Stability of two interfering processors with load balancing

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ABSTRACT

We examine the stability of two interfering processors with service rates depending on the number of users present of each of the classes and subject to static or dynamic load balancing. Such models arise in several contexts, especially in wireless networks, or multiprocessing. In case of static load balancing, we extend existing stability results by deriving Lyapunov functions that are connected to the solutions of one dimensional Poisson equation. We then characterize the optimal static load balancing. The Lyapunov function found for the static load balancing is used to derive the exact stability condition of an interesting class of dynamic load balancing policies. We show that for certain properties of the state-dependent service rates, simple dynamic load balancing schemes improve the stability condition.

Categories and Subject Descriptors

D.4.8 [**Performance**]: Queueing Theory

General Terms

Performance

Keywords

Stability, Dynamic load balancing, State-dependent service rates, Lyapunov functions.

1. INTRODUCTION

We consider a queueing system with two servers defining two classes of customers and the special feature that the service rates of the servers depend on the number of users present of each class. Such models arise in various settings, particularly in wireless communications, manufacturing systems, and multiprocessing. In wireless networks for instance, the transmission rates in a given cell depend on the activity states of the surrounding cells due to the impact of interference. Thus, the transmission rates in a given cell increase when the interfering cells have no traffic to serve [2]. In these scenarios, the total service rate is hence not maintained where the cells are simultaneously non-empty, so the system will typically not be work-conserving. The service rates as functions of the number of customers become even more complicated in wireless data networks when channel-aware scheduling is employed: the total service rate available to all customers of a given class can be increasing in the total number of customers of this class, due to multiuser diversity [9].

Multiprocessing is the use of two or more central processing units (CPUs) within a single computer system. CPUs can be used symmetrically or can be dedicated to a certain type of task (for instance the execution of the kernel mode code). In both cases, the sharing of a common memory access introduces a reduction of capacity, when CPUs work simultaneously.

In the above examples, the service rates of the corresponding queuing model may depend on which queues are empty, or more generally on the exact number of users or tasks present at each queue. In the first case, the model is equivalent to the so-called coupled processors (or weakly coupled processors) system as considered in [7, 6].

In the present paper, we examine the stability properties of the model described above, with the additional assumptions that the arriving customers can be dynamically directed to one of the queue. Dynamic means here that the routing decision is based on the number of customers present at each queue. Under the usual assumptions of exponentially distributed service times and Poisson arrivals, the process describing the number of customers in the network is Markov and stability refers in the sequel to its positive recurrence.

The stability of multidimensional processes corresponding to interfering processors with static load balancing have been recently studied in [5]. The authors focused on the case where the service rates of the various classes are monotone functions in the sense that they decrease with the number of users of the various classes, and the arrival intensities do not depend on the state of the network. These monotonicity assumptions are satisfied in a variety of situations, such as the examples described above. The method used to prove stability in [5], (as in other contexts in [11, 13]) is essentially stochastic domination and requires some monotonicity properties of the studied Markov process. When the arrival rates depend on the number of customers present in the network (i.e., in the presence of dynamic load balancing), these monotonicity properties are generally not verified anymore and the latter techniques cannot be employed.

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A main contribution of this article is to overcome these technical difficulties by finding Lyapunov functions for this type of model. These Lyapunov functions are connected to the solutions of Poisson equations of dimension 1 which explains the difficulty to intuitively find them. This allows to strengthen the stability results previously derived for a system with two classes of customers and to attack the stability problem of systems with dynamic load balancing. Finding a Lyapunov function not only allows to characterize whether the process is positive recurrent but can also give indications on the convergence to its stationary measure (see [10] for more details). On the other hand, our technique allows to partially release the assumption of monotonicity of the service rates and to find necessary and/or sufficient conditions of stability for non-monotone service rates.

Building on these results, we first characterize the optimal static load balancing and the maximum arrival intensity that the network can put up with when the arrival intensities are fixed. We show in particular that even for symmetric state dependent service rates, the optimal static policy can be to send the arriving customers to each queue with unequal positive probabilities being the roots of certain functions. We also give a simple upper bound on the maximum intensity leading to a stable system. This bound is attained in the case of the weakly coupled processors, which shows that a dynamic load balancing cannot improve the stability of a static load balancing when the service rates depend only on whether the queues are empty or not. This result can be more generally interpreted in terms of some properties of the stability region in case of static load balancing.

We then turn our attention to a class of dynamic load balancing depending only on the number of customers present of one of the classes. This assumption makes the analysis considerably simpler and tractable. Using the Lyapunov functions derived for the static case, we derive the stability conditions of such schemes and further characterize the optimal scheme of this class. Depending on the service rates, we find that such dynamic schemes might improve the stability conditions over static schemes. This result contrasts with the case of static (non-state dependent) service rates, where dynamic load balancing does not improve the stability condition (see for instance [8]). To the best of our knowledge, a similar result has not been observed previously in queuing networks. We illustrate the results on a simple example. These results suggest that dynamic load balancing of a switch curve type might significantly improve the stability condition for certain service rates.

The remainder of the paper is organized as follows. In Section 2, we present a detailed model description and introduce notation. In Section 3, we derive Lyapunov functions for the model with static load balancing. and extend existing stability results. In Section 4, we characterize the optimal static load balancing in terms of the service rates properties. In Section 5, we give a simple bound on the maximal intensity leading to a stable system. Stochastic comparisons are established to interpret the result. In Section 6, the class of decoupled load balancing is introduced. We characterize the optimal load balancing within that class and give an example of gain of stability via dynamically balancing the load. Section 7 concludes with some final observations and suggestions for further research.

2. MODEL

We consider an open network with 2 nodes. $x = (x_1, x_2) \in \mathbb{N}^2$ denotes the number of customers in each node. Customers arrive subject to a Poisson process of intensity ν and are directed with an intensity $\lambda_i(x)$ towards node $i = 1, 2$ where they require an exponentially distributed service time. We assume that the arrival process and the service times are independent and the service times are i.i.d. By definition of the $\lambda_i(x)$:

$$\forall x, \sum_{i=1}^{2} \lambda_i(x) = \nu.$$

Let X be the stochastic process describing the state of the system on a state space $x \in \mathbb{N}^2$. Denote e_i, $i = 1, 2$ the unit vectors in $\mathbb{N}^2$. Following from the statistical assumptions made on the traffic, X is a multi-dimensional birth and death process with transition rates, $i = 1, 2$:

$$q(x, x - e_i) = \phi_i(x),$$
$$q(x, x + e_i) = \lambda_i(x).$$

2.1 Assumptions on the service rates

In the following, we consider a model M with given service rates ϕ having the following properties:

1. ϕ is monotone in the sense that $\phi_i(x)$ is decreasing in x_j when $x_i > 0$, $\forall j \neq i$. In other words, adding a customer in queue j decreases the capacity of queue $i \neq j$, (when there are customers of class i to be served).
2. $\phi_i(.)$ has a limit when x_1 or/and x_2 go to infinity and the limits are uniform in the sense that,

$$\lim_{x_1\to\infty} \lim_{x_2\to\infty} \phi_i(x) = \lim_{x_2\to\infty} \lim_{x_1\to\infty} \phi_i(x), i = 1, 2.$$

   This assumption is sometimes referred to as 'asymptotically spatially homogeneous' [4].

An interesting particular case is when the service rates are symmetric and do not depend on their own coordinate. We hence define the symmetric instance (M') of our model verifying the additional assumptions:

$$\phi_i(x) = \phi_i(x_j) = \phi(x_j),\ j \neq i.$$

2.2 Static and dynamic allocations

A routing scheme is a choice of rates $\lambda(\cdot)$. We consider in the following routing schemes that depend only on the number of customers in the network at the time of the routing decision (the residual service times and the past history do not influence the decisions taken). Note that the routing can be probabilistic (at time t customers are directed to queue i with probability $\frac{\lambda_i(x)}{\nu}$), or deterministic (if $\lambda_i(x) = \nu$ or $\lambda_i(x) = 0$, $\forall x$.) The policy is said to be **static** if $\lambda(\cdot)$ does not depend x. Otherwise the policy is called **dynamic**.

We study the stability properties of this system and more specifically address the following issues:

1. What is the maximal stability condition of the system i.e., for a given ϕ, what is the maximum ν for which there exists $\lambda(\cdot)$ such that the process X with intensity $\nu - \epsilon$ is stable (positive recurrent), for all $\epsilon > 0$.

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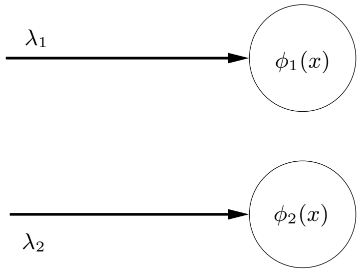

Figure 1: Interfering processors with static load balancing

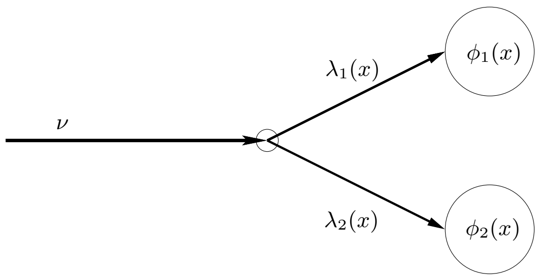

Figure 2: Interfering processors with dynamic load balancing

We denote this maximal intensity $\nu^{\max}$. This question appears difficult both theoretically and numerically. A first step is thus to find theoretical bounds on $\nu^{\max}$.

2. Define similarly $\nu_{\mathcal{S}}^{\max}$ the maximum ν for which there exists a *static* λ, such that the process X with intensity $\nu-\epsilon$ is stable (positive recurrent), for all $\epsilon > 0$. How to calculate $\nu_{\mathcal{S}}^{\max}$ and what is the corresponding optimal static load balancing?

3. When do we have $\nu^{\max} = \nu_{\mathcal{S}}^{\max}$, and $\nu^{\max} > \nu_{\mathcal{S}}^{\max}$?

3. STABILITY CONDITIONS FOR STATIC LOAD BALANCING

We first need to introduce notations to describe the rates of the system when the number of customers tend to infinity. In the sequel, we denote

$$C_i = \lim_{x_1,x_2} \phi_i(x),\ i = 1, 2,$$

the limit of the service rates when both queues tend to infinity, and

$$\phi_1^1(x_1) = \lim_{x_2\to\infty} \phi_1(x),$$
$$\phi_2^1(x_1) = \lim_{x_2\to\infty} \phi_2(x),$$
$$\phi_1^2(x_2) = \lim_{x_1\to\infty} \phi_1(x),$$
$$\phi_2^2(x_2) = \lim_{x_1\to\infty} \phi_2(x),$$

the limit of the service rates, when one of the queue tends to infinity. In a more compact form,

$$\phi_j^i(x_i) = \lim_{x_k\to\infty, k\neq i} \phi_j(x).$$

We also define what can be informally seen as an average departure rate for the system, when customers of class $j \neq i$ tends to infinity, while the number of class-i customers is a stable process. Using a scalar product notation, define:

$$\langle \phi_j^i, \pi^i \rangle = \sum_{x_i} \phi_j^i(x_i)\pi^i(x_i), \quad j \neq i,$$

where π^i, when it exists, is the probability distribution given by (c is the normalization constant):

$$\pi^i(x_i) = c \prod_{z=1}^{x_i} \frac{\lambda_i}{\phi_i^i(z)}, \quad j \neq i. \tag{1}$$

When the load balancing is static and ϕ is decreasing (as defined in subsection 2.1), the stability region of the system was derived in [5].

Theorem 1 ([5]).
If ϕ is decreasing, the stability region can be described as the set of (λ_1, λ_2) such that either

$$\lambda_1 < C_1 \quad \textit{and} \quad \lambda_2 < \langle \phi_2^1, \pi^1 \rangle, \tag{2}$$
$$\textit{or } \lambda_2 < C_2 \quad \textit{and} \quad \lambda_1 < \langle \phi_1^2, \pi^2 \rangle. \tag{3}$$

It is remarkable that the shape of the stability region is generally not simple and might be different from the rate region defined as the convex hull of the service rates $(\phi(x))_{x\in\mathcal{X}}$.

In [5], the use of stochastic comparisons allowed to derive the conditions under which the process X is positive recurrent. However, stronger stability results[1] may be established if a Lyapunov function is known. In the following Theorem, we show how to find Lyapunov functions for the system by solving Poisson equations of dimension 1. For that purpose, suppose without loss of generality that $\lambda_i + \phi_i(x) < 1,\ i = 1, 2,\ \forall x$ and define P_1 the Markovian kernel with transitions:

$$\begin{aligned} p_1(x_1, x_1+1) &= \lambda_1, \\ p_1(x_1, x_1-1) &= \phi_1^1(x_1), \\ p_1(x_1, x_1) &= 1 - (\lambda_1 + \phi_1^1(x_1)). \end{aligned}$$

Define symmetrically P_2. Note that the Markov chain with generator P_i correspond to a uniformized discrete time version of the continuous time birth and death process with rates λ_i and $\phi_i^i(x_i)$. π^1 and π^2 are hence the stationary measures associated with P_1 and P_2 (when they exist) and were already defined by (1).

Theorem 2. *Suppose ϕ decreasing and assume that (λ_1, λ_2) verify:*

$$\lambda_1 < C_1 \quad \textit{and} \quad \lambda_2 < \langle \phi_2^1, \pi^1 \rangle, \tag{4}$$
$$\textit{or } \lambda_2 < C_2 \quad \textit{and} \quad \lambda_1 < \langle \phi_1^2, \pi^2 \rangle. \tag{5}$$

[1] convergence of the process to its stationary measure in a stronger sense and f-ergodicity results, see [10].

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Then, a Lyapunov function for the system is given by:

$$\begin{aligned}
F(x) &= 1_{\lambda_1<C_1}F_1(x) + 1_{\lambda_2<C_2}F_2(x),\\
&\textit{with}\\
F_1(x) &= \psi^{x_1} + \gamma_1[x_2 + V_1(x_1)],\\
F_2(x) &= \psi'^{x_2} + \gamma_2[x_1 + V_2(x_2)].
\end{aligned}$$

With $\psi > 1, \psi' > 1, \gamma_1 > 0, \gamma_2 > 0$ some constants to be chosen and V_1 and V_2 some functions defined as the solutions of the following Poisson equations. If $\lambda_1 < C_1$, V_1 is the bounded solution of:

$$(I - P_1)V_1 = \lambda_2 - \phi_2^1 + \epsilon$$

where ϵ is defined as:

$$\epsilon = -\langle \lambda_2 - \phi_2^1, \pi^1\rangle. \tag{6}$$

V_2 is defined symmetrically.

Proof:
First, remark that if $\lambda_1 < C_1$ and $\lambda_2 < C_2$, then the process is geometrically ergodic because each coordinate X_i is dominated by a geometrically ergodic Markov process.

Hence, assume $\lambda_1 < C_1$ and $\lambda_2 \geq C_2$. (The symmetric case is equivalent.) Remark that under this condition, π^1 is well defined. Let $\psi > 1$ and $\epsilon' > 0$ such that $\epsilon' = \frac{C_1}{\psi} - \lambda_1$. Consider the (one-dimensional) Poisson equation:

$$\begin{aligned}
\lambda_1(V(x_1+1) - V(x_1)) + \phi_1^1(x_1)(V(x_1-1) - V(x_1))\\
= -(\lambda_2 - \phi_2^1(x_1) + \epsilon). \quad (7)
\end{aligned}$$

or written differently:

$$(I - P_1)V = \lambda_2 - \phi_2^1 + \epsilon,$$

with P_1 the corresponding Markovian kernel defined previously[2]. Because $\lambda_1 < C_1$, it is not difficult to see that the Markov process associated with the kernel P_1 is geometrically ergodic. We can therefore apply the results of [1]: there exists a bounded solution to equation (7) if condition $\lambda_1 < C_1$ is fulfilled together with:

$$\langle \lambda_2 - \phi_2^1 + \epsilon, \pi^1\rangle = 0,$$

which we have supposed in condition (6). Let V be such a solution.

Using the existence of uniform limits for ϕ_2, we have that for all $\delta > 0$, there exists x_2^* such that $\forall x_1$, and $\forall x_2 \geq x_2^*$:

$$|\phi_2(x) - \phi_2^1(x_1)| \leq \delta.$$

Let $\delta < \epsilon'$ and consider the following Lyapunov function:

$$F(x) = \psi^{x_1} + \gamma_1(x_2 + V_1(x_1)),$$

with γ_1 a strictly positive constant to be chosen. Since $\psi > 1$, $F(x) \to \infty$, when $|x| \to \infty$ and F is hence a proper Lyapunov function. The drift of F (for the original process X) is defined as:

$$\Delta F(x) = \sum_{y\in\mathbb{N}^2} q(x,y)(F(y) - F(x)),$$

[2] In applied probability, the Poisson equation is often written as $(I - P_1)V = r - c$, with r a reward function and c a constant given by $c = \sum_x \pi^1(x_1)r(x_1)$. Here the drift $\lambda_2 - \phi_2^1$ plays the role of the reward function and ϵ of the constant.

which gives:

$$\Delta F(x) = (\lambda_1 - \frac{\phi_1(x)}{\psi})(\psi-1)\psi^{x_1} + \gamma_1(\lambda_2 - \phi_2(x) + \Delta V(x_1)),$$

with $\Delta V(x_1) = \lambda_1[V_1(x_1+1) - V_1(x_1)] + \phi_1(x)[V_1(x_1-1) - V_1(x_1)]$.

-Suppose first that $x_2 < x_2^*$. Using the monotonicity of ϕ and the existence of limits for ϕ_1^1, there exists x_1^* such that for all $x_1 > x_1^*$,

$$\frac{\phi_1(x)}{\psi} \geq \frac{\phi_1^1(x_1)}{\psi} \geq \frac{C_1}{\psi} - \epsilon'/2,$$

which leads to

$$\lambda_1 - \frac{\phi_1(x)}{\psi} \leq \lambda_1 - \frac{\phi_1(x)}{\psi} + \epsilon'/2 \leq -\epsilon'/2.$$

Hence:

$$\Delta F(x) \leq -\epsilon'/2(\psi - 1)\psi^{x_1} + \gamma_1(\lambda_2 - \phi_2(x) + \Delta V(x_1)).$$

V being bounded and all transitions being bounded, $(\lambda_2 - \phi_2(x) + \Delta V(x_1))$ is bounded by a positive constant K and we can then choose x_1^* such that $\forall x_1 > x_1^*$, $\psi^{x_1}(\psi - 1)\epsilon'/2 - K\gamma_1 > \epsilon$, which leads to $\Delta F(x) \leq -\epsilon$.

- Consider now the case $x_2 \geq x_2^*$. Since $\phi_1(x) \geq \phi_1^1(x_1)$ and $\phi_1^1(x_1)$ converges to C_1, the term $\Delta\psi^{x_1} = (\lambda_1 - \frac{\phi_1(x)}{\psi})(\psi - 1)\psi^{x_1} \leq (\lambda_1 - \frac{\phi_1^1(x_1)}{\psi})(\psi-1)\psi^{x_1}$ is bounded by a constant K' independent of x_2. (Recall that for $x_1 > x_1^*$, $\lambda_1 - \frac{\phi_1^1(x_1)}{\psi} < 0$). Hence

$$\begin{aligned}
\Delta F(x) &\leq K' + \gamma_1(\lambda_2 - \phi_2(x) + \Delta V(x_1))\\
&\leq \delta + K' + \gamma_1((P_1 - I)V(x_1) + \lambda_2 - \phi_2^1(x_1)),\\
&= \delta + K' - \gamma_1\epsilon.
\end{aligned}$$

Choosing γ_1 such that $\delta + K - \gamma_1\epsilon \leq -\epsilon$, we have shown that the drift of F_1 is negative outside a finite set. □

Looking at the latter proof, we can partially release the assumption of monotonicity of ϕ. Define

$$C_1' = \lim_{x_1\to\infty} \inf_{x_2} \phi_1(x), \tag{8}$$

$$C_2' = \lim_{x_2\to\infty} \inf_{x_1} \phi_2(x), \tag{9}$$

Note that we might however obtain a gap between sufficient and necessary condition of stability due to the fact than the C_i' and C_i might not coincide in general. We give here only sufficient conditions of stability.

THEOREM 3. *Assume that (λ_1, λ_2) verify:*

$$\begin{aligned}
\lambda_1 < C_1' \quad &\textit{and} \quad \lambda_2 < \langle\phi_2^1, \pi^1\rangle,\\
\textit{or } \lambda_2 < C_2' \quad &\textit{and} \quad \lambda_1 < \langle\phi_1^2, \pi^2\rangle.
\end{aligned}$$

Then X is positive recurrent.

Proof: The proof follows from the Lyapunov-Foster criterion [12] with the Lyapunov function defined in the previous theorem. The monotonicity assumption used in the latter argument is that for all ϵ', for x_1 sufficiently large, for all x_2, $\phi_1(x) > C_1 + \epsilon'$, and hence $\lambda_1 - \frac{\phi_1(x)}{\psi} \leq -\epsilon'$, for large x_1. This can hence be replaced by assumption (8) since for large x_1, $\phi_1(x) \geq C_1' + \epsilon'$. □

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4. OPTIMAL STATIC LOAD BALANCING

Using the stability conditions for the system with fixed arrival rates, and maximizing the sum of arrival rates $\lambda_1 + \lambda_2 = \nu$, we directly get the following characterization of the maximum intensity $\nu_{\mathcal{S}}^{\max}$:

PROPOSITION 1. *The maximal intensity leading to a stable system when arrival rates are fixed is given by:*

$$\begin{aligned}\nu_{\mathcal{S}}^{\max} = \\ &\max[\max_{0\le\lambda_2\le C_2}(\lambda_2+\langle\phi_2^1,\pi^1\rangle), \max_{0\le\lambda_1\le C_1}(\lambda_1+\langle\phi_1^2,\pi^2\rangle)],\\ &=\max[\phi_1(1,0),\phi_2(0,1),\max_{0<\lambda_2\le C_2}(\lambda_2+\langle\phi_2^1,\pi^1\rangle),\\ &\max_{0<\lambda_1\le C_1}(\lambda_1+\langle\phi_1^2,\pi^2\rangle)].\end{aligned}$$

Writing $\lambda_1 = \eta\nu$ and writing explicitly the dependence of π^i with respect to η, the optimal load balancing is given by:

$$\begin{aligned}(\lambda_1^*,\lambda_2^*) =&\nu(\eta,1-\eta),\\ =&\arg\max_{\eta\in[0,1]}[\nu(1-\eta)+\langle\phi_2^1,\pi^1(\eta)\rangle),\\ &\nu\eta+\langle\phi_1^2,\pi^2(1-\eta)\rangle)].\end{aligned}$$

In the particular simpler instance (M'), writing $C_i = C$, the optimal static load balancing further simplifies to:

PROPOSITION 2. *Consider the instance (M') where*

$$\phi_i(x) = \phi_i(x_j) = \phi(x_j),\ j \neq i.$$

$$\begin{aligned}\nu_{\mathcal{S}}^{\max} &= \max_{\lambda_1\in[0,C]}\left(\lambda_1+\sum_{n=0}^{\infty}\phi(n)(\frac{\lambda_1}{C})^n(1-\frac{\lambda_1}{C})\right), &(10)\\ &= \max_{\eta\in[0,1]}\left[\eta C+\sum_{n=0}^{\infty}\phi(n)\eta^n(1-\eta)\right], &(11)\\ &= \max_{\eta\in[0,1]} G(\eta). &(12)\end{aligned}$$

where $G(\eta) = \eta C + (1-\eta)\tilde{\phi}(\eta)$ and $\tilde{\phi}(\eta)$ is the $z-$transform of ϕ at point η.

This Proposition can be interpreted in the following way: when class 2 is close to be unstable, and class one is stable, the rate offered to class 2 is the linear combination of two 'virtual' rates C and $\tilde{\delta}(\eta)$, the weights of this combination being η and $1-\eta$.

In the case of purely interfering processors (model (M')), we can further characterize the optimal static solution in terms of η^*, the value of η maximizing G and of the sequence $\delta(n) = (n+1)(\phi(n) - \phi(n+1))$, $\tilde{\delta}$ being its transform.

PROPOSITION 3. *Consider the instance (M') where*

$$\phi_i(x) = \phi_i(x_j) = \phi(x_j),\ j \neq i.$$

We have the three following cases:

1. *If $C \le \phi(0)-\phi(1)$, then the optimal static load balancing consists in sending all the incoming traffic to only one queue, i.e., $\eta^* = 0$.*
2. *Let $C > \phi(0) - \phi(1)$ and suppose that the equation*
$$\tilde{\delta}(\eta) = C,$$
has a solution $\eta^* \in]0,1[$. Then it is optimal to balance the load unevenly between the two servers, $\eta^* C$ being the intensity of customers directed to queue 1.
3. *Let $C > \phi(0) - \phi(1)$ and suppose that the equation*
$$\tilde{\delta}(\eta) = C,$$
has no solution on $]0,1[$, then to load equally the two processors is optimal.

Proof: We have $G'(\eta) = C - \tilde{\delta}(\eta)$. As ϕ is decreasing, $\delta(n) \ge 0$ for all n and hence $\tilde{\delta}$ is an increasing function. If $C - \tilde{\delta}(0) \le 0$, then $G'(\eta) \le 0$, and its maximum is attained for $\eta = 0$. If $C - \tilde{\delta}(0) \ge 0$, then $G'(0) \ge 0$ for $\eta \ge \eta'$, with η' the solution of $\tilde{\delta}(\eta') = C$. □

We illustrate the result on the following example.

EXAMPLE 4 ('GEOMETRICALLY' COUPLED PROCESSORS). *Consider the symmetric allocation :*

$$\phi_i(x) = C + a^{x_j},\ j \neq i,$$

with $a < 1$. $\nu_{\mathcal{S}}^{\max}$ is given by:

$$\begin{aligned}\nu_{\mathcal{S}}^{\max} &= \max_{\eta\in[0,1]}\left[\eta C+\sum_{n=0}^{\infty}(a^n+C)\eta^n(1-\eta)\right],\\ &= \max_{\eta\in[0,1]}[C+\eta C+(1-\eta)\frac{1}{1-a\eta}].\end{aligned}$$

This leads to the following cases:

1. *If $C \le 1-a$, then the optimal static load balancing consists in sending all the incoming traffic to only one queue, i.e., $\eta^* = 0$, $\nu_{\mathcal{S}}^{\max} = 1 + C$.*
2. *If $C > 1-a$ and $(1-a)C < 1$, then it is optimal to balance the load unevenly between the two servers with:*
$$\eta^* = 1/a(1-\sqrt{\frac{1-a}{C}}),$$
which gives
$$\nu_{\mathcal{S}}^{\max} = \frac{1}{a}(1+(1+a)C-2\sqrt{C(1-a)}).$$
3. *If $C > 1-a$ and $(1-a)C \ge 1$, then to load equally the two processors is optimal, i.e., $\eta^* = 1$ and $\nu_{\mathcal{S}}^{\max} = 2C$.*

The figure 3 and 4 and 5 illustrate the different scenarios for the optimal static load balancing. The stability region of the system with static load balancing is the union of two regions defined respectively by conditions (2) and (3). The functions defining the frontiers of these regions are:

$$L_1 : \eta \mapsto \langle\phi_2^1,\pi^1(\eta)\rangle, \text{defined on } [0, C_1],$$

$$\text{and } L_2 : \eta \mapsto \langle\phi_1^2,\pi^2(\eta)\rangle,\ \text{defined on } [0, C_2].$$

If these functions are sub-linear, i.e.,

$$L_1(\eta) \le \eta\phi_2^1(0) + (1-\eta)C_1,$$

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$$L_2(\eta) \le \eta\phi_1^2(0) + (1-\eta)C_2,$$

then the best static load balancing is to send all the traffic to one queue or half of the traffic to both. This the case for weakly coupled processors for which L_1 and L_2 are linear as shown in Figure 3. More generally, this is the case if L_1 and L_2 are convex functions.

On the other hand, if there exist values of η for which $\langle \phi_j^i, \pi^i(\eta)\rangle \ge \eta\phi_j^i(0) + (1-\eta)C_i$, then a non-trivial static load balancing, (i.e., different from sending all the traffic to one queue or half of the traffic to both queues) is optimal. This the case on Figure 4 where L_1 and L_2 are concave functions or in Figure 5 where L_1 and L_2 are neither convex nor concave.

In the following section, a simple bound on the maximum intensity that the system can support is explained, given further insights on the cases where the 2 frontiers are convex functions.

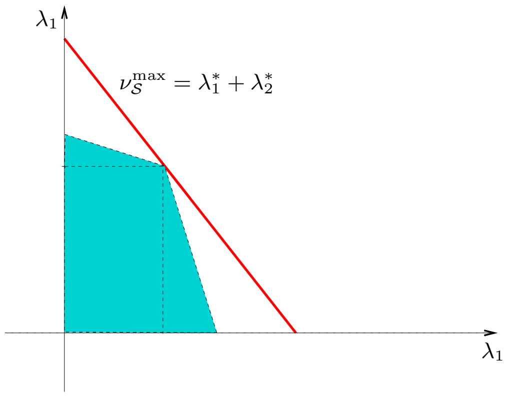

Figure 3: Stability region of weakly coupled processors with static load balancing, the frontiers of the stability regions are linear

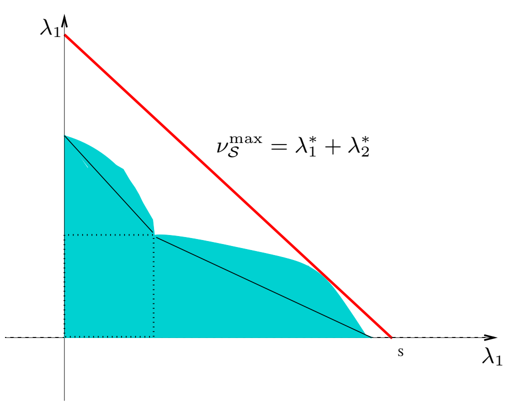

Figure 4: Stability region of geometrically coupled processors with static load balancing, the frontiers of the stability regions are concave

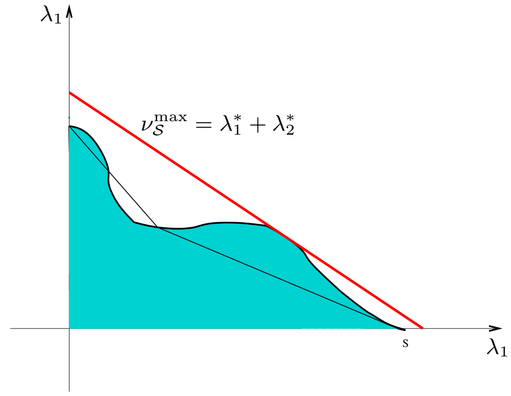

Figure 5: Stability region of coupled processors with general frontiers

5. A SIMPLE UPPER BOUND

In this section, we consider an arbitrary number of queues. Define $S(x) = \sum_{i=1}^K \phi_i(x)$. An upper bound of $\nu^{\max}$ is obtained by replacing the system by a unique queue with total service rates $\bar{S} = \max_x S(x)$.

PROPOSITION 4. *We have the following upper bound:*

$$\nu^{\max} \le \bar{S}$$

Proof: Consider a system such that $\nu > \bar{S}$. Consider the Lyapunov function $F(x) = x_1 + x_2$. $\Delta(x) = \nu - \sum_{i=1}^K \phi_i(x) \ge \nu - \bar{S} > 0$. Using the Foster criterion, the system is unstable. □

This simple result can actually be better understood and refined using stochastic comparisons. Denote Y the process corresponding to a M/M/1 queue with transitions ν and $\bar{S}$.

PROPOSITION 5. *Y and $|X|$ are stochastically comparable:*

$$Y(t) \le_{st} |X|(t), \forall t.$$

Proof: We construct the following coupling. Consider the Markov process $(\tilde{X}, \tilde{Y})$ having the following transitions:

$$\begin{aligned}
(\tilde{X}, \tilde{Y}) &\rightarrow (\tilde{X} + e_i, \tilde{Y} + 1) \text{ with rate } \lambda_i(x),\ \forall i.\\
(\tilde{X}, \tilde{Y}) &\rightarrow (\tilde{X} - e_i, \tilde{Y} - 1) \text{ with rate } \phi_i(x),\ \forall i\\
(\tilde{X}, \tilde{Y}) &\rightarrow (\tilde{X}, \tilde{Y} - 1) \text{ with rate } \bar{S} - S(x).
\end{aligned}$$

The marginals of $(\tilde{X}, \tilde{Y})$ have the same laws as X and Y which concludes the proof. □

In the case of 'weakly' coupled processors or more generally in the case where the functions L_1 and L_2 are convex, then this bound allows to conclude than no dynamic load balancing can improve stability upon static load balancing.

EXAMPLE 5 (2 WEAKLY COUPLED PROCESSORS). *Consider $\phi_i(x) = 1$ if $x_j = 0$, $j \ne i$ and $\phi_i(x) = a_i$ if $x_j > 0$, $j \ne i$. We get:*

$$\nu_S^{\max} = \bar{S} = \max(1, a_1 + a_2).$$

It follows that

$$\nu_S^{\max} = \nu^{\max} = \bar{S}.$$

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On the other hand, in the case of 'geometrically' coupled processors, this bound is strictly bigger than $\nu_{\mathcal{S}}^{\max}$. This thus suggest that there is a possible improvement of the stability condition using a dynamic scheme.

EXAMPLE 6 ('GEOMETRICALLY' COUPLED PROCESSORS). *Consider the symmetric allocation $\phi_i(x) = C + a^{x_j}$, $j \neq i$ with $a < 1$. For this allocation:*

$$\bar{S} = \max(1 + C, 2a + 2C).$$

It follows that if $C > 1 - a$, then $\nu_{\mathcal{S}}^{\max} < \bar{S}$.

6. DECOUPLED DYNAMIC LOAD BALANCING

In this section, we study some load balancing schemes depending on the number of customers in one queue only, i.e.,

$$\lambda_1(x) = \lambda_1(x_1),$$

$$\lambda_2(x) = \lambda_2(x_1) = \nu - \lambda_1(x_1).$$

In the following, we characterize the optimal schemes within this class and we show in the following that this class of routing already allows a gain of stability for certain capacity allocation functions.

6.1 Stability condition

As previously, define (when it exists) the following distribution:

$$\pi^1(x_1) = \frac{\prod_{z=1}^{x_1} \frac{\lambda_1(z)}{\phi_1^1(z)}}{\sum_{x_1} \prod_{z=1}^{x_1} \frac{\lambda_1(z)}{\phi_1^1(z)}}.$$

THEOREM 7. *Under the following conditions:*

$$\limsup_{x_1} \lambda_1(x_1) < C_1 \tag{13}$$

$$\text{and } \nu < \langle \lambda_1 + \phi_2^1, \pi^1 \rangle, \tag{14}$$

the process X is stable.

Proof: The proof uses a Lyapunov function exactly along the lines of Theorem 2. It is therefore postponed to the Appendix. □

6.2 Stochastic comparisons

In the specific case of purely interfering processors, (i.e., $\phi_i(x) = \phi_j(x_j)$), and when λ_1 is a decreasing function, the stability results can once again be interpreted in terms of a dominant system via appropriate stochastic comparisons. This also leads to general performance bounds. Note however that the monotonicity of the Markov processes is not verified anymore in the general case, which makes this technique inapplicable. Consider the process $\tilde{X}$ having the transitions:

$$\begin{aligned}
\tilde{q}(x, x - e_1) &= C_1, \\
\tilde{q}(x, x - e_2) &= \phi_2(x_1), \\
\tilde{q}(x, x + e_1) &= \lambda_1(x_1), \\
\tilde{q}(x, x + e_2) &= \nu - \lambda_1(x_1),
\end{aligned}$$

Given the considered monotonicity of the transitions, we have the following comparison between the original process (with the considered decoupled transition) and $\tilde{X}$:

PROPOSITION 1. *X and $\tilde{X}$ are stochastically comparable with the coordinate-wise partial order, i.e., for ordered initial conditions:*

$$X(t) \leq_{st} \tilde{X}(t), \forall t.$$

Proof: We construct the following coupling, i.e. a Markov process $(Y, \tilde{Y})$ preserving $Y(t) \leq \tilde{Y}(t)$ on its trajectories and having marginals with laws equal to the laws of X and $\tilde{X}$. The coupled transitions are given by:

$$\begin{array}{ccl}
(Y, \tilde{Y}) & \to & (Y + e_1, \tilde{Y} + e_1), \\
\text{with rate} & & \lambda_1(Y_1) \text{ if } Y_1 = \tilde{Y}_1, \\
(Y, \tilde{Y}) & \to & (Y, \tilde{Y} + e_2), \\
\text{with rate} & & \lambda_1(Y_1) - \lambda_1(\tilde{Y}_1) \geq 0 \text{ if } Y_1 \leq \tilde{Y}_1, \\
(Y, \tilde{Y}) & \to & (Y + e_2, \tilde{Y} + e_2), \\
\text{with rate} & & 1 - \lambda_1(Y_1) \text{ if } Y_1 \leq \tilde{Y}_1, \\
& & \\
(Y, \tilde{Y}) & \to & (Y - e_1, \tilde{Y} - e_1), \\
\text{with rate} & & C_1, \\
(Y, \tilde{Y}) & \to & (Y - e_1, \tilde{Y}), \\
\text{with rate} & & \phi_1(Y_2) - C_1 \geq 0, \\
(Y, \tilde{Y}) & \to & (Y - e_2, \tilde{Y} - e_2), \\
\text{with rate} & & \phi_2(\tilde{Y}_1), \\
(Y, \tilde{Y}) & \to & (Y - e_2, \tilde{Y}), \\
\text{with rate} & & \phi_2(Y_1) - \phi_2(\tilde{Y}_1) \geq 0.
\end{array}$$

□

6.3 Optimal decoupled load balancing

We define a specific policy that we call 'simple' by the parameter $N \in \mathbb{N}$ with:

$$\lambda_1^N(x_1) = \nu, \; x_1 \leq N - 1,$$

$$\lambda_1^N(x_1) = 0, \; x_1 \geq N.$$

The stability condition given by equation (13) is depends on λ_1 through the stationary measure π^1. Let $\pi^{1,N} = \pi^N$ the stationary measure associated with a simple routing of parameter N. We show in the following Proposition that the stationary measure π^1 associated with any routing $\lambda_1(\cdot)$ can be expressed as a convex combination of stationary measures of simple load balancing. Similar ideas were used to derive the optimal insensitive load balancing in [3].

PROPOSITION 6.
Let π^1 be the stationary measure associated with an admissible (stable) routing $\lambda_1(\cdot)$. There exists positive weights $(\alpha(N))_{N \geq 0}$, summing to one ($|\alpha|_1 = 1$) such that:

$$\pi(x) = \sum_{N \geq 0} \alpha(N) \pi^N(x). \tag{15}$$

Proof: The reversibility of the stationary measure of a one dimensional birth and death process implies that π^1 verifies:

$$\lambda_1(x_1) = \frac{\pi^1(x_1 + 1)}{\pi^1(x_1)} \phi_1^1(x_1), \tag{16}$$

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Consider an admissible π^1. Because of the routing constraints, $\lambda_1(x_1) \leq \nu$, hence we have for all x_1:

$$\pi^1(x_1+1) \leq \nu\pi^1(x_1)\phi_1^1(x_1).$$

π^1 is hence equivalently defined by a function β through:

$$\pi^1(x_1) = \beta(x_1) + \frac{1}{\nu}\frac{\pi^1(x_1+1)}{\phi_1^1(x_1)}.$$

Let $\alpha(N) = \frac{\beta(N)}{\pi^N(N)}$. Define the stationary measure $\pi'(x_1) = \sum_{N\geq 0}\frac{\beta(N)}{\pi^N(N)}\pi^N(x_1) = \sum_N \alpha(N)\pi^N(x)$. Using the definition of the simple routing we get that:

$$\begin{aligned}
\pi'(x_1) &= \beta(x_1) + \sum_{N\geq x_1+1}\alpha(N)\pi^N(x_1),\\
\pi'(x_1) &= \beta(x_1) + \sum_{N\geq x_1+1}\alpha(N)\frac{1}{\nu}\frac{\pi^N(x_1+1)}{\phi_1^1(x_1)},\\
\pi'(x_1) &= \beta(x_1) + \frac{1}{\phi_1^1(x_1)\nu}\sum_{N\geq x_1+1}\alpha(N)\pi^N(x_1+1),\\
\pi'(x_1) &= \beta(x_1) + \frac{1}{\phi_1^1(x_1)\nu}\pi'(x_1+1).
\end{aligned}$$

As π' and π^1 follow the same recursive equations, there are equal. Moreover, because each π^N sums to 1, $|\alpha|_1 = 1$. □

The next Proposition characterize the stability condition derived for a given intensity ν using the previous Proposition.

Proposition 7. *If the couple ν, λ_1 lead to a stable system with stationary measure π^{ν,λ_1}, then:*

$$\nu \leq \langle \delta^1, \pi^{\nu,\lambda_1}\rangle,$$

where $\delta^1(n) = \phi_1^1(n) + \phi_2^1(n)$.

Proof: Given that $\lambda_1(x_1) = \phi_1^1(x_1)\frac{\pi(x_1+1)}{\pi(x_1)}$, the left term of condition (14) can be written $\sum_{x_1}(\pi(x_1+1)\phi_1^1(x_1+1) + \phi_2^1(x_1)\pi(x_1))$. (Remark also that $\phi_1^1(0) = 0$). □

Of course, a symmetric situation where the routing depends only on x_2 would lead to the symmetric stability condition:

$$\nu \leq \langle \delta^2, \pi^{\nu,\lambda_2}\rangle,$$

The previous Proposition has an easy and important interpretation. The maximal arrival intensity equals the departure rate of the system when class 2 goes to infinity, which is the sum of the service rates $\phi_1^1(n) + \phi_2^1(n)$, weighted by the probability than $X_1 = n$.

We now aim at maximizing the left term of (14) by choosing an appropriate λ_1. This can be done with the help of Proposition 6. Define ν^N as the stability condition of a simple routing with parameter N, and $\nu_{\mathcal{D}}^{\max}$ the maximum routing intensity in the class of one dimensional routing. Further define:

$$N^* = \arg\max_{N\geq 0}\{\langle \delta^1, \pi^{\nu,\lambda_1}\rangle, \langle \delta^2, \pi^{\nu,\lambda_2}\rangle\}.$$

The following Theorem is the main result of this Section:

Theorem 8. *We have the two following cases. If $0 < N^* < +\infty$, then a dynamic simple routing is optimal and $\nu_{\mathcal{D}}^{\max}$ is the maximal solution of one of the polynomial equations of degree $N+1$:*

$$x^{N+1} = \sum_{n=0}^{N}\phi_2^1(n)\phi_1^1(n+1)\dots\phi_1^1(N)x^n, \tag{17}$$

$$x^{N+1} = \sum_{n=0}^{N}\phi_1^2(n)\phi_2^2(n+1)\dots\phi_2^2(N)x^n, \tag{18}$$

Otherwise, if $N^* = 0$ or $N^* = +\infty$, i.e., the supremum of the solutions of (17) and (18) are not attained for a finite N or are attained for $N = 0$, then a static load balancing is optimal.

Proof: Fix a routing λ_1, an intensity ν and let π^1 the associated stationary measure. Using the previous Propositions, the considered system is stable if:

$$\nu \leq \sum_n \delta^1(n)\pi^1(n).$$

Hence,

$$\nu_{\mathcal{D}}^{\max} \leq \sum_n \delta^1(n)\sum_N \alpha(N)\pi^N(n) \leq \sup_N\langle\delta^1, \pi^N\rangle. \tag{19}$$

Introduce the function $g(\eta, N) = \langle\delta^1, \pi^{\eta,N}\rangle$. Using classical theorems on power series, if $\eta > C_1$, then $g(\eta, N) \to \delta^1(0) \leq \nu_{\mathcal{S}}^{\max}$, when $N \to \infty$. Hence if the supremum of $\langle\delta^1, \pi^{\nu^{\max},N}\rangle$ is attained for $N = \infty$, then

$$\nu_{\mathcal{D}}^{\max} = \max_{\eta\leq C_1}\langle\delta^1, \pi^{\eta,\infty}\rangle = \nu_{\mathcal{S}}^{\max}.$$

Similarly, if the supremum is attained for $N = 0$, then a static load balancing corresponding to sending all the incoming traffic to only queue is optimal. If the supremum is attained for a finite value $0 < N^* < \infty$, we can further simplify equation (14) to get:

$$\nu\pi^{N,\nu}(N) \leq \sum_{n=0}^{N}\phi_2^1(n)\pi^N(n),$$

which rewrites:

$$\nu^{N+1} \leq \sum_{n=0}^{N}\phi_2^1(n)\phi_1^1(n+1)\dots\phi_1^1(N)\nu^n.$$

Let $g(\nu) = \nu^{N+1} - \sum_{n\geq 0}\phi_2^1(n)\phi_1^1(n+1)\dots\phi_1^1(N)\nu^n$. Since $g(0) < 0$, g is continuous and diverging to infinity, g has at least one positive real root. Provided that the maximal root is greater than $C_1 + C_2$, $\nu_{\mathcal{D}}^{\max}$ is hence the maximal solution of the equation:

$$\sum_{n\geq 0}\phi_2^1(n)\phi_1^1(n+1)\dots\phi_1^1(N)x^n = x^{N+1}.$$

Equation (18) is given by the symmetric situation where $\lambda_1(x) = \nu - \lambda_2(x_2)$. □

It follows from the proof of the previous Theorem that:

Corollary 1. *If the optimal static load balancing is not given by $\lambda \in \{(\nu, 0), (0, \nu), (\nu/2, \nu/2)\}$, then $\nu^{\max} \geq \nu_{\mathcal{D}}^{\max} > \nu_{\mathcal{S}}^{\max}$.*

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We come back again to our main example and identify the different cases in the light of the results of this section:

EXAMPLE 9. *Consider the symmetric allocation :*

$$\phi_i(x) = C + a^{x_j}, \ j \neq i, \ \text{with } a < 1.$$

If $C > 1-a$ and $(1-a)C < 1$, the optimal static load balancing is given by $\eta^* = 1/a(1-\sqrt{\frac{1-a}{C}}) \neq 0, 1$ and the optimal decoupled load balancing corresponds to a simple load balancing with a finite parameter N, $\nu_{\mathcal{D}}^{\max}$ being Cy with y the root of:

$$Cy^{N+1} = \frac{1-y^{N+1}}{1-y} + \frac{1-(ay)^{N+1}}{1-ay}.$$

The results are illustrated on Figures 5 to 9 for different C and a. For $a = 0.8$ and $C = 0.9$, we observe a gain of around 10 percent:

$$\nu_{\mathcal{D}}^{\max} \simeq 1.11 \nu_{\mathcal{S}}^{\max}.$$

Note that the dynamic scheme employed uses only the state of the first queue as information the system. This suggest that more sophisticated schemes like a switch-curve type routing might improve very significantly the stability condition.

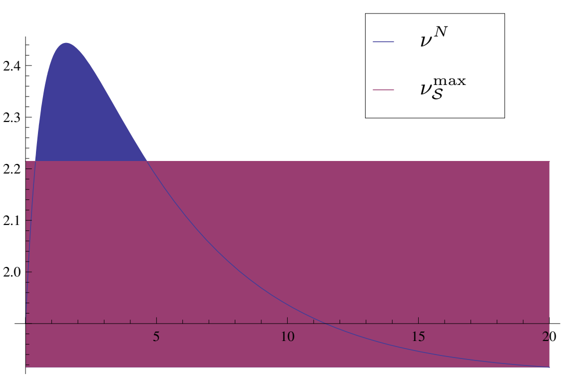

Figure 6: ν^N and $\nu_{\mathcal{S}}^{\max}$, when N varies, $a = 0.8$, $C = 0.9$.

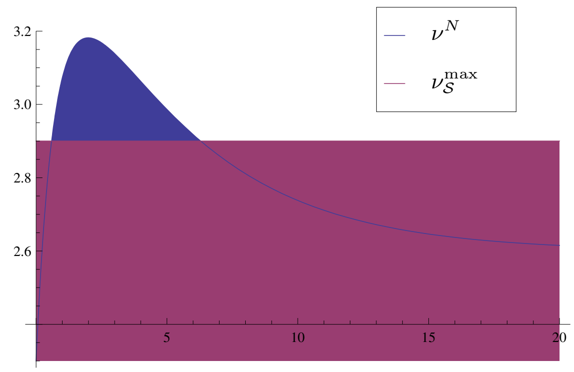

Figure 7: ν^N and $\nu_{\mathcal{S}}^{\max}$, when N varies, $a = 0.8$, $C = 1.3$.

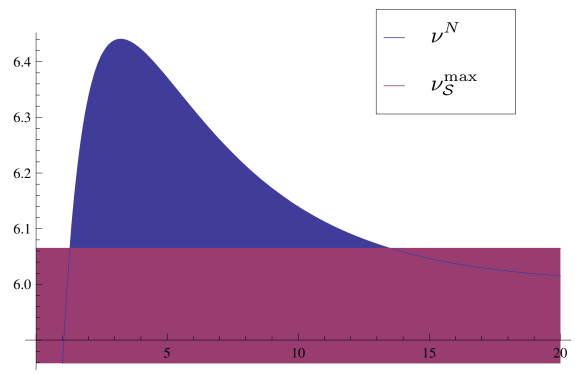

Figure 8: ν^N and $\nu_{\mathcal{S}}^{\max}$, when N varies, $a = 0.8$, $C = 3$.

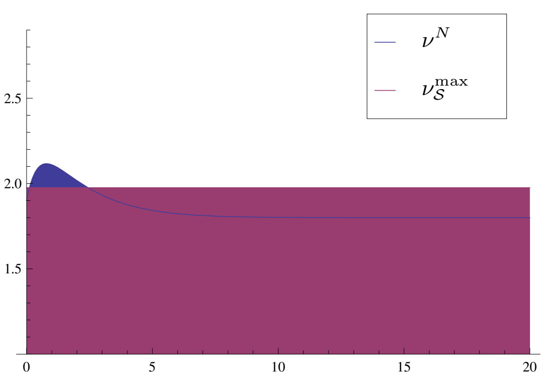

Figure 9: ν^N and $\nu_{\mathcal{S}}^{\max}$, when N varies, $a = 0.4$, $C = 1.3$.

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7. CONCLUSION

We have found Lyapunov functions for a system of two coupled processors with static or decoupled dynamic load balancing. We then have characterized the optimal static load balancing and the optimal decoupled load balancing and have shown that dynamic load balancing may improve stability. We expect to generalize these results to higher dimensions in a near future. Also, these results suggest that a switch-curve type dynamic load balancing is generally optimal and might improve significantly the stability condition for certain service rates. An interesting future line of research is to jointly optimize the service rates and the load balancing scheme, which might allow to draw stronger conclusions.

ACKNOWLEDGMENT

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APPENDIX

Proof of Theorem 7:

Under condition (13), π^1 is well defined. Then choose $\psi > 1$ such that $\lambda_1 - \frac{C_1}{\psi} < -\epsilon'$. Then, fix $\epsilon = \nu - \sum_{x_1}(\lambda_1(x_1) + \phi_2^1(x_1))\pi^1(x_1)$, and consider the (one-dimensional) Poisson equation:

$$\lambda_1(x_1)(V(x_1+1) - V(x_1)) + \phi_1^1(x_1)(V(x_1-1) - V(x_1)) = -(\nu - \lambda_1(x_1) - \phi_2^1(x_1) + \epsilon). \quad (20)$$

or written differently:

$$(I - P_1)V = \lambda_2 - \phi_2^1 + \epsilon,$$

$$(I - P_1)V = \nu - \lambda_1 - \phi_2^1 + \epsilon,$$

with P_1 the corresponding Markovian kernel. Applying again the results of [1], there exists a bounded solution to this Poisson equation if condition (13) is fulfilled together with $\langle \nu - \lambda_1 + \phi_2^1 + \epsilon, \pi^1 \rangle = 0$, which we have supposed. Let then V be such a solution.

Using the existence of uniform limits for ϕ_2, we have: For all $\delta > 0$, there exists x_2^* such that $\forall x_1$, and $\forall x_2 \geq x_2^*$:

$$|\phi_2(x) - \phi_2^1(x_1)| \leq \delta.$$

Let $\delta < \epsilon'$ and consider the following Lyapunov function:

$$F(x) = \psi^{x_1} + \gamma_1(x_2 + V(x_1)),$$

with γ_1 a strictly positive constant to be chosen. Since $\psi > 1$, $F(x) \to \infty$, when $|x| \to \infty$ and F is hence a proper Lyapunov function. The drift of F is:

$$\Delta F(x) = (\lambda_1(x_1) - \frac{\phi_1(x)}{\psi})(\psi - 1)\psi^{x_1} + \gamma_1(\lambda_2 - \phi_2(x) + \Delta V(x_1)),$$

-if $x_2 < x_2^*$ and $x_1 > x_1^*$,

$$\Delta F(x) \leq -\epsilon'(\psi - 1)\psi^{x_1} + \gamma_1(\lambda_2(x_1) - \phi_2(x) + \Delta V(x_1)).$$

V being bounded and all transitions being bounded, $(\lambda_2(x_1) - \phi_2(x) + \Delta V(x_1))$ is bounded by a positive constant K and we can then choose x_1^* such that $\forall x_1 > x_1^*$, $\psi^{x_1}(\psi - 1)\epsilon' - K\gamma_1 > \epsilon$ which leads to $\Delta F(x) \leq -\epsilon$.

- If $x_2 \geq x_2^*$, using that:

$$\Delta \psi^{x_1} \leq K'$$

$$\begin{aligned}
\Delta F(x) &\leq K' + \gamma_1(\lambda_2(x_1) - \phi_2(x) + \Delta V(x_1)) \\
&\leq \delta + K' + \gamma_1((P_1 - I)V(x_1) + \nu - \lambda_1(x_1) - \phi_2^1(x_1)), \\
&= \delta + K' - \gamma_1 \epsilon.
\end{aligned}$$

Choosing γ_1 such that $\delta + \nu(\psi - 1) - \gamma_1 \epsilon \leq -\epsilon$, we have shown that the drift of F is negative outside a finite set. The positive recurrence of X then follows from the Lyapunov-Foster criterion. □

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