i^- p_{i-j}) Low \\
&\quad + \sum_k \Gamma_{\{d,s\}\{s\}}^{k,i} p_{i-j} Low,
\end{aligned}$$

$$\begin{aligned}
E_{\{d,s\}\{s\}}^{ij} &= (\mu_i p_{ij-} + \lambda_i^- p_{i-j-}) Low \\
&\quad + \sum_k \Gamma_{\{d,s\}\{s\}}^{k,i} p_{i-j-} Low.
\end{aligned}$$

After substitution in the system, one obtains that g_i has a geometric distribution with rate ρ_i such that:

$$\rho_i = \frac{\lambda_i + \sum_j \Gamma^{ji}_{\{d,s\}\{a\}}}{\mu_i + \lambda_i^- + \sum_j \Gamma^{ji}_{\{d,s\}\{s\}}},$$

where

$$\Gamma^{ji}_{\{d,s\}\{a\}} = \mu_j \rho_j p_{ji} + p_{j-i} \rho_j (\lambda_j^- + \sum_k \Gamma^{kj}_{\{d,s\}\{s\}}),$$

$$\Gamma^{ji}_{\{d,s\}\{s\}} = \mu_j \rho_j p_{ji-} + p_{j-i-} \rho_j (\lambda_j^- + \sum_k \Gamma^{kj}_{\{d,s\}\{s\}}).$$

REMARK 5.1. *This is a generalization of the example introduced in [12]. The network consists of 3 queues. Customers arrive to queue i from the outside with rate λ_i. The service rate at queue i is μ_i. The routing matrix is as follow: a customer completing service at queue 1 arrives to queue 2 as a signal, this signal will delete a customer at queue 2 and then moves to queue 3 as a signal. The signal will delete a customer at queue 3 and moves to queue 2 as a signal. This will repeat until queue 2 or queue 3 will be empty. In this case, the routing matrix is given by:*

$$p_{12-} = p_{2-3-} = p_{3-2-} = d_2 = d_3 = 1.$$

Hence, g_i has a geometric distribution with rate ρ_i

$$\rho_1 = \frac{\lambda_1}{\mu_1}, \quad \rho_2 = \frac{\lambda_2}{\mu_2 + \Gamma^{12} + \Gamma^{32}}, \quad \rho_3 = \frac{\lambda_3}{\mu_3 + \Gamma^{23}},$$

where

$$\Gamma^{12} = \mu_1 \rho_1, \quad \Gamma^{32} = \rho_3 \Gamma^{23}, \quad \Gamma^{23} = \rho_2 (\Gamma^{12} + \Gamma^{32}).$$

And we obtain the usual flow equations.

5.4 Network with positive signals

In [2], positive signals were introduced. A positive signal will add a customer to queue where it arrives. The matrix corresponding is Up. If we want to use Theorem 4.1, the assumption that g_i is an eigenvector of matrices E leads us to this hypothesis that: when the queue is empty, there will be a positive signals activated. Consider a network of N queues. The service rate in queue i is μ_i. Customers arrive to queue i from the outside with rate λ_i. A departure from queue i arrives to queue j as a customer with probability p_{ij}, as a positive signal with probability p_{ij+}, or leaves the network with probability d_i. A positive signal arrives to queue i from the outside with rate λ_i^+. A positive signal arrives to queue i will add a customer to queue i and leaves to queue j as a customer with probability p_{i+j}, as a positive signal with probability p_{i+j+}. One has:

$$\mathcal{T} = \{a, d, s^+, e\}, \quad \mathcal{T}^1 = \{a\}, \quad \mathcal{T}^2 = \{s^+\},$$
$$\text{Prev}(\mathcal{T}^1) = \text{Prev}(\mathcal{T}^2) = \{d, s^+, e\},$$

where "s^+" and "e" stand for positive signal and "empty". Matrices F_i which describe the local transitions are:

$$F_i = \lambda_i Up + \mu_i d_i Low.$$

Synchronizations are described by:

$$E_0^{(i,d),(j,a)} = \mu_i p_{ij} Low, \quad E_0^{(i,d),(j,s^+)} = \mu_i p_{ij+} Low,$$
$$E_0^{(i,s^+),(j,a)} = \lambda_i^+ p_{i+j} Up,$$
$$E_0^{(i,s^+),(j,s^+)} = \lambda_i^+ p_{i+j+} Up,$$
$$E_0^{(i,e),(j,a)} = C_{ar}^i I^0 p_{i+j}, \quad E_0^{(i,e),(j,s^+)} = C_{ar}^i I^0 p_{i+j+},$$
$$E^{(i,s^+),(j,a)} = p_{i+j} Up, \quad E^{(i,s^+),(j,s^+)} = p_{i+j+} Up,$$
$$E_0^{i,a} = Up,$$

where C_{ar}^i is the active rate when the queue i is empty. Matrices $E^{j,\mathcal{T}^a}$ are given by: $E^{j,a} = E^{j,s^+} = Up$. Now consider Assumption (11). We can restate it in terms of eigenvector: g_i is an eigenvector of both matrices:

$$\begin{aligned}
E^{ij}_{\{d,s^+,e\}\{a\}} &= \mu_i p_{ij} Low + \lambda_i^+ p_{i+j} Up \\
&\quad + C_{ar}^i I^0 p_{i+j} \\
&\quad + \sum_k \Gamma^{ki}_{\{d,s^+,e\}\{s^+\}} p_{i+j} Up,
\end{aligned}$$

$$\begin{aligned}
E^{ij}_{\{d,s^+e\}\{s^+\}} &= \mu_i p_{ij+} Low + \lambda_i^+ p_{i+j+} Up \\
&\quad + C_{ar}^i I^0 p_{i+j+} \\
&\quad + \sum_k \Gamma^{ki}_{\{d,s^+,e\}\{s^+\}} p_{i+j+} Up.
\end{aligned}$$

After substitution one obtains that g_i has a geometric distribution with rate ρ_i:

$$\rho_i = \frac{\lambda_i + \sum_j \Gamma^{ji}_{\{d,s^+,e\}\{a\}} + \lambda_i^+ + \sum_j \Gamma^{ji}_{\{d,s^+,e\}\{s^+\}}}{\mu_i},$$

where

$$\Gamma^{ji}_{\{d,s^+,e\}\{a\}} = \mu_j \rho_j p_{ji} + p_{j+i} C_{ar}^j,$$
$$\Gamma^{ji}_{\{d,s^+,e\}\{s^+\}} = \mu_j \rho_j p_{ji+} + p_{j+i+} C_{ar}^j,$$

and $C_{ar}^i = \rho_i^{-1} (\lambda_i^+ + \sum_k \Gamma^{ki}_{\{d,s^+,e\}\{s^+\}})$. Again this is equivalent to a flow equation.

5.5 Network with signals and jump over blocking

We now consider networks with negative signals, positive signals and "jump over blocking". Sets of type $\mathcal{T}$, $\mathcal{T}^a$ and Prev are defined by:

$$\mathcal{T} = \{a, d, f, s, e, s^+, f^+\},$$
$$\mathcal{T}^1 = \{a, f\}, \quad \mathcal{T}^2 = \{s\}, \quad \mathcal{T}^3 = \{s^+, f^+\},$$
$$\text{Prev}(\mathcal{T}^1) = \text{Prev}(\mathcal{T}^2) = \text{Prev}(\mathcal{T}^3) = \mathcal{T} \setminus \{a\},$$

where "f" stands for "full", f and f^+ are the "full state" of a queue corresponding to an arrival of a "normal customer" and a "positive signal", respectively.

The distribution g_i has a geometric distribution with rate ρ_i which is not dependent of the buffer size. When the buffer is infinite we still have the same value for ρ_i. The result is the combination of the ones obtained in the two previous sections: networks with negative signals and positive signals.

5.6 Domino synchronization

The SAN with Domino synchronization introduced in [9] is a special case of the synchronization multiple steps. The set of type $\mathcal{T}$ contain 3 elements: $\mathcal{T} = \{1, 2, 3\}$. The type 1 is associated with the first step, which is the Master, type 2

with the second step (the Slave) and type 3 for the last step denoted as the Relay. Domino synchronization were defined in [9] with matrices D, E and T to represent respectively the effect of the synchronization on the Master, the Slave and the Relay. We can easily translate into the notations used here to model multiple step synchronizations:

$$D^r - \sigma(D^r) = E_0^{(master,1)(slave,2)},$$
$$E_1^r = E^{(slave,2),(relay,3)}, \quad E_2^r = E_0^{(slave,2)},$$
$$T^r = E_0^{(relay,3)}.$$

Sets $\mathcal{T}^a$ and Prev are given by:

$$\mathcal{T}^1 = \{2\}, \ \mathcal{T}^2 = \{3\}, \ \text{Prev}(\mathcal{T}^1) = \{1\}, \ \text{Prev}(\mathcal{T}^2) = \{2\}.$$

And matrices $E_{Prev(\mathcal{T}^a)\mathcal{T}^a}^{ij}$ are obtained easily:

$$E_{\{1\}\{2\}}^{master,slave} = D^r - \sigma(D^r),$$
$$E_{\{2\}\{3\}}^{slave,relay} = \sum_{r'|slave(r')=slave} \Gamma^{r'} E_1^r,$$

and one has $\Gamma^r = \Gamma_{\{1\}\{2\}}^{master,slave}$, $\Omega^r = \frac{E_{\{2\}\{3\}}^{slave,relay}}{\sum_{r'|slave(r')=slaver} \Gamma^{r'}}$. Then, the domino synchronization is just one simple case of a synchronization with multiple steps.

6. CONCLUDING REMARKS

As SANs are very close to Stochastic Process Algebra (SPA), many results proved on SAN apply also on SPA. We want to emphasis that some of the examples presented in section 5 may clearly be proved by Reversed Compound Agent Theorem (RCAT) Theorem [15] which is based on the Performance Evaluation Process Algebra (PEPA) SPA even if it is applied to Gnetworks of queues. But our objective here is to present a new type of proof which is purely algebraic while most of the results on SPA look for a probabilistic interpretation of the sufficient assumptions for product form. The theorem we prove here allows to generalize many results on queues with customers and signals. But the post important result is a simple algebraic proof based on tensor. As the tensor representation is not limited to SAN (see for instance [6, 16]) we hope that this approach will lead to new research activities on the link between tensor representation and closed form solutions. Another approach is to find tensor representation for Zero Automatic queues [3].

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APPENDIX
Proof of the theorem 4.1

As mentioned before, we will prove that $\pi Q = \pi\sigma(Q)$. The idea is similar to the proof in section 4.2. Remember that

$$\mathcal{S}(n) = \{r \in \mathcal{S} \mid |\text{List}(r)| = n\}$$

the set of all synchronizations with length n. Matrix $Q(n)$ is still given by

$$Q(n) = \sum_{r \in \mathcal{S}(n)} M_1^r \otimes \cdots \otimes M_N^r,$$

however $\widehat{Q}(n)$ and $\widetilde{Q}(n)$ will be defined in a slightly different way. In the following definitions, for synchronization r in $\mathcal{S}(n)$, $\text{List}(r) = \{(l_1,t_1),\cdots,(l_n,t_n)\}$. Now let us define:

$$\widehat{Q}(n) = \sum_{r \in \mathcal{S}(n)} M_1^r \otimes \cdots \otimes \widehat{M}_{l_1}^r \cdots \otimes M_N^r,$$

$$\widetilde{Q}(n) = \sum_{r \in \mathcal{S}(n)} M_1^r \otimes \cdots \otimes \widetilde{M}_{l_n}^r \cdots \otimes M_N^r.$$

where

$$\widehat{M}_{l_1}^r = \sum_{k \neq l_1} \sum_{t_1 \in \mathcal{T}^a} \Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{kl_1} E^{(l_1,t_1),(l_2,t_2)} \frac{M_{l_1}^r}{M_{l_1}^r(1)},$$

$$\widetilde{M}_{l_n}^r = \frac{M_{l_n}^r}{M_{l_n}^r(|\text{Order}(l_n,r)|)} \sum_{k \neq l_n, a \leq T} E^{(l_n,t_n),(k,\mathcal{T}^a)}.$$

Equations (11) and $g_i E_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} = \Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} g_i$ imply that:

$$
\begin{aligned}
\pi Q(2) &= \sum_{(i,u),(j,v)} g_i E_0^{(i,u)(j,v)} \otimes g_j E_0^{(j,v)} \\
&= \sum_{i,j,\mathcal{T}^a} \Big[g_i E_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} \otimes g_j E_0^{(j,\mathcal{T}^a)} \\
&\quad - \sum_{\substack{u \in \text{Prev}(\mathcal{T}^a) \\ u \in \mathcal{T}^b, k \neq i}} g_i \Gamma_{\text{Prev}(\mathcal{T}^b)\mathcal{T}^b}^{ki} E^{(i,u),(j,\mathcal{T}^a)} \otimes g_j E_0^{(j,\mathcal{T}^a)} \Big] \\
&= \sum_{i,j,\mathcal{T}^a} g_i \Gamma_{\text{Prev}(\mathcal{T}^a),\mathcal{T}^a}^{ij} \otimes g_j E_0^{(j,\mathcal{T}^a)} - \pi\widehat{Q}(2). \quad (18)
\end{aligned}
$$

For $n \geq 2$, it is now more complicated to prove equation (14),

$$
\begin{aligned}
\pi\widehat{Q}(n) &= \sum_{|List(r)|=n} \otimes_{i \neq l_1} g_i M_i^r \\
&\quad \otimes g_{l_1} \sum_{\substack{k \neq l_1 \\ t_1 \in \mathcal{T}^a}} \Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{kl_1} E^{(l_1,t_1)(l_2,t_2)} \\
&\quad\quad\quad\quad \times M_{l_1}^r(2) \cdots M_{l_1}^r(m) \\
&= \sum_{\substack{|List(r)|=n \\ k \neq l_1, t_1 \in \mathcal{T}^a}} \otimes_{\substack{i \neq k \\ i \neq l_1}} g_i M_i^r \\
&\quad \otimes g_k E_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{kl_1} M_k^r \\
&\quad \otimes g_{l_1} E^{(l_1,t_1)(l_2,t_2)} M_{l_1}^r(2) \cdots M_{l_1}^r(m) \\
&= \pi\widehat{Q}(n+1) + \pi Q(n+1), \quad (19)
\end{aligned}
$$

where $\text{List}(r) = \{(l_1,t_1),\cdots,(l_n,t_n)\}$ and $m = |\text{Order}(l_1,r)|$. Equations (18) and (19) yield that:

$$\pi Q = \sum_i g_i F^i + \sum_{i,j,\mathcal{T}^a} g_i \Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} \otimes g_j E_0^{(j,\mathcal{T}^a)}. \quad (20)$$

Using equations (7) and (6) to evaluate the second component of the right-hand side of (20) we get:

$$
\begin{aligned}
R &= \sum_{i,j,\mathcal{T}^a} g_i \Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} \otimes g_j E_0^{(j,\mathcal{T}^a)} \\
&= \sum_{i,j,\mathcal{T}^a} g_i \Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} \otimes g_j (E_0^{(j,\mathcal{T}^a)} - \sigma(E_0^{(j,\mathcal{T}^a)})) \\
&\quad + \sum_{i,j,\mathcal{T}^a} g_i (E_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} - \sigma(E_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij})) \\
&\quad + \sum_{i,j,\mathcal{T}^a} g_j \Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} \sigma(E_0^{(j,\mathcal{T}^a)}) \\
&\quad + \sum_{i,j,\mathcal{T}^a} g_i \sigma(E_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij}) - \sum_{i,j,\mathcal{T}^a} g_j \Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} \\
&= \sum_i g_i(H_i - F_i + \sigma(F_i)) - \sum_{i,j,\mathcal{T}^a} g_j \Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} \\
&\quad + \sum_{i,j,\mathcal{T}^a} g_j \Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} \sigma(E_0^{(j,\mathcal{T}^a)}) \\
&\quad + \sum_{i,j,\mathcal{T}^a} g_i \sigma\Big(\sum_{u \in \text{Prev}(\mathcal{T}^a)} E_0^{(i,u),(j,\mathcal{T}^a)} \\
&\quad\quad + \sum_{k \neq i} \sum_{\substack{u \in \text{Prev}(\mathcal{T}^b) \\ u \in \mathcal{T}^b}} \Gamma_{\text{Prev}(\mathcal{T}^b)\mathcal{T}^b}^{ki} E^{(i,u),(j,\mathcal{T}^a)} \Big) \\
&= \sum_i g_i(-F_i + \sigma(F_i)) - \sum_{i,j,\mathcal{T}^a} g_j \Gamma_{\text{Prev}(\mathcal{T}^a)\mathcal{T}^a}^{ij} \\
&\quad \times \Big[Id - \sigma\big(E_0^{(j,\mathcal{T}^a)} + \sum_{\substack{k \neq j \\ v \in \mathcal{T}^a, b \leq T}} E^{(j,v)(k,\mathcal{T}^b)}\big) \Big] \\
&\quad + \sum_{\substack{i,j,\mathcal{T}^a \\ u \in \text{Prev}(\mathcal{T}^a)}} g_i \sigma(E_0^{(i,u),(j,\mathcal{T}^a)}) \otimes g_j \sigma(E^{(j,\mathcal{T}^a)}) \\
&= \sum_i g_i(-F_i + \sigma(F_i)) + \pi\sigma(Q(2)) + \pi\sigma(\widetilde{Q}(2)). \quad (21)
\end{aligned}
$$

For $n \geq 2$, using Equations (8), (9) and the definition of $E^{(l_n,t_n),k}$, one obtains:

$$
\begin{aligned}
\pi\sigma(\widetilde{Q}(n)) &= \sum_{|List(r)|=n} \otimes_{i \neq l_n} g_i \sigma(M_i^r) \\
&\quad \otimes g_{l_n} \sigma(M_{l_n}^r(1) \cdots M_{l_n}^r(m-1) \\
&\quad \times \sum_{\substack{k \neq l_n \\ a \leq T}} E^{(l_n,t_n)(k,\mathcal{T}^a)}) \\
&= \sum_{\substack{|List(r)|=n \\ k \neq l_n, a \leq T}} \otimes_{i \notin \{k,l_n\}} g_i \sigma(M_i^r) \otimes g_{l_n} \\
&\quad \sigma(M_{l_n}^r(1) \cdots M_{l_n}^r(m-1) E^{(l_n,t_n)(k,\mathcal{T}^a)}) \\
&\quad \otimes g_k \sigma(M_k^r E^{(l_n,t_n),k}) \\
&= \pi\sigma(Q(n+1)) + \pi\sigma(\widetilde{Q}(n+1)). \quad (22)
\end{aligned}
$$

where $\text{List}(r) = \{(l_1,t_1),\cdots,(l_n,t_n)\}$ and $m = |\text{Order}(l_n,r)|$. Equations (20),(21) and (22) imply:

$$\pi Q = \pi\sigma(Q).$$

And the proof is completed.