Efficiency Loss in a Cournot Oligopoly with Convex Market Demand

John N. Tsitsiklis and Yunjian Xu*

Laboratory or Information and Decision Systems, MIT, Cambridge, MA, 02139, USA
{jnt,yunjian}@mit.edu

Abstract. We consider a Cournot oligopoly model where multiple suppliers (oligopolists) compete by choosing quantities. We compare the social welfare achieved at a Cournot equilibrium to the maximum possible, for the case where the inverse market demand function is convex. We establish a lower bound on the efficiency of Cournot equilibria in terms of a scalar parameter derived from the inverse demand function. Our results provide nontrivial quantitative bounds on the loss of social welfare and aggregate profit for several convex inverse demand functions that appear in the economics literature.

Keywords: Price of anarchy, Cournot oligopoly, revenue management.

1 Introduction

In a book on oligopoly theory (see Chapter 2.4 of [6]), Friedman raises an interesting question on the relation between Cournot equilibria and competitive equilibria: "is the Cournot equilibrium close, in some reasonable sense, to the competitive equilibrium?" While a competitive equilibrium is generally socially optimal, a Cournot (Nash) equilibrium can yield arbitrarily high efficiency loss in general [8]. The concept of efficiency loss is intimately related to the concept of "price of anarchy," advanced by Koutsoupias and Papadimitriou in a seminal paper [11]; it provides a natural measure of the difference between a Cournot equilibrium and a socially optimal competitive equilibrium.

For Cournot oligopoly with affine demand functions, various efficiency bounds have been reported in recent works [9][10]. Convex demand functions, such as the negative exponential and the constant elasticity demand curves, have been widely used in oligopoly analysis and marketing research [2,4,14]. The efficiency loss in a Cournot oligopoly with some specific forms of convex inverse demand functions[1] has received some recent attention. For a particular form of convex

* This research was supported in part by the National Science Foundation under grant CMMI-0856063 and by a Graduate Fellowship from Shell.

[1] Since a demand function is generally nonincreasing, the convexity of a demand function implies that the corresponding inverse demand function is also convex. For a Cournot oligopoly model with non-concave inverse demand functions, existence results for Cournot equilibria can be found in [12,1].

V. Krishnamurthy et al. (Eds.): GameNets 2012, LNICST 105, pp. 63–76, 2012.

inverse demand functions, i.e., $p(q) = \alpha - \beta q^{\gamma}$, the authors of [3] show that when $\gamma > 0$, the worst case efficiency loss occurs when an efficient supplier has to share the market with infinitely many inefficient suppliers. The authors of [7] consider a class of inverse demand functions that solve a certain differential equation (for example, constant elasticity inverse demand functions belong to this class), and establish efficiency lower bounds that depend on equilibrium market shares, the market demand, and the number of suppliers.

For Cournot oligopolies with general convex and nonincreasing demand functions, we establish a lower bound on the efficiency of Cournot equilibria in terms of a scalar parameter c/d derived from the inverse demand function, namely, the ratio of the slope of the inverse demand function at the Cournot equilibrium, c, to the average slope of the inverse demand function between the Cournot equilibrium and a social optimum, d. For convex and nonincreasing inverse demand functions, we have $c \geq d$; for affine inverse demand functions, we have $c/d = 1$. In the latter case, our efficiency bound is $f(1) = 2/3$, which is consistent with the bound derived in [9]. More generally, the ratio c/d can be viewed as a measure of nonlinearity of the inverse demand function.

The rest of the paper is organized as follows. In the next section, we formulate the model and provide some mathematical preliminaries on Cournot equilibria that will be useful later, including the fact that efficiency lower bounds can be obtained by restricting to linear cost functions. In Section 3, we consider affine inverse demand functions and derive a refined lower bound on the efficiency of Cournot equilibria that depends on a small amount of ex post information. We also show this bound to be tight. In Section 4, we consider a more general model, involving convex inverse demand functions. We show that for convex inverse demand functions, and for the purpose of studying the worst case efficiency loss, it suffices to restrict to a special class of piecewise linear inverse demand functions. This leads to the main result of the paper, a lower bound on the efficiency of Cournot equilibria (Theorem 2). Based on this theorem, in Section 5 we derive a corollary that provides an efficiency lower bound that can be calculated without detailed information on Cournot equilibria, and apply it to various commonly encountered convex inverse demand functions. Finally, in Section 6, we make some brief concluding remarks. Most proofs are omitted and can be found in an extended version of the paper [13].

2 Formulation and Preliminaries

In this section, we first define the Cournot competition model that we study, and introduce several main assumptions that we will be working with. In Section 2.1, we present conditions for a nonnegative vector to be a social optimum or a Cournot equilibrium. Then, in Section 2.2, we define the efficiency of a Cournot equilibrium. In Sections 2.3 and 2.4, we derive some properties of Cournot equilibria that will be useful later, but which may also be of some independent interest. For example, we show that the worst case efficiency occurs when the cost functions are linear.

We consider a market for a single homogeneous good with inverse demand function $p : [0,\infty) \to [0,\infty)$ and N suppliers. Supplier $n \in \{1,2,\ldots,N\}$ has a cost function $C_n : [0,\infty) \to [0,\infty)$. Each supplier n chooses a nonnegative real number x_n, which is the amount of the good to be supplied by her. The **strategy profile** $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ results in a total supply denoted by $X = \sum_{n=1}^{N} x_n$, and a corresponding market price $p(X)$. The payoff to supplier n is

$$\pi_n(x_n, \mathbf{x}_{-n}) = x_n p(X) - C_n(x_n),$$

where we have used the standard notation $\mathbf{x}_{-n}$ to indicate the vector $\mathbf{x}$ with the component x_n omitted. In the sequel, we will use $\partial_- p$ and $\partial_+ p$ to denote the left and right derivatives of p, respectively.

Assumption 1. *For any n, the cost function $C_n : [0,\infty) \to [0,\infty)$ is convex, continuous, and nondecreasing on $[0,\infty)$, and continuously differentiable on $(0,\infty)$. Furthermore, $C_n(0) = 0$.*

Assumption 2. *The inverse demand function $p : [0,\infty) \to [0,\infty)$ is continuous, nonnegative, and nonincreasing, with $p(0) > 0$. Its right derivative at 0 exists and at every $q > 0$, its left and right derivatives also exist.*

Note that we do not yet assume that the inverse demand function is convex. The reason is that some of the results to be derived in this section are valid even in the absence of such a convexity assumption. Note also that some parts of our assumptions are redundant, but are included for easy reference. For example, if $C_n(\cdot)$ is convex and nonnegative, with $C_n(0) = 0$, then it is automatically continuous and nondecreasing.

Definition 1. *The **optimal social welfare** is the optimal objective value in the following optimization problem,*

$$\begin{aligned} \text{maximize} \quad & \int_0^X p(q)\,dq - \sum_{n=1}^{N} C_n(x_n) \\ \text{subject to} \quad & x_n \geq 0, \qquad n = 1,2,\ldots,N, \end{aligned} \tag{1}$$

where $X = \sum_{n=1}^{N} x_n$.

In the above definition, $\int_0^X p(q)\,dq$ is the aggregate consumer surplus and $\sum_{n=1}^{N} C_n(x_n)$ is the total cost of the suppliers. For a model with a nonincreasing continuous inverse demand function and continuous convex cost functions, the following assumption guarantees the existence of an optimal solution to (1).

Assumption 3. *There exists some $R > 0$ such that $p(R) \leq \min_n\{C_n'(0)\}$.*

The social optimization problem (1) may admit multiple optimal solutions. However, they must all result in the same price. We note that the differentiability of the cost functions is crucial for this result to hold.

Proposition 1. *Suppose that Assumptions 1 and 2 hold. All optimal solutions to (1) result in the same price.*

2.1 Optimality and Equilibrium Conditions

We observe that under Assumption 1 and 2, the objective function in (1) is concave. Hence, we have the following *necessary and sufficient* conditions for a vector $\mathbf{x}^S$ to achieve the optimal social welfare:

$$\begin{cases} C_n'(x_n^S) = p\left(X^S\right), & \text{if } x_n^S > 0, \\ C_n'(0) \geq p\left(X^S\right), & \text{if } x_n^S = 0, \end{cases} \tag{2}$$

where $X^S = \sum_{n=1}^{N} x_n^S$.

We have the following equilibrium conditions for a strategy profile $\mathbf{x}$. In particular, under Assumptions 1 and 2, if $\mathbf{x}$ is a Cournot equilibrium, then

$$C_n'(x_n) \leq p\left(X\right) + x_n \cdot \partial_- p\left(X\right), \quad \text{if } x_n > 0, \tag{3}$$

$$C_n'(x_n) \geq p\left(X\right) + x_n \cdot \partial_+ p\left(X\right), \tag{4}$$

where again $X = \sum_{n=1}^{N} x_n$. Note, however, that in the absence of further assumptions, the payoff of supplier n need not be a concave function of x_n and these conditions are, in general, not sufficient.

We will say that a nonnegative vector $\mathbf{x}$ is a **Cournot candidate** if it satisfies the necessary conditions (3)-(4). Note that for a given model, the set of Cournot equilibria is a subset of the set of Cournot candidates. Most of the results obtained in this section, including the efficiency lower bound in Proposition 5, apply to all Cournot candidates.

For convex inverse demand functions, the necessary conditions (3)-(4) can be further refined.

Proposition 2. *Suppose that Assumptions 1 and 2 hold, and that the inverse demand function $p(\cdot)$ is convex. If $\mathbf{x}$ is a Cournot candidate with $X = \sum_{n=1}^{N} x_n > 0$, then $p(\cdot)$ must be differentiable at X, i.e.,*

$$\partial_- p\left(X\right) = \partial_+ p\left(X\right).$$

Because of the above proposition, when Assumptions 1 and 2 hold and the inverse demand function is convex, we have the following necessary (and, by definition, sufficient) conditions for a nonzero vector $\mathbf{x}$ to be a Cournot candidate:

$$\begin{cases} C_n'(x_n) = p\left(X\right) + x_n p'(X), & \text{if } x_n > 0, \\ C_n'(0) \geq p\left(X\right) + x_n p'(X), & \text{if } x_n = 0. \end{cases} \tag{5}$$

2.2 Efficiency of Cournot Equilibria

As shown in [5], if $p(0) > \min_n\{C_n'(0)\}$, then the aggregate supply at a Cournot equilibrium is positive; see Proposition 3 below for a slight generalization. If on the other hand $p(0) \leq \min_n\{C_n'(0)\}$, then the model is uninteresting, because no supplier has an incentive to produce and the optimal social welfare is zero. This motivates the assumption that follows.

Assumption 4. *The price at zero supply is larger than the minimum marginal cost of the suppliers, i.e.,*

$$p(0) > \min_n \{C_n'(0)\}.$$

Proposition 3. *Suppose that Assumptions 1, 2, and 4 hold. If* $\mathbf{x}$ *is a Cournot candidate, then* $X > 0$.

Under Assumption 4, at least one supplier has an incentive to choose a positive quantity, which leads us to the next result.

Proposition 4. *Suppose that Assumptions 1-4 hold. Then, the social welfare achieved at a Cournot candidate, as well as the optimal social welfare [cf. (1)], are positive.*

We now define the efficiency of a nonnegative vector $\mathbf{x}$ as the ratio of the social welfare that it achieves to the optimal social welfare.

Definition 2. *Suppose that Assumptions 1-4 hold. The* ***efficiency*** *of a nonnegative vector* $\mathbf{x} = (x_1, \ldots, x_N)$ *is defined as*

$$\gamma(\mathbf{x}) = \frac{\displaystyle\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)}{\displaystyle\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N C_n(x_n^S)}, \tag{6}$$

where $\mathbf{x}^S = (x_1^S, \ldots, x_N^S)$ *is an optimal solution of the optimization problem in (1) and* $X^S = \sum_{n=1}^N x_n^S$.

We note that $\gamma(\mathbf{x})$ is well defined: because of Assumption 4 and Proposition 4, the denominator on the right-hand side of (6) is guaranteed to be positive. Furthermore, even if there are multiple socially optimal solutions $\mathbf{x}^S$, the value of the denominator is the same for all such $\mathbf{x}^S$. Note that $\gamma(\mathbf{x}) \leq 1$ for every nonnegative vector $\mathbf{x}$. Furthermore, if $\mathbf{x}$ is a Cournot candidate, then $\gamma(\mathbf{x}) > 0$, by Proposition 4.

2.3 Restricting to Linear Cost Functions

Proposition 5. *Suppose that Assumptions 1-4 hold and that* $p(\cdot)$ *is convex. Let* $\mathbf{x}$ *be a Cournot candidate which is not socially optimal, and let* $\alpha_n = C_n'(x_n)$. *Consider a modified model in which we replace the cost function of each supplier* n *by a new function* $\overline{C}_n(\cdot)$, *defined by*

$$\overline{C}_n(x) = \alpha_n x, \qquad \forall\ x \geq 0.$$

Then, for the modified model, Assumptions 1-4 still hold, the vector $\mathbf{x}$ *is a Cournot candidate, and its efficiency, denoted by* $\overline{\gamma}(\mathbf{x})$, *satisfies* $0 < \overline{\gamma}(\mathbf{x}) \leq \gamma(\mathbf{x})$.

If $\mathbf{x}$ is a Cournot equilibrium, then it satisfies Eqs. (3)-(4), and therefore is a Cournot candidate. Hence, Proposition 5 applies to all Cournot equilibria that are not socially optimal. We note that if a Cournot candidate $\mathbf{x}$ is socially optimal for the original model, then the optimal social welfare in the modified model could be zero, in which case $\gamma(\mathbf{x}) = 1$, but $\overline{\gamma}(\mathbf{x})$ is undefined; see the example that follows.

Example 1. Consider a model involving two suppliers ($N = 2$). The cost function of supplier n is $C_n(x) = x^2$, for $n = 1, 2$. The inverse demand function is constant, with $p(q) = 1$ for any $q \geq 0$. It is not hard to see that the vector $(1/2, 1/2)$ is a Cournot candidate, which is also socially optimal. In the modified model, we have $\overline{C}_n(x) = x$, for $n = 1, 2$. The optimal social welfare achieved in the modified model is zero. □

To lower bound the efficiency of a Cournot equilibrium in the original model, it suffices to lower bound the efficiency achieved at a worst Cournot candidate for a modified model. Accordingly, and for the purpose of deriving lower bounds, we can (and will) restrict to the case of linear cost functions, and study the worst case efficiency over all Cournot candidates.

2.4 Other Properties of Cournot Candidates

In this subsection, we collect a few useful and intuitive properties of Cournot candidates. We show that at a Cournot candidate there are two possibilities: either $p(X) > p(X^S)$ and $X < X^S$, or $p(X) = p(X^S)$ (Proposition 6); in the latter case, under the additional assumption that $p(\cdot)$ is convex, a Cournot candidate is socially optimal (Proposition 7). In either case, imperfect competition can never result in a price that is less than the socially optimal price.

Proposition 6. *Suppose that Assumptions 1-4 hold. Let $\mathbf{x}$ and $\mathbf{x}^S$ be a Cournot candidate and an optimal solution to (1), respectively. If $p(X) \neq p(X^S)$, then $p(X) > p(X^S)$ and $X < X^S$.*

For the case where $p(X) = p(X^S)$, Proposition 6 does not provide any comparison between X and X^S. While one usually has $X < X^S$ (imperfect competition results in lower quantities), it is also possible that $X > X^S$, as in the following example.

Example 2. Consider a model involving two suppliers ($N = 2$). The cost function of each supplier is linear, with slope equal to 1. The inverse demand function is convex, of the form

$$p(q) = \begin{cases} 2 - q, & \text{if } 0 \leq q \leq 1, \\ 1, & \text{if } 1 < q. \end{cases}$$

It is not hard to see that any nonnegative vector $\mathbf{x}^S$ that satisfies $x_1^S + x_2^S \geq 1$ is socially optimal; $x_1^S = x_2^S = 1/2$ is one such vector. On the other hand, it can be verified that $x_1 = x_2 = 1$ is a Cournot equilibrium. Hence, in this example, $2 = X > X^S = 1$. □

Proposition 7. *Suppose that Assumptions 1-4 hold and that the inverse demand function is convex. Let* $\mathbf{x}$ *and* $\mathbf{x}^S$ *be a Cournot candidate and an optimal solution to (1), respectively. If* $p(X) = p(X^S)$*, then* $p'(X) = 0$ *and* $\gamma(\mathbf{x}) = 1$.

Proposition 1 shows that all social optima lead to a unique "socially optimal" price. Combining with Proposition 7, we conclude that if $p(\cdot)$ is convex, a Cournot candidate is socially optimal if and only if it results in the socially optimal price.

2.5 Concave Inverse Demand Functions

In this section, we argue that the case of concave inverse demand functions is fundamentally different. For this reason, the study of the concave case would require a very different line of analysis, and is not considered further in this paper.

According to Proposition 7, if the inverse demand function is convex and if the price at a Cournot equilibrium equals the price at a socially optimal point, then the Cournot equilibrium is socially optimal. For nonconvex inverse demand functions, this is not necessarily true: a socially optimal price can be associated with a socially suboptimal Cournot equilibrium, as demonstrated by the following example.

Example 3. Consider a model involving two suppliers ($N = 2$), with $C_1(x) = x$ and $C_2(x) = x^2$. The inverse demand function is concave on the interval where it is positive, of the form

$$p(q) = \begin{cases} 1, & \text{if } \ 0 \leq q \leq 1, \\ \max\{0, -M(q-1)+1\}, & \text{if } \ 1 < q, \end{cases}$$

where $M > 2$. It is not hard to see that the vector $(0.5, 0.5)$ satisfies the optimality conditions in (2), and is therefore socially optimal. We now argue that $(1/M, 1-1/M)$ is a Cournot equilibrium. Given the action $x_2 = 1/M$ of supplier 2, any action on the interval $[0, 1-1/M]$ is a best response for supplier 1. Given the action $x_1 = 1 - (1/M)$ of supplier 1, a simple calculation shows that

$$\arg \max_{x \in [0,\infty)} \left\{ x \cdot p(x + 1 - 1/M) - x^2 \right\} = 1/M.$$

Hence, $(1/M, 1 - 1/M)$ is a Cournot equilibrium. Note that $X = X^S = 1$, so that $p(X) = p(X^S)$. However, the optimal social welfare is 0.25, while the social welfare achieved at the Cournot equilibrium is $1/M - 1/M^2$. By considering arbitrarily large M, the corresponding efficiency can be made arbitrarily small. □

The preceding example shows that arbitrarily high efficiency losses are possible, even if $X = X^S$. The possibility of inefficient allocations even when the price is the correct one opens up the possibility of substantial inefficiencies that are hard to bound.

3 Affine Inverse Demand Functions

In this section, we establish an efficiency lower bound for Cournot oligopoly models with affine inverse demand functions, of the form:

$$p(q) = \begin{cases} b - aq, & \text{if } 0 \le q \le b/a, \\ 0, & \text{if } b/a < q, \end{cases} \tag{7}$$

where a and b are positive constants.

Theorem 1. *Suppose that Assumption 1 holds (convex cost functions), and that the inverse demand function is affine, of the form (7). Suppose also that* $b > \min_n\{C_n'(0)\}$ *(Assumption 4). Let* $\mathbf{x}$ *be a Cournot equilibrium, and let* $\alpha_n = C_n'(x_n)$*. Let also*

$$\beta = \frac{aX}{b - \min_n\{\alpha_n\}},$$

If $X > b/a$*, then* $\mathbf{x}$ *is socially optimal. Otherwise:*

(a) We have $1/2 \le \beta < 1$*.*
(b) The efficiency of $\mathbf{x}$ *satisfies,*

$$\gamma(\mathbf{x}) \ge g(\beta) = 3\beta^2 - 4\beta + 2.$$

(c) The bound in part (b) is tight. That is, for every $\beta \in [1/2, 1)$ *and every* $\epsilon > 0$*, there exists a model with a Cournot equilibrium whose efficiency is no more than* $g(\beta)+\epsilon$*.*
(d) The function $g(\beta)$ *is minimized at* $\beta = 2/3$ *and the worst case efficiency is* $2/3$*.*

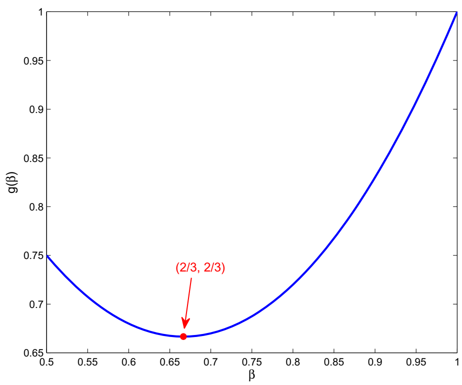

Fig. 1. A tight lower bound on the efficiency of Cournot equilibria for the case of affine inverse demand functions

The lower bound $g(\beta)$ is illustrated in Fig. 1. For the special case where all the cost functions are linear, of the form $C_n(x_n) = \alpha_n$, Theorem 1 has an interesting interpretation. We first note that $\beta = X/X^S$, which is the ratio of the aggregate supply at the Cournot equilibrium to that at a social optimum. Clearly, if β is close to 1 we expect the efficiency loss due to the difference $X^S - X$ to be small. However, efficiency losses may also arise if the total supply at a Cournot equilibrium is not provided by the most efficient suppliers. Our result shows that, for the affine case, β can be used to lower bound the total efficiency loss due to this second factor as well. Somewhat surprisingly, the worst case efficiency also tends to be somewhat better for low β, that is, when β approaches 1/2, as compared to intermediate values ($\beta \approx 2/3$).

4 Convex Inverse Demand Functions

In this section, we first show that in order to study the worst-case efficiency of Cournot equilibria, it suffices to consider a particular form of piecewise linear inverse demand functions. We then introduce the main result of this section, an efficiency lower bound that holds for Cournot oligopoly models with convex inverse demand functions.

Proposition 8. *Suppose that Assumptions 1-4 hold, and that the inverse demand function is convex. Let* $\mathbf{x}$ *and* $\mathbf{x}^S$ *be a Cournot candidate and an optimal solution to (1), respectively. Assume that* $p(X) \neq p(X^S)$ *and let* $c = |p'(X)|$. *Consider a modified model in which we replace the inverse demand function by a new function* $p^0(\cdot)$, *defined by*

$$p^0(q) = \begin{cases} -c(q - X) + p(X), & \text{if} \quad 0 \leq q \leq X, \\ \max\left\{0, \frac{p(X^S)-p(X)}{X^S-X}(q - X) + p(X)\right\}, & \text{if} \quad X < q. \end{cases} \tag{8}$$

Then, for the modified model, with inverse demand function $p^0(\cdot)$, *the vector* $\mathbf{x}^S$ *remains socially optimal, and the efficiency of* $\mathbf{x}$, *denoted by* $\gamma^0(\mathbf{x})$, *satisfies*

$$\gamma^0(\mathbf{x}) \leq \gamma(\mathbf{x}).$$

Proof. Proof Since $p(X) \neq p(X^S)$, Proposition 6 implies that $X < X^S$, so that $p^0(\cdot)$ is well defined. Since the necessary and sufficient optimality conditions in (2) only involve the value of the inverse demand function at X^S, which has been unchanged, the vector $\mathbf{x}^S$ remains socially optimal for the modified model.

Let

$$A = \int_0^X p^0(q)\, dq, \quad B = \int_X^{X^S} p(q)\, dq,$$

and

$$C = \int_X^{X^S} (p^0(q) - p(q))\, dq, \quad D = \int_0^X (p(q) - p^0(q))\, dq.$$

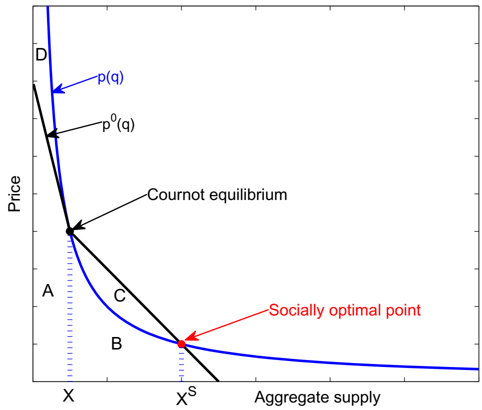

Fig. 2. The efficiency of a Cournot equilibrium cannot increase if we replace the inverse demand function by the piecewise linear function $p^0(\cdot)$. The function $p^0(\cdot)$ is tangent to the inverse demand function $p(\cdot)$ at the equilibrium point, and connects the Cournot equilibrium point with the socially optimal point.

See Fig. 2 for an illustration of $p(\cdot)$ and a graphical interpretation of A, B, C, D. Note that since $p(\cdot)$ is convex, we have $C \geq 0$ and $D \geq 0$. The efficiency of $\mathbf{x}$ in the original model with inverse demand function $p(\cdot)$, is

$$0 < \gamma(\mathbf{x}) = \frac{A + D - \sum_{n=1}^{N} C_n(x_n)}{A + B + D - \sum_{n=1}^{N} C_n(x_n^S)} \leq 1,$$

where the first inequality is true because the social welfare achieved at any Cournot candidate is positive (Proposition 4). The efficiency of $\mathbf{x}$ in the modified model is

$$\gamma^0(\mathbf{x}) = \frac{A - \sum_{n=1}^{N} C_n(x_n)}{A + B + C - \sum_{n=1}^{N} C_n(x_n^S)}.$$

Note that the denominators in the above formulas for $\gamma(\mathbf{x})$ and $\gamma^0(\mathbf{x})$ are all positive, by Proposition 4.

If $A - \sum_{n=1}^{N} C_n(x_n) \leq 0$, then $\gamma^0(\mathbf{x}) \leq 0$ and the result is clearly true. We can therefore assume that $A - \sum_{n=1}^{N} C_n(x_n) > 0$. We then have

$$0 < \gamma^0(\mathbf{x}) = \frac{A - \sum\limits_{n=1}^{N} C_n(x_n)}{A + B + C - \sum\limits_{n=1}^{N} C_n(x_n^S)} \leq \frac{A + D - \sum\limits_{n=1}^{N} C_n(x_n)}{A + B + C + D - \sum\limits_{n=1}^{N} C_n(x_n^S)}$$

$$\leq \frac{A + D - \sum\limits_{n=1}^{N} C_n(x_n)}{A + B + D - \sum\limits_{n=1}^{N} C_n(x_n^S)} = \gamma(\mathbf{x}) \leq 1,$$

which proves the desired result. □

Note that unless $p(\cdot)$ happens to be linear on the interval $[X, X^S]$, the function $p^0(\cdot)$ is not differentiable at X and, according to Proposition 2, $\mathbf{x}$ cannot be a Cournot candidate for the modified model. Nevertheless, $p^0(\cdot)$ can still be used to derive a lower bound on the efficiency of Cournot candidates in the original model.

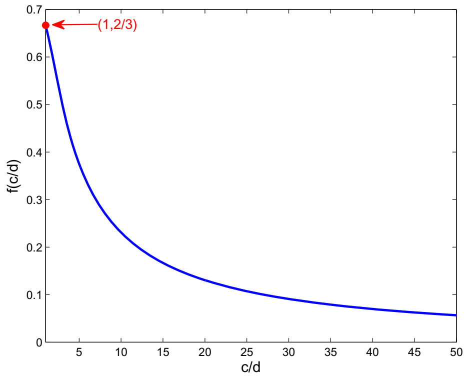

Fig. 3. Plot of the lower bound on the efficiency of a Cournot equilibrium in a Cournot oligopoly with convex inverse demand functions, as a function of the ratio c/d

Theorem 2. *Suppose that Assumptions 1-4 hold, and that the inverse demand function is convex. Let* $\mathbf{x}$ *and* $\mathbf{x}^S$ *be a Cournot equilibrium and a solution to (1), respectively. Then, the following hold.*

(a) If $p(X) = p(X^S)$*, then* $\gamma(\mathbf{x}) = 1$.
(b) If $p(X) \neq p(X^S)$*, let* $c = |p'(X)|$, $d = |(p(X^S) - p(X))/(X^S - X)|$*, and* $\overline{c} = c/d$*. We have* $\overline{c} \geq 1$ *and*

$$1 > \gamma(\mathbf{x}) \geq f(\overline{c}) = \frac{\phi^2 + 2}{\phi^2 + 2\phi + \overline{c}}, \tag{9}$$

where

$$\phi = \max\left\{\frac{2 - \overline{c} + \sqrt{\overline{c}^2 - 4\overline{c} + 12}}{2}, 1\right\}.$$

Remark 1. We do not know whether the lower bound in Theorem 2 is tight. The difficulty in proving tightness is due to the fact that the vector $\mathbf{x}$ need not be a Cournot equilibrium in the modified model. □

The lower bound established in part (b) is depicted in Fig. 2. If $p(\cdot)$ is affine, then $\overline{c} = c/d = 1$. From (9), it can be verified that $f(1) = 2/3$, which agrees with the lower bound in [9] for the affine case. We note that the lower bound $f(\overline{c})$ is monotonically decreasing in $\overline{c}$, over the domain $[1, \infty)$. When $\overline{c} \in [1, 3)$, ϕ is at least 1, and monotonically decreasing in $\overline{c}$. When $\overline{c} \geq 3$, $\phi = 1$.

5 Corollaries and Applications

For a given inverse demand function $p(\cdot)$, the lower bound derived in Theorem 2 requires some knowledge on the Cournot candidate and the social optimum, namely, the aggregate supplies X and X^S. We will derive an efficiency lower bound that does not require knowledge of X and X^S, and apply it to various convex inverse demand functions that have been considered in the economics literature.

Corollary 1. *Suppose that Assumptions 1-4 hold and that $p(\cdot)$ is convex. Let*[2]

$$s = \inf\{q \mid p(q) = \min_n C_n'(0)\}, \quad t = \inf\left\{q \;\middle|\; \min_n C_n'(q) \geq p(q) + q\partial_+ p(q)\right\}. \tag{10}$$

If $\partial_- p(s) < 0$, then the efficiency of a Cournot candidate is at least $f\left(\partial_+ p(t)/\partial_- p(s)\right)$.

Note that if there exists a "best" supplier n such that $C_n'(x) \leq C_m'(x)$, for any other supplier m and any $x > 0$, then the parameters s and t depend only on $p(\cdot)$ and $C_n'(\cdot)$.

Example 4. Suppose that Assumptions 1, 3, and 4 hold, and that there is a best supplier, whose cost function is linear with a slope $c \geq 0$. Consider inverse demand functions of the form (cf. Eq. (6) in [2])

$$p(q) = \max\{0, \alpha - \beta \log q\}, \qquad 0 < q, \tag{11}$$

where α and β are positive constants.[3] Through a simple calculation we obtain

$$s = \exp\left(\frac{\alpha - c}{\beta}\right), \qquad t = \exp\left(\frac{\alpha - \beta - c}{\beta}\right).$$

From Corollary 1 we obtain that for every Cournot equilibrium $\mathbf{x}$,

$$\gamma(\mathbf{x}) \geq f\left(\frac{\exp\left((\alpha - c)/\beta\right)}{\exp\left((\alpha - \beta - c)/\beta\right)}\right) = f\left(\exp(1)\right) \geq 0.5237. \tag{12}$$

Now we argue that the efficiency lower bound (12) holds even without the assumption that there is a best supplier associated with a linear cost function. From Proposition 5, the efficiency of any Cournot equilibrium $\mathbf{x}$ will not increase if the cost function of each supplier n is replaced by

$$\overline{C}_n(x) = C_n'(x_n)x, \qquad \forall x \geq 0.$$

[2] Under Assumption 3, the existence of the real numbers defined in (10) is guaranteed.

[3] In fact, $p(0)$ is undefined. This turns out to not be an issue: for a small enough $\epsilon > 0$, we can guarantee that no supplier chooses a quantity below ϵ. Furthermore, $\lim_{\epsilon \downarrow 0} \int_0^\epsilon p(q)\, dq = 0$. For this reason, the details of the inverse demand function in the vicinity of zero are immaterial as far as the chosen quantities or the resulting social welfare are concerned.

Let $c = \min_n C_n'(x_n)\}$. Since the efficiency lower bound in (12) holds for the modified model with linear cost functions, it applies whenever the inverse demand function is of the form (11). □

Example 5. Suppose that Assumptions 1, 3, and 4 hold, and that there is a best supplier, whose cost function is linear with a slope $c \geq 0$. Consider inverse demand functions of the form (cf. Eq. (5) in [2])

$$p(q) = \max\{\alpha - \beta q^\delta, 0\}, \qquad 0 < \delta \leq 1, \tag{13}$$

where α and β are positive constants. Note that if $\delta = 1$, then $p(\cdot)$ is affine; if $0 < \delta \leq 1$, then $p(\cdot)$ is convex. Assumption 4 implies that $\alpha > \chi$. Through a simple calculation we have

$$s = \left(\frac{\alpha - c}{\beta}\right)^{1/\delta}, \quad t = \left(\frac{\alpha - c}{\beta(\delta + 1)}\right)^{1/\delta}.$$

From Corollary 1 we know that for every Cournot equilibrium $\mathbf{x}$,

$$\gamma(\mathbf{x}) \geq f\left(\frac{-\beta\delta t^{\delta-1}}{-\beta\delta s^{\delta-1}}\right) = f\left((\delta + 1)^{\frac{1-\delta}{\delta}}\right).$$

Using the argument in Example 4, we conclude that this lower bound also applies to the case of general convex cost functions. □

6 Conclusion

It is well known that Cournot oligopoly can yield arbitrarily high efficiency loss in general; for details, see [8]. For Cournot oligopoly with convex market demand and cost functions, results such as those provided in Theorem 2 show that the efficiency loss of a Cournot equilibrium can be bounded away from zero by a function of a scalar parameter that captures quantitative properties of the inverse demand function. With additional information on the cost functions, the efficiency lower bounds can be further refined. Our results apply to various convex inverse demand functions that have been considered in the economics literature.

References

1. Amir, R.: Cournot oligopoly and the theory of supermodular games. Games Econ. Behav. 15, 132–148 (1996)
2. Bulow, J., Pfleiderer, P.: A note on the effect of cost changes on prices. J. Political Econ. 91(1), 182–185 (1983)
3. Corchon, L.C.: Welfare losses under Cournot competition. International J. of Industrial Organization 26(5), 1120–1131 (2008)
4. Fabingeryand, M., Weyl, G.: Apt Demand: A flexible, tractable adjustable-pass-through class of demand functions (2009), `http://isites.harvard.edu/fs/docs/icb.topic482110.files/Fabinger.pdf`

5. Friedman, J.W.: Oligopoly and the Theory of Games. North-Holland, Amsterdam (1977)
6. Friedman, J.: Oligopoly Theory. Cambridge University Press (1983)
7. Guo, X., Yang, H.: The Price of Anarchy of Cournot Oligopoly. In: Deng, X., Ye, Y. (eds.) WINE 2005. LNCS, vol. 3828, pp. 246–257. Springer, Heidelberg (2005)
8. Johari, R.: Efficiency loss in market mechanisms for resource allocation, Ph.D. dissertation, Mass. Inst. Technol., Cambridge, MA, USA (2004)
9. Johari, R., Tsitsiklis, J.N.: Efficiency loss in Cournot games, MIT Lab. Inf. Decision Syst., Cambridge, MA, USA. Technical report 2639 (2005), `http://web.mit.edu/jnt/www/Papers/R-05-cournot-tr.pdf`
10. Kluberg, J., Perakis, G.: Generalized quantity competition for multiple products and loss of efficiency. In: Allerton Conf. Comm., Control, Comput., Monticello, IL, USA (2008)
11. Koutsoupias, E., Papadimitriou, C.H.: Worst-case equilibria. Computer Sci. Review 3(2), 65–69 (1999)
12. Novshek, W.: On the existence of Cournot equilibrium. Review of Econ. Studies 52(1), 85–98 (1985)
13. Tsitsiklis, J.N., Xu, Y.: Efficiency loss in a Cournot oligopoly with convex market demand (2012), `http://arxiv.org/abs/1203.6675`
14. Tyagi, R.: A characterization of retailer response to manufacturer trade deals. J. of Marketing Res. 36(4), 510–516 (1999)