Fair resource allocation in wireless networks in the presence of a jammer*

Eitan Altman
INRIA BP93
06902 Sophia Antipolis
France
Altman@sophia.inria.fr

Konstantin Avrachenkov
INRIA BP93
06902 Sophia Antipolis
France
kavratch@sophia.inria.fr

Andrey Garnaev
St. Petersburg State University
St. Petersburg, Russia
agarnaev@rambler.ru

ABSTRACT

We consider jamming in wireless networks in the framework of zero-sum games with α-utility functions. The base station has to distribute the power fairly among the users in the presence of a jammer. The jammer in turn tries to distribute its power among the channels to produce as much harm as possible. The Shannon capacity and SNIR optimization are particular cases of the proposed more general α-fairness SNIR based utility functions. Specifically, we consider two α-fairness utility functions, based on SNIR and Shifted SNIR. This game can also be viewed as a minimax problem against the nature. We show that the game has the unique equilibrium and investigate its properties. In particular, in several important cases we present the equilibrium strategies and the Jain's fairness index in closed form. It turns out that there is an important difference between SNIR and Shifted SNIR α-fairness utility functions. In the case of the SNIR based utility function all users obtain non-zero powers when $\alpha > 0$. On contrary, when the Shifted SNIR based utility function is used, some users with bad channel conditions might not receive any power at all. We have also detected a surprising non-monotone behaviour of the Jain's fairness index in the case of the SNIR based utility function.

Categories and Subject Descriptors

C.2.1 [**Network Architecture and Design**]: Wireless communication

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Fairness, wireless network, jamming, zero-sum game

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1. INTRODUCTION

Fairness concepts have been playing a central role in networking. In the ATM standards [9], the maxmin fairness and its weighted versions appear as the way to allocate throughput to connections using the ABR (Available Bit Rate) best effort service. The proportional fairness concept (which agrees with the definition of Nash bargaining [5]) has been introduced in [3, 2]. Later it was implemented in wireless communications (e.g. in the Qualcomm High Data Rate (HDR) scheduler) as a way to allocate throughputs (through time slots); it has also been shown to correspond to the way that some versions of the TCP Internet Protocol share bottleneck capacities [4]. A unifying mathematical formulation to fair throughput assignment (which we call the "α-fairness") has been proposed in [6], see [8] for a generalization. In the present work we use the concept of "α-fairness" to allocate power resource in the presence of a jammer. The goal of the base station is to maximize the α-fairness utility function with respect to the SNIR and the shifted SNIR and the jammer on contrary wants to minimize this utility function. Thus, the problem can be considered as a zero-sum game.

Let us specify the problem formulation. There is a base station which needs to allocate the power resource $\bar{P}$ to n users. We assume that for each user there is a channel and there is an interference among the channels. The pure strategy of the base station is $P = (P_1, \dots, P_n)$ where $P_i \geq 0$ for $i \in [1, n]$ and $\sum_{i=1}^n P_i = \bar{P}$ where $\bar{P} > 0$ for $i \in [1, n]$. The component P_i can be interpreted as the power level dedicated to user i. The pure strategy of the jammer is $J = (J_1, \dots, J_n)$ where $J_i \geq 0$ for $i \in [1, n]$ and $\sum_{i=1}^n J_i = \bar{J}$ where $\bar{J} > 0$. We consider two payoffs. The first payoff is the α-fairness utility function of the SNIRs

$$v(P, J) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \left(\frac{g_i P_i}{N_i^0 + h_i J_i} \right)^{1-\alpha} \text{ for } \alpha \neq 1 \qquad (1)$$

and

$$v(P, J) = \sum_{i=1}^{n} \ln \left(\frac{g_i P_i}{N_i^0 + h_i J_i} \right) \text{ for } \alpha = 1, \qquad (2)$$

where $\alpha \geq 0$, N_i^0 is the power level of the uncontrolled noise in the channel i, $g_i > 0$ and $h_i > 0$ are corresponding fading channel gains for the user and the jammer. We assume that all the fading channel gains g_i, h_i and the power level of the uncontrolled noise N_i^0 for $i \in [1, n]$ as well as the total power resource $\bar{P}$ of the base station and the total noise $\bar{J}$ induced by the jammer are fixed and known to both players. We consider zero-sum game, so the payoff to Jammer is

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$-v(P,J)$.

The second payoff is the α-fairness utility function of the shifted SNRs

$$v(P,J)=\frac{1}{1-\alpha}\sum_{i=1}^{n}\left(\left(1+\frac{g_iP_i}{N_i^0+h_iJ_i}\right)^{1-\alpha}-1\right). \quad (3)$$

We use the α-fairness with the parameter α belonging to the interval $[0,2]$. The use of the α-fairness utility function allows us to treat in the same universal framework several important particular cases. When $\alpha=0$ both SNIR and shifted SNIR versions give the same solution which corresponds to the SNIR sum maximization. When $\alpha=1$, the SNIR formulation corresponds to the proportional fair assignment of SNIRs and the shifted SNIR formulation corresponds to the Shannon capacity maximization. When $\alpha=2$, the shifted SNIR formulation corresponds to the delay minimization. We note that the present formulation can also be viewed as a minimax problem in which the Base Station uses the best strategy against the worst possible environment conditions.

The structure of the paper is as follows: In Section 2 we study the SNIR α-fairness formulation. In Section 3 we study the shifted SNIR α-fairness formulation. In Section 4 we study the case $\alpha=0$, which corresponds to the linear utility function. Then, in Section 5 we provide numerical examples which illustrate the theoretical results of Sections 2, 3 and 4.

2. THE SNIR α-FAIRNESS FORMULATION

In this section we investigate the SNIR α-fairness formulation with the payoff given by (1) and (2). We will prove that the game has the unique Nash equilibrium. Moreover, we are able to find equilibrium strategies in a closed form.

It is clear that $v(P,J)$ is concave in P for any α and $-v(P,J)$ is concave in J for $\alpha\le 2$ since

$$\frac{\partial^2 v}{\partial P^2}=-\frac{g_i\alpha_i}{P_i(N_i^0+h_iJ_i)}\left(\frac{N_i^0+h_iJ_i}{g_iP_i}\right)^{\alpha}$$

and

$$\frac{\partial^2 v}{\partial J^2}=(2-\alpha)\frac{P_ih_i^2\alpha_i}{(N_i^0+h_iJ_i)^3}\left(\frac{N_i^0+h_iJ_i}{g_iP_i}\right)^{\alpha}.$$

Thus, we shall assume that $0<\alpha\le 2$. Then by [7] the game has the unique equilibrium. Our aim is to describe the equilibrium in details, namely to find it in a closed form. For this purpose we will apply the Kuhn-Tucker Theorem which implies the following result.

LEMMA 1. *Let $\alpha\in(0,2]$ then (P,J) is the equilibrium if and only if there are ω and ν (Lagrange multipliers) such that*

$$\frac{1}{P_i}\left(\frac{g_iP_i}{N_i^0+h_iJ_i}\right)^{1-\alpha}\begin{cases}=\omega, & P_i>0,\\ \le\omega, & P_i=0,\end{cases} \quad (4)$$

$$\frac{h_i}{N_i^0+h_iJ_i}\left(\frac{g_iP_i}{N_i^0+h_iJ_i}\right)^{1-\alpha}\begin{cases}=\nu, & J_i>0,\\ \le\nu, & J_i=0.\end{cases} \quad (5)$$

Since $P_i>0$ at least for one i by (4) and (5) the Lagrange multipliers have to be positive and also since $\alpha\le 2$ then $P_i>0$ for all $i\in[1,n]$.

Applying Lemma 1 we can describe the structure of the optimal solution more precisely as it is given in the following Theorem.

THEOREM 1. *For $\alpha\in(0,2]$ each equilibrium is of the form $(P(\omega,\nu),J(\omega,\nu))$ for some positive ω and ν where for $i\in[1,n]$:*

$$P_i(\omega,\nu)=\begin{cases}\frac{1}{\omega}\left(\frac{\nu g_i}{\omega h_i}\right)^{1-\alpha}, & i\in I\left(\frac{1}{\nu}\left(\frac{\nu}{\omega}\right)^{1-\alpha}\right),\\ \frac{1}{\omega^{1/\alpha}}\left(\frac{g_i}{N_i^0}\right)^{1/\alpha-1}, & i\notin I\left(\frac{1}{\nu}\left(\frac{\nu}{\omega}\right)^{1-\alpha}\right),\end{cases} \quad (6)$$

$$J_i(\omega,\nu)=\left[\frac{1}{\nu}\left(\frac{\nu g_i}{\omega h_i}\right)^{1-\alpha}-\frac{N_i^0}{h_i}\right]_+, \quad (7)$$

where

$$I(\tau)=\left\{i\in[1,n]:\frac{N_i^0}{h_i}\left(\frac{h_i}{g_i}\right)^{1-\alpha}\le\tau\right\}.$$

Theorem 1 reduces the problem of finding the optimal solution to the problem of finding two parameters ω and ν such that:

$$H_P(\omega,\nu):=\sum_{i=1}^{n}P_i(\omega,\nu)=\bar{P} \quad (8)$$

and

$$H_J(\omega,\nu):=\sum_{i=1}^{n}J_i(\omega,\nu)=\bar{J}. \quad (9)$$

We can write (7) as follows:

$$J_i(\omega,\nu)=\tilde{J}_i(\tau)=\left[\tau\left(\frac{g_i}{h_i}\right)^{1-\alpha}-\frac{N_i^0}{h_i}\right]_+, \quad (10)$$

with

$$\tau=\frac{1}{\nu}\left(\frac{\nu}{\omega}\right)^{1-\alpha}$$

Then, equation (9) turns into an equation for the water-filling problem:

$$\begin{aligned}H_J(\omega,\nu)&=\tilde{H}_J(\tau)\\&=\sum_{i=1}^{n}\left[\tau\left(\frac{g_i}{h_i}\right)^{1-\alpha}-\frac{N_i^0}{h_i}\right]_+=\bar{J}.\end{aligned} \quad (11)$$

Thus, in the equilibrium τ has to be equal to τ_* which is the unique root of the equation (11). And consequently,

$$\nu=\tau_*^{-1/\alpha}\omega^{(\alpha-1)/\alpha}. \quad (12)$$

We would like to note that the number of non-zero terms in equation (11) can be found in a finite number of operations by the procedure proposed in [1].

To find ω and ν we note that by (7) the equation (9) turns into the following one:

$$A\nu^{1-\alpha}/\omega^{2-\alpha}+B\omega^{-1/\alpha}=\bar{P},$$

with

$$A=\sum_{i\in I(\tau_*)}(g_i/h_i)^{1-\alpha},\quad B=\sum_{i\in[1,n]\setminus I(\tau_*)}(g_i/N_i^0)^{1-1/\alpha}.$$

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Using (12) we get a one parameter equation for ω

$$\left(A\tau_*^{(\alpha-1)/\alpha}+B\right)\omega^{-1/\alpha}=\bar{P},$$

which implies that

$$\omega=\left(\frac{A\tau_*^{(\alpha-1)/\alpha}+B}{\bar{P}}\right)^{\alpha}. \tag{13}$$

Thus, we have obtained the following result.

THEOREM 2. *The α fairness game with jamming for $\alpha \in (0,2]$ and $\alpha \neq 1$ has the unique equilibrium strategy given by (6) for the Base Station and given by (7) for the jammer where $\tau=\tau_*$ is given as the root of the water filling equation (11) and ω and ν are given by (13) and (12), respectively.*

We would like to note that $P_i(\omega,\nu)\to 0$ for $i\notin I(\tau_*)$ as $\alpha\to 0$ which corresponds to the linear case considered separately in Section 4.

For the case $\alpha=1$ the situation simplifies essentially.

THEOREM 3. *The α fairness game with jamming where $\alpha=1$ has the unique equilibrium. Besides, the Base Station equilibrium strategy is uniform one, namely $P_i=\bar{P}/n$, $i\in[1,n]$ meanwhile the jammer equilibrium strategy is the water filling one, namely $J_i(\tau)=[\tau-N_i^0/h_i]_+$, $i\in[1,n]$ where τ is the unique root of the equation $\sum_{i=1}^n[\tau-N_i^0/h_i]_+=\bar{J}$.*

Furthermore, for the Base Station equilibrium strategy we can get the Jain's fairness index in a closed form as follows:

$$\mathcal{J}=\frac{\bar{P}^2/n}{\dfrac{\nu^{2-2\alpha}}{\omega^{2-4\alpha}}\displaystyle\sum_{i\in I(\tau_*)}\left(\frac{g_i}{h_i}\right)^{2-2\alpha}+\frac{1}{\omega^{2/\alpha}}\sum_{i\notin I(\tau_*)}\left(\frac{g_i}{h_i}\right)^{2/\alpha-2}}.$$

3. THE SHIFTED SNIR α-FAIRNESS FORMULATION

In this section we consider the payoff which is an α-fairness utility function of the shifted SNIR (3). One advantage of this payoff function is that the particular case $\alpha=1$ corresponds to the Shannon capacity maximization. In this section, we consider the case $\alpha>0$. The case $\alpha=0$ corresponds to the linear utility function and will be considered separately in Section 4.

We shall show that the game has the unique Nash equilibrium. Moreover, based on the Kuhn-Tucker Theorem we will present the equilibrium in a closed form depending on two parameters (Lagrange multipliers), in that way reducing the original optimization problem to a problem of finding solution of two non-linear equations with two parameters. Because of some monotonicity properties of this solution we can further simplify the two parameter problem to a one parameter problem with monotonous equation which finally will allow us to produce an efficient algorithm for finding the equilibrium of the original game-theoretical problem.

In Lemma 2 and Theorem 4 we described the structure of the equilibrium. In Lemmas 3–7 some monotonous properties of the equilibrium are presented. In Lemma 8 and Theorem 5 the uniqueness of the equilibrium is proved and an algorithm for the calculation of the equilibrium is presented.

It is clear that $v(P,J)$ is concave on P for any α and $-v(P,J)$ is concave on J for $\alpha\le 2$ since

$$\frac{\partial^2 v}{\partial P^2}=-\alpha g_i^2\frac{(N_i^0+h_iJ_i)^{\alpha-1}}{(N_i^0+h_iJ_i+g_iP_i)^{\alpha+1}}$$

and

$$\frac{\partial^2 v}{\partial J^2}=\alpha P_i g_i h_i^2\frac{(N_i^0+h_iJ_i)^{\alpha-3}}{(N_i^0+h_iJ_i+g_iP_i)^{\alpha+1}}\times(2N_i^0+2h_iJ_i+(2-\alpha)g_iP_i).$$

Thus, we shall assume that $\alpha\le 2$. By [7] the game has the unique Nash equilibrium. We shall investigate its structure and produce an algorithm for its calculation. the following result follows from the application of the Kuhn-Tucker Theorem.

LEMMA 2. *Let $\alpha\le 2$ then (P,J) is the equilibrium if and only if there are ω and ν (Lagrange multipliers) such that*

$$\frac{g_i}{(N_i^0+h_iJ_i+g_iP_i)^{\alpha}(N_i^0+h_iJ_i)^{1-\alpha}}\begin{cases}=\omega, & P_i>0,\\ \le\omega, & P_i=0,\end{cases} \tag{14}$$

$$\frac{g_ih_iP_i}{(N_i^0+h_iJ_i+g_iP_i)^{\alpha}(N_i^0+h_iJ_i)^{2-\alpha}}\begin{cases}=\nu, & J_i>0,\\ \le\nu, & J_i=0.\end{cases} \tag{15}$$

First note that the Lagrange multipliers have to be positive. That $\omega>0$ follows from (14). Since $P_i>0$ at least for one i, then by (15), also $\nu>0$.

Applying Lemma 2 we can describe the structure of the optimal solution more precisely as it is given in the following Theorem.

THEOREM 4. *Each equilibrium is of the form $(P(\omega,\nu),J(\omega,\nu))$ for some positive ω and ν where for $i\in[1,n]$:*

$$P_i(\omega,\nu)=\begin{cases}\left(\dfrac{\omega h_i}{\omega h_i+\nu g_i}\right)^{\alpha}\dfrac{g_i\nu}{h_i\omega^2}, & i\in I_{11}(\omega,\nu),\\ \dfrac{N_i^0}{g_i}\left(\left(\dfrac{g_i}{\omega N_i^0}\right)^{1/\alpha}-1\right), & i\in I_{10}(\omega,\nu),\\ 0, & i\in I_{00}(\omega,\nu),\end{cases}$$

$$J_i(\omega,\nu)=\begin{cases}\dfrac{g_i}{h_i}\left(\dfrac{1}{\omega}\left(\dfrac{\omega h_i}{\omega h_i+\nu g_i}\right)^{\alpha}-\dfrac{N_i^0}{g_i}\right), & i\in I_{11}(\omega,\nu),\\ 0, & i\in I_{00}(\omega,\nu)\\ & \cup I_{10}(\omega,\nu),\end{cases}$$

and

$$I_{11}(\omega,\nu)=\left\{i\in[1,n]:\omega\left(\frac{\omega h_i+\nu g_i}{\omega h_i}\right)^{\alpha}<\frac{g_i}{N_i^0}\right\},$$

$$I_{10}(\omega,\nu)=\left\{i\in[1,n]:\omega<\frac{g_i}{N_i^0}\le\omega\left(\frac{\omega h_i+\nu g_i}{\omega h_i}\right)^{\alpha}\right\},$$

$$I_{00}(\omega,\nu)=\left\{i\in[1,n]:\frac{g_i}{N_i^0}\le\omega\right\}.$$

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Theorem 4 reduces the problem of finding the optimal solution to the problem of finding two parameters ω and ν such that the following conditions hold:

$$H_P(\omega,\nu)=\bar{P}, \quad H_J(\omega,\nu)=\bar{J}, \tag{16}$$

where

$$H_P(\omega,\nu)=\sum_{i=1}^{n}P_i(\omega,\nu), \quad H_J(\omega,\nu)=\sum_{i=1}^{n}J_i(\omega,\nu).$$

The strategies $P(\omega,\nu)$ and $J(\omega,\nu)$ have some continuity and monotonicity properties, formulated in the following Lemma which will allow us to produce a simple algorithm for finding the optimal ω and ν and also to prove that they are unique.

LEMMA 3. *The strategies $P(\omega,\nu)$ and $J(\omega,\nu)$ have the following monotonicity properties:*
(a) $J_i(\omega,\nu)$ for $i\in[1,n]$ and $H_J(\omega,\nu)$ is strictly decreasing in ν while they are positive,
(b) if $\alpha\le 1$, then $J_i(\omega,\nu)$ for $i\in[1,n]$ and $H_J(\omega,\nu)$ are strictly decreasing in ω while they are positive,
(c) $P_i(\omega,\nu)$ for $i\in[1,n]$ and $H_P(\omega,\nu)$ are strictly decreasing in ω while they are positive,
(d) if $\alpha\le 1$, then $P_i(\omega,\nu)$ for $i\in[1,n]$ and $H_P(\omega,\nu)$ are strictly increasing in ν while they are positive,
(e) $H_P(\omega,\nu)$ and $H_J(\omega,\nu)$ are continuous functions.

In the next two lemmas we show that there is an explicit monotone relation between ω and ν in (16). It will allow to reduce the two parameters problem (16) to a one parameter problem.

LEMMA 4. *For each $\omega\in(0,\bar{\omega}]$ there is unique non-negative $\nu(\omega)$ such that $H_J(\omega,\nu(\omega))=\bar{J}$ where $\bar{\omega}>0$ is the unique root of the equation $H_J(\bar{\omega},0)=\bar{J}$.*

From Lemma 4 and Lemma 3 (a),(d) we have the following result.

LEMMA 5. *$\nu(\omega)$ is continuous strictly decreasing function on $(0,\bar{\omega}]$ such that $\nu(\bar{\omega})=0$ and $\nu(0+)=\infty$.*

LEMMA 6. *The solution of two parameters non-linear system (16) is equivalent to the solution of one parameter non-linear equation*

$$H_P(\omega,\nu(\omega))=\bar{P}. \tag{17}$$

The function $H_P(\omega\nu(\omega))$ has the following properties with respect to ω based on Lemma 5 and Lemma 3(b)-(d).

LEMMA 7. *The function $H_P(\omega,\nu(\omega))$ has the following properties:*
(a) $H_P(\omega,\nu(\omega))$ is continuous for $\omega>0$,
(b) $H_P(\bar{\omega},\nu(\bar{\omega}))=0$,
(c) $H_P(0+,\nu(0+))=+\infty$,
(d) $H_P(\omega,\nu(\omega))$ is strictly decreasing while it is positive and $\alpha\le 1$.

From Lemma 7 we immediately have the following result about solution of (17).

LEMMA 8. *The equation (17) has a positive root ω_* which can be found, for example, by bisection method. If $\alpha\le 1$ then this root is unique.*

Putting together Lemma 8 and Theorem 4 we have the following result giving the optimal solution.

THEOREM 5. *The game has the unique equilibrium $(P(\omega_*,\nu(\omega_*)),J(\omega_*,\nu(\omega_*)))$ for $\alpha\le 1$.*

It is worth to note that for the Shannon capacity case (i.e. $\alpha=1$) the equilibrium strategies have simpler structure. Namely, it is of the form $(P(\omega,\nu),J(\omega,\nu))$ where for $i\in[1,n]$ we have:

$$P_i(\omega,\nu)=\begin{cases}\dfrac{g_i}{\omega h_i+\nu g_i}\dfrac{\nu}{\omega}, & i\in I_{11}(\omega,\nu),\\ \dfrac{1}{\omega}-\dfrac{N_i^0}{g_i}, & i\in I_{10}(\omega,\nu),\\ 0, & i\in I_{00}(\omega,\nu),\end{cases}$$

$$J_i(\omega,\nu)=\begin{cases}\dfrac{g_i}{\omega h_i+\nu g_i}-\dfrac{N_i^0}{h_i}, & i\in I_{11}(\omega,\nu),\\ 0, & i\in I_{00}(\omega,\nu)\\ & \cup I_{10}(\omega,\nu)\end{cases}$$

and

$$I_{11}(\omega,\nu)=\left\{i\in[1,n]:\frac{\omega h_i+\nu g_i}{h_i}<\frac{g_i}{N_i^0}\right\},$$

$$I_{10}(\omega,\nu)=\left\{i\in[1,n]:\omega<\frac{g_i}{N_i^0}\le\frac{\omega h_i+\nu g_i}{h_i}\right\},$$

$$I_{00}(\omega,\nu)=\left\{i\in[1,n]:\frac{g_i}{N_i^0}\le\omega\right\}.$$

In the case $\alpha\in(1,2]$ the function $H_P(\omega,\nu(\omega))$ losses the monotonicity properties so by the above arguments the equation $H_P(\omega,\nu(\omega))=\bar{P}$ could in principle have more that one root. However, using the results of [7] it still has the unique root which supplies NE.

For a particular case when fading coefficients are proportional, namely,

$$g_i=\gamma h_i \text{ for } i\in[1,n]$$

the optimal strategies can be simplified, by introducing a new parameter τ instead of ν, in the following way:

$$P_i(\omega,\tau)=\begin{cases}\frac{1}{\tau}\left(\left(\frac{\tau}{\omega}\right)^{1/\alpha}-1\right), & i\in I_{11}(\tau),\\ \frac{N_i^0}{g_i}\left(\left(\frac{g_i}{\omega N_i^0}\right)^{1/\alpha}-1\right), & i\in I_{10}(\omega,\tau),\\ 0, & i\in I_{00}(\omega),\end{cases} \tag{18}$$

$$J_i(\tau)=\begin{cases}\gamma\left(\frac{1}{\tau}-\frac{N_i^0}{g_i}\right), & i\in I_{11}(\tau),\\ 0, & i\in[1,n]\backslash I_{11}(\tau),\end{cases}$$

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where

$$I_{11}(\tau)=\left\{i\in[1,n]:\tau<\frac{g_i}{N_i^0}\right\},$$

$$I_{10}(\omega,\tau)=\left\{i\in[1,n]:\omega<\frac{g_i}{N_i^0}\le\tau\right\},$$

$$I_{00}(\omega)=\left\{i\in[1,n]:\frac{g_i}{N_i^0}\le\omega\right\}$$

and

$$\tau=\omega\left(1+\frac{\gamma\nu}{\omega}\right)^{\alpha}.$$

So, we have

$$\nu=\frac{\omega}{\gamma}\left(\left(\frac{\tau}{\omega}\right)^{1/\alpha}-1\right).$$

Thus, the optimal jammer strategy has the water filling structure:

$$J_i(\tau)=\gamma\left[\frac{1}{\tau}-\frac{N_i^0}{g_i}\right]_+.$$

and the optimal value of τ_* can be found as the unique positive root of the following the water filling equation:

$$\gamma\sum_{\{i:\tau<g_i/N_i^0\}}\left(\frac{1}{\tau}-\frac{N_i^0}{g_i}\right)=\bar{J}. \tag{19}$$

Then, knowing τ_* we can find the optimal value of ω_* as the unique root in $(0,\tau_*]$ of the equation:

$$\begin{aligned}&\frac{1}{\tau_*}\left(\left(\frac{\tau_*}{\omega}\right)^{1/\alpha}-1\right)\sum_{\{i:\tau_*<g_i/N_i^0\}}1\\&+\sum_{\{i:\omega<g_i/N_i^0\le\tau_*\}}\frac{N_i^0}{g_i}\left(\left(\frac{g_i}{\omega N_i^0}\right)^{1/\alpha}-1\right)=\bar{P}.\end{aligned} \tag{20}$$

The formulas (18) and (19) bring us to an interesting conclusion about possibility to use uniform distribution as an optimal strategy, namely, if jammer tries to jam all the users, it takes place under the following condition:

$$\gamma\sum_{i=1}^{n}\left(\max_j\frac{N_j^0}{g_j}-\frac{N_i^0}{g_i}\right)\ge\bar{J}.$$

then the optimal strategy of the Base Station is to allocate the resources to the users equally. Also, it is very surprising that by (19) the jammer equilibrium strategy does not depend on α, meanwhile by (20) the base station equilibrium strategy of course depends on α.

Let us present an algorithm based on the bisection method, Theorem 5, and Lemmas 7 and 8 for finding the optimal values of ω and ν and the corresponding equilibrium strategies.

Description of the algorithm:

Step 1 Let $\omega^0=\epsilon$ (ϵ is the algorithm's tolerance) and $\omega^1=\bar{\Omega}()$.

Step 2 If $H_P(\omega^0,BS(\omega^0))<\bar{P}$ then $\epsilon=\epsilon/2$ and go to step 1.

Step 3 Set $\omega_c=(\omega^1+\omega^0)/2$.

Step 4 If $\omega^1-\omega^0\le\epsilon$, then $\omega^*=(\omega^1+\omega^0)/2$, $\nu^*=BS(\omega^*)$ and $(P(\omega^*,\nu^*),J(\omega^*,\nu^*))$ is equilibrium and the algorithm is terminated.

Step 5 $\nu_c=BS(\omega_c)$.

Step 6 If $\omega^1-\omega^0>\epsilon$, then, if $H_P(\omega_c,\nu_c)<\bar{T}$ then set $\omega^1=\omega_c$ else set $\omega^0=\omega_c$ and go to Step 3.

Step 7 Let $\omega^1-\omega^0>\epsilon$ and $H_P(\omega_c,\nu_c)=\bar{T}$ then $\omega^*=\omega_c$, $\nu^*=\nu_c$ and $(P(\omega^*,\nu^*),J(\omega^*,\nu^*))$ is equilibrium and the algorithm is terminated.

Function $\nu=BS(\omega)$ (defined for $\omega\in(0,\bar{\Omega}()]$)

Step 1 Set $\nu^0=\epsilon$ and

$$\nu^1=\max_i\left\{\frac{\omega h_i}{g_i}\left(\left(\frac{g_i}{\omega N_i^0}\right)^{1/\alpha}-1\right)\right\}$$

Step 2 Set $\nu_c=(\nu^1+\nu^0)/2$.

Step 3 If $\nu^1-\nu^0\le\epsilon$, then return $(\nu^1+\nu^0)/2$.

Step 4 If $\nu^1-\nu^0>\epsilon$ then, if $H_J(\omega,\nu_c)<\bar{J}$ then set $\nu^1=\nu_c$ else set $\nu^0=\nu_c$ and go to Step 2.

Step 5 Let $\nu^1-\nu^0>\epsilon$ and $H_J(\omega,\nu_c)=\bar{J}$ then return ν_c.

Function $\bar{\omega}=\bar{\Omega}()$

Step 1 Set $\omega^0=\epsilon$, $\omega^1=\max_i\{g_i/N_i^0\}$.

Step 2 Set $\omega_c=(\omega^1+\omega^0)/2$.

Step 3 If $\omega^1-\omega^0\le\epsilon$, then return $(\omega^1+\omega^0)/2$.

Step 4 If $\omega^1-\omega^0>\epsilon$ then, if $H_J(\omega_c,0)<\bar{J}$ then set $\omega^1=\omega_c$ else set $\omega^0=\omega_c$, and go to Step 2.

Step 5 Let $\omega^1-\omega^0>\epsilon$ and $H_J(\omega_c,0)=\bar{J}$ then return ω_c.

where $\epsilon>0$ is the tolerance of finding the optimal ω and ν.

4. LINEAR UTILITY FUNCTION FORMULATION

In this section we consider the jamming game with linear utility function, which corresponds to $\alpha=0$ in both SNIR and Shifted SNIR based utility functions. Thus, we consider the following payoff

$$v(P,J)=\sum_{i=1}^{n}\frac{g_iP_i}{N_i^0+h_iJ_i}.$$

Since $v(P,J)$ is linear in P and concave in J we have the following result describing the optimal solution.

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THEOREM 6. *There is unique equilibrium (P, J) such that P_i and J_i are positive at the same sub-channels. Namely, the equilibrium has the form $(P(\omega, \nu), J(\omega))$ where the jammer strategy $J(\omega)$ has the water filling structure*

$$J_i(\omega) = \frac{g_i}{h_i}\left[\frac{1}{\omega} - \frac{N_i^0}{g_i}\right]_+ \quad \text{for } i \in [1, n]$$

Meanwhile the base station strategy has the form

$$P_i(\omega, \nu) = \begin{cases} \frac{\nu}{\omega^2}\frac{g_i}{h_i} & \text{for } i \in I(\omega), \\ 0 & \text{otherwise,} \end{cases}$$

where $I(\omega) = \{i \in [1, n] : J_i(\omega) > 0\}$ and ω is the unique root of the equation:

$$\sum_{i=1}^{n} \frac{g_i}{h_i}\left[\frac{1}{\omega} - \frac{N_i^0}{g_i}\right]_+ = \bar{J}$$

and

$$\nu = \frac{\bar{P}}{\sum_{i \in I(\omega)} \pi_i g_i / h_i}\omega^2.$$

Without loss of generality we can assume that the channels are arranged such that

$$N_1^0/g_1 \le N_2^0/g_2 \le \ldots \le N_n^0/g_n.$$

Then following the approach developed in [1] for water-filling optimization problem we can present solution in closed form as given in the following theorem.

THEOREM 7. *The solution (P^*, J^*) of the jamming game with linear utility function is given by*

$$J_i^* = \begin{cases} \dfrac{\bar{P} + \sum_{t=1}^{k}(g_t/h_t)(N_t^0 - N_i^0)}{\sum_{t=1}^{k}(\pi_t g_t/h_t)}, & \text{if } i \le k, \\ 0, & \text{if } i > k, \end{cases}$$

$$P_i^* = \begin{cases} \dfrac{\bar{P} g_i/h_i}{\sum_{t=1}^{k} g_t/h_t}, & \text{if } i \le k, \\ 0, & \text{if } i > k, \end{cases}$$

where k can be found from the following conditions:

$$\varphi_k < \bar{P} \le \varphi_{k+1},$$

where

$$\varphi_t = \sum_{i=1}^{t}(g_i/h_i)(N_t^0 - N_i^0) \text{ for } t \in [1, n]$$

and $\varphi_{n+1} = \infty$.

Besides, for the base station equilibrium strategy the Jain's fairness index is given as follows:

$$\mathcal{J} = \frac{1}{n}\frac{\left(\sum_{t=1}^{k}(g_t/h_t)\right)^2}{\sum_{t=1}^{k}(g_t/h_t)^2}.$$

5. NUMERICAL EXAMPLES

In this section we present some numerical examples of the equilibrium strategies and the value of the Jain's fairness index. We consider an important particular case when the jammer is near the base station and there are five users ($n = 5$). In this scenario $h_i = 1$ for all $i \in [1, 5]$. We take $g_i = \kappa^{i-1}$ for $i \in [1, 5]$ where $\kappa = 0.7$ and $N_i^0 = 1$ for $i \in [1, 5]$ In Table 1 we give strategies of the Base Station and the jammer for different value of α and for $\bar{P} = 10$ and $\bar{J} = 1$ for the SNIR and the shifted SNIR based payoffs.

Table 1: Dependence of the strategies on α

α		user and jammer strategies				
0.0	P	5.88	4.12	0.00	0.00	0.00
	J	0.77	0.24	0.00	0.00	0.00
0.5 (SNIR)	P	2.89	2.42	2.03	1.57	1.10
	J	0.58	0.32	0.10	0.00	0.00
0.5 (Shifted SNIR)	P	4.03	3.18	2.47	0.34	0.00
	J	0.67	0.31	0.02	0.00	0.00
1.0 (SNIR)	P	2.00	2.00	2.00	2.00	2.00
	J	0.20	0.20	0.20	0.20	0.20
1.0 (Shifted SNIR)	P	3.03	2.65	2.24	1.64	0.39
	J	0.53	0.34	0.13	0.00	0.00
1.5 (SNIR)	P	1.45	1.64	1.91	2.28	2.72
	J	0.00	0.00	0.10	0.32	0.58
1.5 (Shifted SNIR)	P	2.41	2.31	2.12	1.86	1.31
	J	0.38	0.33	0.22	0.07	0.00
2.0 (SNIR)	P	1.22	1.45	1.74	2.31	3.29
	J	0.00	0.00	0.00	0.24	0.77
2.0 (Shifted SNIR)	P	2.01	2.13	2.12	1.98	1.74
	J	0.21	0.28	0.27	0.19	0.05

In Figure 1 the Jain's fairness index for the Base Station equilibrium strategies for the SNIR and the shifted SNIR utility functions is plotted as a function of the parameter α.

It is interesting that they coincide at two points $\alpha = 0$ and $\alpha = 1.45$ and the Jain's fairness index for SNIR based utility function achieve the maximum around the point $\alpha = 1$.

6. APPENDIX

Proof of Theorem 4. Let (P, J) be an equilibrium.
(a) Let $P_i = 0$ then $J_i = 0$ and $g_i/N_i^0 \le \omega$. So, if $P_i = 0$ and $J_i = 0$ then $i \in I_{00}(\omega, nu)$
(b) Let $P_i > 0$ and $J_i = 0$. Then, by (14)

$$\frac{g_i}{(N_i^0 + g_i P_i)^\alpha (N_i^0)^{1-\alpha}} = \omega.$$

Thus, $\omega < \frac{g_i}{N_i^0}$ and P_i is given by (22). Then, by (15),

$$\begin{aligned} \nu &\ge \frac{g_i h_i P_i}{(N_i^0 + g_i P_i)^\alpha (N_i^0)^{2-\alpha}} = \frac{h_i \omega}{N_i^0} P_i \\ &= \frac{h_i \omega}{g_i}\left(\left(\frac{g_i}{\omega N_i^0}\right)^{1/\alpha} - 1\right). \end{aligned} \tag{21}$$

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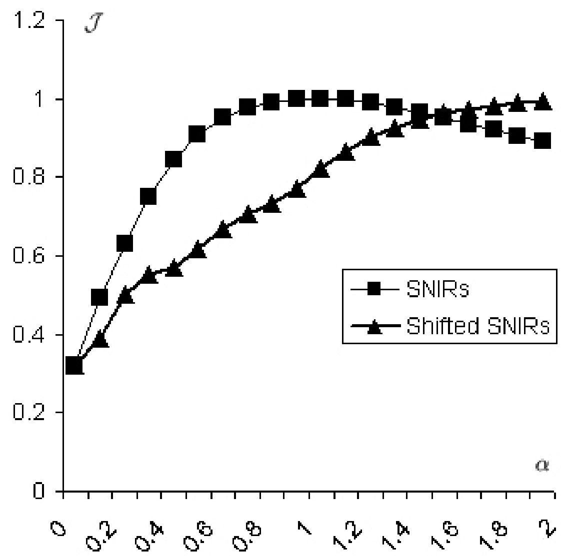

Figure 1: The Jain's fairness index

So,

$$\frac{g_i}{N_i^0} \le \omega \left(\frac{\omega h_i + \nu g_i}{\omega h_i} \right)^{\alpha}.$$

So,

$$P_i = \frac{N_i^0}{g_i} \left(\left(\frac{g_i}{\omega N_i^0} \right)^{1/\alpha} - 1 \right) \tag{22}$$

and $i \in I_{10}(\omega, \nu)$.
(c) If $P_i > 0$ and $J_i > 0$ then, by (14) and (15), we have

$$\frac{g_i}{(N_i^0 + h_i J_i + g_i P_i)^{\alpha} (N_i^0 + h_i J_i)^{1-\alpha}} = \omega, \tag{23}$$

$$\frac{g_i h_i P_i}{(N_i^0 + h_i J_i + g_i P_i)^{\alpha} (N_i^0 + h_i J_i)^{2-\alpha}} = \nu. \tag{24}$$

Dividing (23) by (24) we have

$$\omega h_i P_i = \nu (N_i^0 + h_i J_i)$$

Hence, (23) and (24) can be present in the following equivalent form:

$$\frac{g_i}{N_i^0 + h_i J_i} \left(\frac{\omega h_i}{\omega h_i + \nu g_i} \right)^{\alpha} = \omega,$$

$$\frac{g_i \nu^2}{h_i P_i \omega^2} \left(\frac{\omega h_i}{\omega h_i + \nu g_i} \right)^{\alpha} = \nu.$$

So,

$$J_i = \frac{g_i}{h_i} \left(\frac{1}{\omega} \left(\frac{\omega h_i}{\omega h_i + \nu g_i} \right)^{\alpha} - \frac{N_i^0}{g_i} \right),$$

$$P_i = \left(\frac{\omega h_i}{\omega h_i + \nu g_i} \right)^{\alpha} \frac{g_i \nu}{h_i \omega^2}$$

and $i \in I_{11}(\omega, \nu)$.

Proof of Lemma 3. (a) and (e) are obvious. (b)–(d) follows from the relations:

$$\begin{aligned} \frac{d}{d\nu} P_i(\omega, \nu) &= -\frac{d}{d\omega} J_i(\omega, \nu) \\ &= \frac{g_i}{\omega} \frac{(\omega h_i)^{\alpha-1}}{(\omega h_i + \nu g_i)^{\alpha+1}} ((1-\alpha)\nu g_i + \omega h_i) > 0, \end{aligned}$$

$$\frac{d}{d\omega} P_i(\omega, \nu) = -\frac{g_i \nu}{\omega^2} \frac{(\omega h_i)^{\alpha-1}}{(\omega h_i + \nu g_i)^{\alpha+1}} ((2-\alpha)\nu g_i + 2\omega h_i) < 0,$$

This completes the proof of Lemma 3.

Proof of Lemma 4. It is clear that $H_J(0+, 0) = \infty$ and $H_J(\omega, 0) = 0$ for enough big ω. So, by Lemma 4(a),(d) there is unique $\bar{\omega}$ such that $H_J(\bar{\omega}, 0) = \bar{J}$. So, $H_J(\omega, 0) > \bar{J}$ and $H_J(\omega, \nu) = 0$ for each $\omega < \bar{\omega}$ and enough big ν. Then result follows from Lemma 4(a),(d).

Proof of Lemma 7. (a), (c) and (d) are obvious. (d) follows from the following relations:

$$H_P(\bar{\omega}, \nu(\bar{\omega})) = H_P(\bar{\omega}, 0) = (\text{since } I_{10}(\bar{\omega}, 0) = \emptyset) = 0.$$

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