Consensus in Inventory Games *

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ABSTRACT

This paper studies design, convergence, stability and optimality of a distributed consensus protocol for n-player repeated non cooperative games under incomplete information. Information available to each player concerning the other players' strategies evolves in time. At each stage (time period), the players select myopically their best binary strategy on the basis of a payoff, defined on a single stage, monotonically decreasing with the number of active players. The game is specialized to an inventory application, where fixed costs are shared among all retailers, interested in reordering or not from a common warehouse. As information evolves in time, the number of active players changes too, and then each player adjusts its strategy at each stage based on the updated information. In particular, the authors focus on Pareto optimality as a measure of coordination of reordering strategies, proving that there exists a unique Pareto optimal Nash equilibrium which verifies certain stability conditions. The main contribution of the paper is the design of a consensus protocol allowing the distributed convergence of the strategies to the unique Pareto optimal Nash equilibrium. Results may also be extended to externality games, pollution/congestion games, and cost-sharing games, with the only constraint of being the strategies binary and with threshold structure.

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1. INTRODUCTION

In a competitive environment, decision makers may find convenient to coordinate their strategies to share costs for using resources, services or facilities. For this reason, there is a vast literature on game theory devoted to *mechanism design*, i.e., to the definition of game rules or incentive schemes that induce self-interested players to coordinate their strategies so that they converge to Pareto optimal equilibria [1]. The main contribution of this paper is the design of a *consensus protocol* (see, e.g., [2]) that allows the convergence of strategies to the desired equilibrium by exploiting stability properties of Pareto optimal Nash equilibria. In our consensus protocol each player exchanges a limited amount of information with a subset of other players. We cast this protocol within the *minimal information paradigm* to reduce each player's data exposure to the competitors. We also prove that the use of linear predictors increases the protocol speed of convergence. It is well known that in a game with a uniform set up cost allocation rule, optimal policies have threshold structures. Threshold strategies arise in repeated non cooperative games where, at each stage or time period, the payoff of the players is a monotonic function of the strategies of the others. Monotonicity of payoffs arises also in classical economic problems involving an externality (see the literature on *pollution/congestion games* or *externality games*) or in *cost-sharing games* [3] modelling the sharing of airport facilities or telephone systems, drilling for oil, cooperative farming, and fishing.

As motivating example, we consider a multi-retailer inventory application. The players, namely different competing retailers, share a common warehouse (or supplier) and cannot hold any private inventory from stage to stage, i.e., inventory left in excess at one stage is no longer utilizable in the future. The latter fact prevents the retailers from having large replenishments and stocks. Such a situation occurs when dealing with perishable goods as, for instance, the newspapers. The players aim at coordinating joint orders thus to share fixed transportation costs. As typical of repeated games, the retailers act myopically, that is, at each stage, they choose their best strategy on the basis of a payoff defined on single stage.

The rest of the paper is organized as follows. In Section 2, we develop the game theoretic model of the inventory system and formally state the problem. In Section 3, we prove that the desired Pareto optimal Nash equilibrium exists and is unique. In Section 4 we prove some stability properties of the the Pareto optimal Nash equilibria. In Section 5, we design a distributed protocol that allows the convergence

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of the strategies to the Pareto optimal Nash equilibrium. In Section 6, we analyze the speed of convergence of the protocol. In Section 7, we introduce a numerical example. Finally, in Section 8, we draw some conclusions.

2. THE INVENTORY GAME

In this section, we introduce the application, chosen from the inventory field, which motivates our study. However, the obtained results may also apply to other examples where strategies are binary and have a threshold structure. Hereafter, we indicate with the same symbol i both the generic player and the associated index. We consider a set of n players $\Gamma = \{1, \ldots, n\}$ where each player may exchange information only with a subset of neighbor players. More formally, we assume that the set Γ induces a single component graph $G = (\Gamma, E)$ whose edgeset E includes all the non oriented couples (i, j) of players that exchange information with each other. In this context, we define the neighborhood of a player i the set $N_i = \{j : (i, j) \in E\} \cup \{i\}$. At each stage k, each player i faces a customer demand and must decide whether to fulfill it or to pay a penalty p_i (see it, for instance, as a missed revenue); the unfilled demand is lost. Differently, we can review penalty p_i as the cost incurred by the player when, rather than participating in the game, it fulfills the demand by turning to a different supplier. We call *active* player the one who decides to meet the demand. The active players receive the items required by their customer from the common warehouse and equally divide a fixed transportation cost K. More formally, we define the function $s_i(k) \in S_i = \{0, 1\}$ as the strategy of player i, for each player $i \in \Gamma$. We indicate $s(k) = \{s_1(k), \ldots, s_n(k)\}$ as the vector of the players' strategies and $s_{-i} = \{s_1(k), \ldots, s_{i-1}(k), s_{i+1}(k), \ldots, s_n(k)\}$ as the vector of strategies of players $j \neq i$. At stage k, $s_i(k)$ is equal to 1 if player i meets the demand and equal to 0 otherwise. Then $s_i(k)$ has the following payoff defined on single stage

$$J_i(s_i(k), s_{-i}(k)) = \frac{K}{1+\|s_{-i}(k)\|_1} s_i(k) + (1 - s_i(k))p_i, \tag{1}$$

where $\|s_{-i}(k)\|_1$ is trivially equal to the number of active players other than i. At stage k, player i processes two types of public information: *pre-decision information*, $x_i(k)$, received from the neighbor players in N_i, and *post-decision information*, $z_i(k)$, transmitted to the neighbor players. Player i selects its strategy $s_i(k) = \mu_i(x_i(k))$ on the basis of the only pre-decision information. The information evolves according to a *distributed protocol* $\Pi = \{\phi_i,\, h_i,\ i \in \Gamma\}$ defined by the following dynamic equations:

$$x_i(k+1) = \phi_i(z_j(k) \text{ for all } j \in N_i) \tag{2}$$

$$z_i(k) = h_i(s_i(k), s_i(k-1), x_i(k)), \tag{3}$$

where the functions $\phi_i(\cdot)$ and $h_i(\cdot)$ are to be designed in Section 5. In the above context, *complete information* means that each player i knows the other players' strategies $s_{-i}(k)$ and optimizes repeatedly over stages its payoff (1) choosing as *best response* the following threshold strategy

$$s_i(k) = (\|s_{-i}(k)\|_1 \geq l_i), \tag{4}$$

where the threshold l_i is equal to $\frac{K}{p_i} - 1$ and $(\|s_{-i}(k)\|_1 \geq l_i)$ is a boolean function that returns 1 if its argument holds true, 0 otherwise. *Incomplete information* means that player i may only estimate the number $\|s_{-i}(k)\|_1$ of all other active players. In the rest of the paper, being $\hat{\chi}_i(k)$ such an estimate, the best response strategy (4) slightly modifies as

$$s_i(k) = (\hat{\chi}_i(k) \geq l_i). \tag{5}$$

We consider the following problem.

Problem Given the n-player repeated inventory game with binary strategies $s_i(k) = \{0, 1\}$ and payoffs (1), determine a distributed protocol $\Pi = \{\phi, h_i\ i \in \Gamma\}$ as in (2), (3) that allows the convergence of strategies (5) to a Pareto optimal Nash equilibrium s^*, if exists.

Observe that all results presented in the rest of the paper require only that the strategies are binary and have a threshold structure. Therefore the structure of the payoff can be relaxed as long as the best responses maintain a threshold structure as defined in (4)-(5).

3. A PARETO OPTIMAL NASH EQUILIBRIUM

In this section, we prove that a Pareto optimal Nash equilibrium exists. To this end, here and in the rest of the paper, we make, without loss of generality, the following assumptions:

Assumption 1
The set Γ of players is ordered so that $l_1 \leq l_2 \leq \ldots \leq l_n$.

Assumption 2
There may exist other players $i = n+1, n+2, \ldots$ not included in Γ, all of them with thresholds $l_i = \infty$.

Assumption 3
The players in the empty subset of Γ have thresholds $l_i = -\infty$.

The last assumption is obviously artificial, but simplifies the proofs of most results in the rest of the paper. Indeed, such an assumption allows us to prove the theorems without the necessity of introducing different arguments in the case when the set of active players is empty.

3.1 Existence of Nash Equilibria

In a Nash equilibrium $s^\star = \{s_1^\star, \ldots, s_n^\star\}$, each player i selects a strategy $s_i^\star$ such that

$$J_i(s_i^\star,\, s_{-i}^\star) \leq J_i(s_i,\, s_{-i}^\star) \quad \text{for all } s_i \in S_i,\ i \in \Gamma. \tag{6}$$

Hence, from (4), we obtain the following equilibrium conditions

$$s_i^\star = (\|s_{-i}^\star\|_1 \geq l_i), \quad \text{for all } i \in \Gamma. \tag{7}$$

On the basis of (7), we can state the following necessary conditions on the existence of a Nash Equilibrium.

Lemma 1
If $s^\star$ is a Nash equilibrium then:

i) if player i is active, namely $s_i^\star = 1$, then all the preceding players $1, \ldots, i-1$ are also active, i.e., $s_1^\star = \ldots = s_{i-1}^\star = 1$;

ii) if player i is not active, namely $s_i^\star = 0$, then neither all successive players $i+1, \ldots, n$ are active, i.e., $s_{i+1}^\star = \ldots = s_n^\star = 0$.

Let us now introduce two definitions. *Definition* A set $C \subseteq \Gamma$ is *compatible* if $l_i \leq |C| - 1$ for all $i \in C$.

In a compatible set C, each player finds convenient to meet the demand if all other players in C do the same.

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Definition A set $C \subseteq \Gamma$ of cardinality $|C| = r$ is *complete* if it contains all the first r players, with $r \geq 0$, i.e., $C = \{1, \ldots, r\}$.

Note that $C = \emptyset$ is both a compatible and a complete set.

Theorem 1
The vector of strategies $s^\star$, defined as

$$s_i^\star = \begin{cases} 1 & \text{if} \quad i \in C \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

is a Nash equilibrium if and only if the set $C = \{1, \ldots, r\} \subseteq \Gamma$ is both compatible and complete and the following condition holds

$$l_{r+1} > r. \tag{9}$$

From Theorem 1 we derive the following corollary.

Corollary 1
(Existence of Nash equilibria) There always exists a Nash equilibrium

$$s_i^\star = \begin{cases} 1 & \text{if} \quad i \in \overline{C} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where $\overline{C}$ is the maximal compatible set.

Observe that, if $\overline{C}$ is the maximal compatible set, it trivially holds

$$r = |\overline{C}| = \max_{\lambda} \{\lambda \in \{1, \ldots, n\} : \, l_\lambda < \lambda\}. \tag{11}$$

3.2 Pareto Optimal Nash equilibrium associated to $\overline{C}$

A vector of strategies $\hat{s} = \{\hat{s}_1, \ldots, \hat{s}_n\}$ is Pareto optimal if there is no other vector of strategies s such that

$$J_i(s_i, \, s_{-i}) \leq J_i(\hat{s}_i, \, \hat{s}_{-i}) \quad \text{for all } i \in \Gamma, \tag{12}$$

where the inequality is satisfied by at least one player.

Theorem 2
Let $s^\star$ be the Nash equilibrium associated to the maximal compatible set $\overline{C}$. If $p_i \neq \frac{K}{|\overline{C}|}$ for all $i \in \overline{C}$, then

- *Pareto optimality.* The vector of strategies $s^\star$ is Pareto optimal;
- *Uniqueness.* The vector of strategies $s^\star$ is the unique Pareto optimal Nash equilibrium.

Observe that if and only if $p_i = \frac{K}{|\overline{C}|}$ for all i, there exists two Pareto optimal Nash equilibria with equal payoff. They are associated respectively to the maximal compatible set $\overline{C}$ and to the empty set. In the rest of the paper, only the equilibrium $s^\star$ associated to the maximal compatible set $\overline{C}$ will be called the *Pareto optimal Nash equilibrium.*

4. STABILITY OF NASH EQUILIBRIA

In this section, we prove the stability of the Pareto optimal Nash equilibrium under the hypothesis that at each stage k, each player i knows the number of active players at the previous stage and then sets $\hat{\chi}_i(k) = \|s_{-i}(k-1)\|_1$. On the basis of this preliminary analysis, in the next section, we will be able to study the convergence properties of the repeated inventory game. Under the above hypothesis on $\hat{\chi}_i(k)$, the best response strategy (5) yields the following dynamic model

$$s_i(k) = (\|s_{-i}(k-1)\|_1 \geq l_i) \quad \text{for all } i \in \Gamma. \tag{13}$$

Given an equilibrium $s^\star$ and the associated complete compatible set $C = \{1, \ldots, r\}$, we define a *positive (negative) perturbation* at stage 0, the vector $\Delta s(0) = s(0) - s^\star \geq 0$ ($\Delta s(0) \leq 0$). In other words, a positive (negative) perturbation is a change of strategies of a subset of players $P = \{i \in \Gamma \setminus C : \Delta s_i(0) = 1\}$ ($P = \{i \in C : \Delta s_i(0) = -1\}$), called *perturbed set.* The cardinality of the perturbed set $|P| = \|\Delta s(0)\|_1$ is the number of players that join the set C (leave the set C). In addition, a positive (negative) perturbation $\overline{\Delta s}(0)$ is *maximal* when $\|\overline{\Delta s}(0)\|_1 = |\Gamma \setminus C|$, ($\|\overline{\Delta s}(0)\|_1 = |C|$). In this last case, all the players in $\Gamma \setminus C$, (C) change strategy.

A Nash equilibrium $s^\star$ is called *stable with respect to positive perturbations* if there exists a scalar $\delta > 0$ and $\bar{k} > 0$ such that if $\|\Delta s(0)\|_1 \leq \delta$, then $s^k = s^\star$ for all $k \geq \bar{k}$. Analogously a Nash equilibrium $s^\star$ is called *maximally stable with respect to positive perturbations* if it is stable with respect to the maximal positive perturbation $\overline{\Delta s}(0)$.

In the following, we introduce some theorems concerning the stability of Nash equilibria.

Theorem 3
Consider a Nash equilibrium $s^\star$ associated to a set $C = \{1, \ldots, r\}$. The vector of strategies $s^\star$ is stable with respect to positive perturbations $\Delta s(0) : ||\Delta s(0)||_1 = j - r - 1$ if all players $i \notin C$, with $r < i \leq j$, have thresholds $l_i \geq i$. In addition, if there exists a player $\hat{j} = arg \min\{i \in \Gamma \setminus C : l_i < i\}$. The vector of strategies $s^\star$ is not stable with respect to positive perturbations $\Delta s(0) : ||\Delta s(0)||_1 = \hat{j} - r$.

It is immediate to observe that, when player j exists, it must be $j \geq r + 2$, since it must necessarily be $l_{r+1} \geq r + 1$ from condition (9) of Theorem 1. In addition, $l_j = j - 1$ since for all i such that $r < i < j$ there holds $l_i \geq i$ by minimality of j.

Given a Nash equilibrium $s^\star$ and assuming the existence of a player $j = arg \min\{i \in \Gamma \setminus C : l_i < i\}$, Theorem 3 establishes that $s^\star$ is stable with respect to positive perturbation $\Delta s(0)$ if $||\Delta s(0)||_1 < j - r - 1$ and is not stable if $||\Delta s(0)||_1 \geq j - r$.

Assuming that there exists player $j = arg \min\{i \in \Gamma \setminus C : l_i < i\}$, the following theorem addresses the case $||\Delta s(0)||_1 = j - r - 1$. It must be noted that if player j does not exist, results from Theorem 3 apply and stability of the equilibrium is guaranteed.

Theorem 4
Consider a Nash equilibrium $s^\star$ associated to a set $C = \{1, \ldots, r\}$. Assume that there exists a player $j = arg \min\{i \in \Gamma \setminus C : l_i < i\}$ and let $\hat{i} = arg \min\{i \in \Gamma \setminus C : l_i = j - 1\}$. The vector of strategies $s^\star$ is not stable with respect to positive perturbations $\Delta s(0) : ||\Delta s(0)||_1 = j - r - 1$ iff at least one of the following conditions holds:
i) there exist players $j+1, \ldots, 2j - \hat{i} - 1$ with threshold equal to $j - 1$,
ii) there exist players $j + 1, \ldots, 2j - \hat{i}$.

Now, we specialize the previous theorems to the Pareto optimal Nash equilibrium.

Corollary 2
The unique Pareto optimal Nash equilibrium is maximally stable with respect to positive perturbations.

Let us conclude this section remarking that the Pareto optimal Nash equilibrium may not be globally stable with respect to negative perturbations. It is straightforward to prove this fact when, e.g., several Nash equilibria exist.

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5. A CONSENSUS PROTOCOL

In this section, we exploit the stability properties introduced in the previous section to design a protocol $\widehat{\Pi} = \{\phi_i,\, h_i,\ i \in \Gamma\}$ that allows the distributed convergence of the strategies to the Pareto optimal Nash equilibrium. Consider the graph G induced by the set of players Γ as defined in Section 2. Let L be the Laplacian matrix of G and use L_{ij} and $L_{i\bullet}$ to denote respectively the i, j entry and the i-th row of L. Let us consider the *almost-linear* protocol $\widehat{\Pi}$ defined by the following dynamics:

$$x_i(k+1) = z_i(k) + \alpha \sum_{j \in N_i} L_{ij} z_j(k) + \delta_T(k) \quad (14)$$

$$z_i(k) = x_i(k) + s_i(k) - s_i(k-1) \quad \text{for all } k \geq 1 \quad (15)$$

$$z_i(0) = x_i(0) = s_i(0) \quad (16)$$

where α is a negative scalar such that the eigenvalues of the matrix $(I + \alpha L)$ are inside the unit circle, except for the largest one that is equal to one. We will show that the pre-decision information $x_i(k)$ in (14) is a local estimate of the percentage of the active players at each stage $k-1$. Almost linearity is due to the non linear correcting term $\delta_T(k)$ acting any T stages in (14). This term describes the use of linear predictors, which will be discussed in Section 6. Throughout this section, we disregard this term by assuming $\delta_T(k)$ constantly equal to 0. The post-decision information $z_i(k)$ in (15) updates the estimate in the light of the strategy $s_i(k)$.

In the following we introduce two lemmas. The first one states that, at each stage k, the average value $Avg(x(k)) = \frac{\sum_i x_i(k)}{n}$ is the percentage of active players at the previous stage $k-1$. The second lemma states that if no player changes its strategy for a sufficient number of stages the pre-decision information $x_i(k)$ converges to the $Avg(x(k))$. For this last reason, protocol $\widehat{\Pi}$ may also be referred to as an *average consensus protocol* (see, e.g., [2]).

Now, let us initially rewrite the dynamic of the pre-decision information (14) for $k \geq 1$ as

$$\begin{aligned} x(k+1) = (I + \alpha L)(x(k) + s(k) - s(k-1)) = \\ s(k) + \textstyle\sum_{r=0}^{k} (I + \alpha L)^{k-r} \alpha L s(r). \end{aligned} \quad (17)$$

To obtain the second term of (17) we substitute in (14) the value of $z_i(k)$ in (15). Then we observe that from (14) it holds $x(1) = s(0) + \alpha L s(0)$ hence, by induction, if we assume $x(k) = s(k-1) + \alpha \sum_{r=0}^{k-1} (I + \alpha L)^{(k-1)-r} L s(r)$, we obtain the last term of (17).

Lemma 2

Given the dynamic of the pre-decision and the post-decision information vectors as described in (14), (15) and (16) at each stage k, the following condition holds

$$Avg(x(k)) = \frac{\|x(k)\|_1}{n} = \frac{\|s(k-1)\|_1}{n}. \quad (18)$$

Lemma 3

Consider the dynamic of the pre-decision and the post-decision information vectors as described in (14), (15) and (16) and assume that no player changes strategy from stage r on, then there exists a finite integer $\hat{r} \geq 1$ such that, for player i, it holds $x_i(r + \hat{r}) = \frac{\|s(r+\hat{r}-1)\|_1}{n} = \frac{\|s(r)\|_1}{n}$, i.e., $x_i(r + \hat{r})$ is equal to the percentage of active players at stage r.

In the assumption that no player changes strategy from a generic stage r on, the above arguments guarantee that each player i may estimate the percentage of the active players in a finite number of stages T. Lemma 3 shows that $T \leq \hat{r} - r$. It will be shown in Section 6, that T may be less than $\hat{r} - r$ in presence of linear predictors.

Then, at stage $r + T$ player i estimates the number of all other active players as

$$\begin{aligned} \chi_i(r+T) = \|s_{-i}(r+T-1)\| = \|s_{-i}(r)\| \\ = n x_i(r+T) - s_i(r). \end{aligned} \quad (19)$$

Now, assume that players can change strategy only at stages $\hat{k} = qT$, $q = 0, 1, 2, \ldots$. At stages $\hat{k} \geq 1$, we can generalize (19) as $\chi_i(\hat{k}) = n x_i(\hat{k}) - s_i(\hat{k} - T)$. At stage $\hat{k} = 0$ let the players estimate all the other players active, i.e., $\chi_i(0) = n - 1$.

Theorem 5

The average consensus protocol $\widehat{\Pi}$ defined in (14), (15) and (16) allows the best response strategy (5) to converge in $(n-1)T$ stages to the unique Pareto optimal Nash equilibrium.

Note that the convergence properties of the protocol established in the previous theorem still hold for any initial estimate $z_i(0)$ in (16) that is an upper bound of the $|\overline{C}|$.

6. A-PRIORI INFORMATION AND SPEED OF CONVERGENCE OF THE PROTOCOL

In this section, we determine the values of both α and T as functions of the players' computation capabilities and their knowledge about the structure of graph G. We show that T grows linearly with n when players can use linear predictors and discuss the non linear correcting term $\delta_T(k)$ in (14). Differently, in absence of linear predictors ($\delta_T(k) = 0$ for all k) the players wait that the pre-decision information converges to the desired percentage of currently active players; the number of stages T may become proportional to $n^2 log(n)$ or even to $n^3 log(n)$ depending on the knowledge that players have on the eigenvalues of the Laplacian matrix L.

Throughout this section we recall the hypotheses of Lemma 3, i.e., players are interested in determining the value of $Avg(s(r)) = Avg(x(r+1))$ and do not change strategy from stage r on.

6.1 Linear Predictors

With focus on (14) the non-linear correcting term must i) compensate the linear dynamics $-z_i(k) - \alpha \sum_{j \in N_i} L_{ij} z_j(k)$ and ii) correct the estimate of the percentage of active players. For doing so, the non linear correction may take the form

$$\begin{aligned} \delta_T(k) &= -z_i(k) - \alpha \textstyle\sum_{j \in N_i} L_{ij} z_j(k) \\ &+ \rho(x_i(k), x_i(k-1), \ldots, x_i(k-T)). \end{aligned} \quad (20)$$

Now, we show that it is possible to design ρ linearly as follows

$$\begin{aligned} \rho(x_i(r+T), x_i(r+T-1), \ldots, x_i(r)) = \\ \textstyle\sum_{k=0}^{n-1} \gamma_k x_i(r+k), \end{aligned} \quad (21)$$

where γ_k are the coefficients of the characteristic polynomial of the matrix $I + \alpha L$ and therefore depend on the structure of graph G.

The next theorem shows that each player i may determine the value of $Avg(x(r+1))$ in $n-1$ stages.

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Theorem 6
Given the protocol $\widehat{\Pi}$ as in (14), (15) and (16) the number of stages necessary for the generic player to estimate the percentage of active players $\frac{\|s(k)\|_1}{n}$ is $T \leq n-1$, if the players know the characteristic polynomial of the matrix $I + \alpha L$.

An immediate consequence of the above theorem is that, in the worst case, no other distributed protocol may determine the number of active players faster than $\widehat{\Pi}$, provided that players know the characteristic polynomial of the matrix $I + \alpha L$. If G is a path graph, the value of T can never be less than n, since information takes $n-1$ stages to propagate end to end all over the path.
Now, consider the case in which the players have no knowledge on the structure of the graph G, then the values of the parameters γ_k cannot be a priori fixed. The next theorem proves that $2n$ stages are sufficient for the generic player to estimate $Avg(x(r+1))$.

Theorem 7
Given the protocol $\widehat{\Pi}$ as in (14), (15) and (16), the number of stages necessary for the generic player to estimate the percentage of active players $\frac{\|s(k)\|_1}{n}$ is $T \leq 2n$.

6.2 No Predictors

We now compare the previous results with the ones obtainable when no predictors are used.
Lemma 3 states that, in any case, the pre-decision information converges to the desired average value $Avg(s(r)) = Avg(x(r+1))$. We are then interested in deriving after how many stages a player can determine $Avg(x(r+1))$ by rounding the pre-decision information currently available. To this end let us consider the following autonomous discrete time system of order n

$$x(k+1) = (I + \alpha L)x(k). \qquad (22)$$

System (22) describes the evolution of the pre-decision information when players do not change their strategies from stage r on. Actually, equation (22) is trivially equivalent to (17) when the players' strategies are disregarded. Starting from any initial state $x(r+1)$ the system (22) converges to $Avg(x(r+1))$. Then, observe that$Avg(x(r+1))$ must be equal to a multiple of $\frac{1}{n}$ due to its physical meaning. As a consequence, we could choose T as equal to the minimal k such that $|x_i(k+r+1) - Avg(x(r+1))| < \frac{1}{2n}$ for each player i and let the players determining $Avg(x(r+1))$ by simply rounding $x_i(k+r+1)$ to its closest multiple of $\frac{1}{n}$.
To determine the value of T, consider first the modal decomposition of the undriven response of system (22) given by

$$x(k+r+1) = (I + \alpha L)^k x(r+1) = \sum_{i=1}^{n} \beta_i \bar{\lambda}_i^k v_i,$$

where, for $i = 1, \ldots, n$, $\bar{\lambda}_i$ is an eigenvalue of $I + \alpha L$, v_i is the associate eigenvector, and β_i depends on the initial state according to

$$x(r+1) = \sum_{i=1}^{n} \beta_i v_i.$$

Note that since the smallest eigenvalue of L is always $\lambda_1 = 0$, then $\bar{\lambda}_1 = 1$ and hence $\beta_1 v_1 = Avg(x(r+1))$. Note also that $I + \alpha L$ is symmetric then, due to the spectral theorem for Hermitian matrices, all its eigenvectors are ortonormal. Hence, $|\beta_i| = \|v_i' x(r+1)\|_\infty \leq \|v_i\|_\infty \|x(r+1)\|_\infty \leq 1$ since the initial state $x(r+1)$ satisfies $\|x(r+1)\|_\infty \leq 1$. We can now state that (subscript ∞ is dropped)

$$\begin{aligned} &\|x(k+r+1) - Avg(x(r+1))\| = \\ &\|x(k+r+1) - \beta_1 v_1\| = \\ &\|\textstyle\sum_{i=2}^{n} \bar{\lambda}_i^k \beta_i v_i\| \leq \sum_{i=2}^{n} \|\bar{\lambda}_i^k \beta_i v_i\| \leq \sum_{i=2}^{n} |\bar{\lambda}_i^k| |\beta_i| \|v_i\| \leq \\ &\leq |\hat{\lambda}|^k \textstyle\sum_{i=2}^{n} \|v_i\|^2 \|x(r+1)\| \leq |\hat{\lambda}|^k (n-1) \end{aligned}$$

where $\hat{\lambda}$ is the eigenvalue of $I + \alpha L$ with the second greatest absolute value. Indeed, the eigenvalue of $I + \alpha L$ with the greatest absolute value is $\bar{\lambda}_1$.
Given the above arguments a conservative condition on T is to impose $|\hat{\lambda}|^T (n-1) < \frac{1}{2n}$, from which we obtain

$$T \geq \frac{-log(2(n-1)n)}{log(|\hat{\lambda}|)} + 1. \qquad (23)$$

In condition (23) T depends indirectly on the value of α through the eigenvalue $\hat{\lambda}$. In the following we discuss how to choose α in order to minimize T and, at the same time, to guarantee the stability of system (22). In (23), T is minimized if $|\hat{\lambda}|$ is minimum, since $|\hat{\lambda}| < 1$ for system (22) to be stable. Note that $|\hat{\lambda}|$ is equal

$$|\hat{\lambda}| = \max\{|1 + \alpha\lambda_n|, |1 + \alpha\lambda_2|\} \qquad (24)$$

The optimal $\alpha^\star$ is then the solution of the following equation

$$\begin{aligned} &\alpha^\star = \\ &\arg\min_\alpha |\hat{\lambda}| = \arg\min_\alpha \max\{|1 + \alpha\lambda_n|, |1 + \alpha\lambda_2|\} \end{aligned} \qquad (25)$$

It is easy to show that the solutions of the above equation are

$$\alpha^\star = -\frac{2}{\lambda_2 + \lambda_n} \qquad (26)$$

$$\hat{\lambda}^\star = 1 - \frac{2\lambda_2}{\lambda_2 + \lambda_n}. \qquad (27)$$

Consider now the stability of system (22). System (22) is stable if $|\bar{\lambda}_i| < 1$, $i = 2, \ldots, n$, which in turns implies that $|1 + \alpha\lambda_i| < 1$. Since $\alpha < 0$ and $\lambda_i > 0$, the latter condition is certainly satisfied if and only if $1 + \alpha\lambda_n > -1$. From this last inequality, system (22) is stable if and only if $-\frac{2}{\lambda_n} < \alpha < 0$. In this context, note that $-\frac{2}{\lambda_n} < \alpha^\star < 0$.
Let us now introduce the following lemma that collects well-known properties on the eigenvalues λ_2 and λ_n that turn useful in the rest of the section.

Lemma 4
Let $G_1 = (\Gamma, E_1)$ and $G_2 = (\Gamma, E_2)$ be two connected graphs on the same set of vertices Γ, and let $\lambda_2(G_1)$ and $\lambda_2(G_2)$ the second smallest eigenvalues of the Laplacian matrices associated to G_1 and G_2, respectively. Analogously, let $\lambda_n(G_1)$ and $\lambda_n(G_2)$ the greatest eigenvalues of the Laplacian matrices associated to G_1 and G_2, respectively. Then, the following properties hold

1. $\lambda_2(G_1) \leq \lambda_2(G_2)$, if $E_1 \subseteq E_2$;
2. $\lambda_n(G_1) \leq \lambda_n(G_2)$, if $E_1 \subseteq E_2$;
3. $\lambda_n(G_1) = \lambda_n(G_2) = n$, if G_1 is complete graph;
4. $\lambda_2(G_1) = 2(1 - cos(\frac{\pi}{n}))$, if G_1 is a path graph;

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Table 1: Players' thresholds and initial strategies

players	1	2	3	4	5	6	7	8
l_i	5	∞	∞	∞	2	1	4	1
$s_i(0)$	1	0	0	0	1	1	1	1

5. $\lambda_n(G_1) = 2(1 + cos(\frac{\pi}{n}))$, if G_1 is a path graph.

An immediate consequence of the previous lemma is that, if players know λ_2 and λ_n and the graph G is complete, then $\alpha^\star = -\frac{1}{n}$, $\hat{\lambda}^\star = 0$, and from (23) we have $T = 1$, whereas if G is a path graph, $\alpha^\star = -\frac{1}{2}$, $\hat{\lambda}^\star = cos(\frac{\pi}{n})$, and hence $T \rightarrow \frac{2n^2 log(2(n-1)n)}{\pi^2} + 1$ as n increases. Differently, if players know neither the structure of the graph G nor the eigenvalues λ_2 and λ_n. To guarantee the stability of system (22), condition $-\frac{2}{\lambda_n} < \alpha < 0$ must hold for any possible value of λ_n. By Lemma 4, the largest λ_n occurs when G is a complete graph, where $\lambda_n = n$. Then, α must be chosen within the interval $-\frac{2}{n} < \alpha < 0$. Now, consider a path graph. The fastest convergence occurs for the greatest $|\alpha|$, and when $\alpha \rightarrow -\frac{2}{n}$ we obtain $T \rightarrow \frac{n^3 log(2(n-1)n)}{2\pi^2} + 1$ as n increases.

7. SIMULATION RESULTS

In this section we provide a numerical example and some simulation results for a set Γ of 8 players implementing the designed protocol with and without predictors. We will see that in both cases the strategies converge to the Pareto optimal Nash equilibrium though with different speed of convergence. Fig. 1 reports the induced graph G, whereas Tab. 7 lists the players' thresholds l_i and the initial strategies $s_i(0)$. Note that at $k = 0$ the strategies are not in the Pareto optimal Nash equilibrium $s^* = \{0, 0, 0, 0, 1, 1, 0, 1\}$.

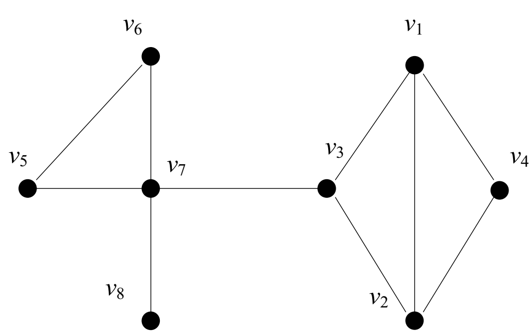

Figure 1: An example of graph G for a set Γ of 8 players

Fig. 2 displays the evolution of the pre-decision information according the protocol $\hat{\Pi}$ defined in (14)-(16) when the players use the linear predictors as in (20)-(21). Fig. 3 shows the evolution of the pre-decision information when the linear predictors are not present.
Both Fig. 2 and Fig. 3 show that at $k = 0$ players $1-5-6-7-8$ are active. At stage $k = T$ all the players estimate the number of active players as equal to 5. Then, player 1 changes strategy from $s_1(T-1) = 1$ to $s_1(T) = 0$ since its estimate is lower than his corresponding threshold $l_1 = 5$ (circles in Fig. 2-3). At $k = 2T$, the players' new estimate is 4 and player 7 changes strategy, too. Finally, at stage $k = 3T$, the players strategies converge to the Pareto optimal Nash equilibrium with $\|s^\star\|_1 = 3$.
The difference between the two figures is that, in Fig. 2 the value of T is 15 whereas in Fig. 3 the value of T is 80.

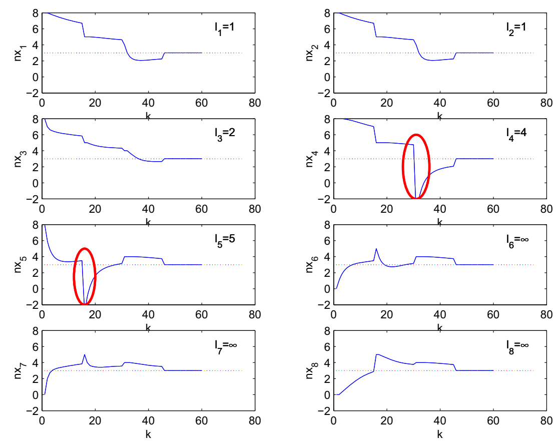

Figure 2: Evolution of $nx_i(k)$ in presence of linear predictors as in (20)-(21). The circles indicate when a player changes strategy.

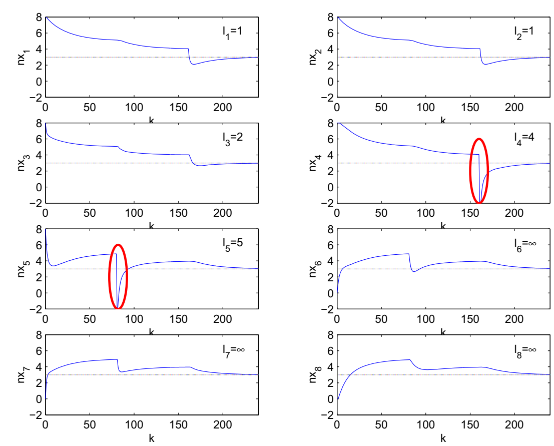

Figure 3: Evolution of $nx_i(k)$ in absence of linear predictors. The circles indicate when a player changes strategy.

8. CONCLUSION

In this paper, we have introduced a consensus protocol to achieve distributed convergence to Pareto optimal Nash equilibria, for a class of repeated non cooperative games under incomplete information. We have considered games with monotonic payoffs and we have specialized them to multi-retailer inventory problems, where transportation or set up costs are shared among all retailers, reordering from

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a common warehouse. The main results concern: i) the existence and the stability of Nash equilibria, ii) the structure of the consensus protocol and its convergence properties. Results may also be extended to externality games, pollution/congestion games, and cost-sharing games, with the only constraint of being the strategies binary and with threshold structure. Further work in this direction would involve the study of information protocols and decision mechanisms in presence of stochastic processes.

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