A Stackelberg Game to Optimize the Distribution of Controls in Transportation Networks

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Abstract. We propose a game theoretic model for the spatial distribution of inspectors on a transportation network. The problem is to spread out the controls so as to enforce the payment of a transit toll. We formulate a linear program to find the control distribution which maximizes the expected toll revenue, and a mixed integer program for the problem of minimizing the number of evaders. Furthermore, we show that the problem of finding an optimal mixed strategy for a coalition of N inspectors can be solved efficiently by a column generation procedure. Finally, we give experimental results from an application to the truck toll on German motorways.

Keywords: Stackelberg game, Polymatrix game, Controls in transportation networks.

1 Introduction

In this article, we study from a theoretical point of view the problem of allocating inspectors to spatial locations of a transportation network, in order to enforce the payment of a transit fee. The question of setting an optimal level of control in transportation networks has been addressed by several authors, but to the best of our knowledge, none of them takes the topology of the network and the spatial distribution of the inspectors into account. Simple game theoretic models have been proposed to model the effect of the control intensity on the behaviour of the users of the network [4], to find an optimal trade-off between the control costs and the revenue from the network fee [1], or to evaluate the effect of giving some information (about the controls) to the users [6]. More recently, an approach to optimize the schedules of inspectors in public transportation networks was proposed by DSB S-tog in Denmark [7]. In contrast to our problem, the authors of the latter article focus on temporal scheduling and assume an evasion rate which does not depend on the control intensity. The present paper is motivated by an application to the enforcement of a truck toll in Germany, which we present next.

Truck toll on German motorways. In 2005 Germany introduced a distance-based toll for commercial trucks weighing twelve tonnes or more in order to

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fund growing investments for maintenance and extensions of motorways. The enforcement of the toll is the responsibility of the German Federal Office for Goods Transport (BAG), who has the task to carry out a network-wide control, with an intensity which is proportional to spatial and time dependent traffic distributions. It is implemented by a combination of 300 automatic stationary gantry bridges and by tours of 300 control vehicles on the entire highway network. In this paper, we present some theoretical work obtained in the framework of our cooperation with the BAG, whose final goal is to develop an optimization tool to schedule the control tours of the inspectors. This real-world problem is subject to a variety of legal constraints, which we handle by mixed integer programming [2]. We propose a game theoretic approach to optimize the spatial distribution of the controls with respect to two different objectives: (i) maximize the (expected) monetary profit of the government; (ii) minimize the number of evaders. The goal of this study is twofold. On the one hand, we want to evaluate the reasonableness of current BAG's methodology (control intensities proportional to traffic volumes). On the other hand, we plan to use in a follow-up work the distributions computed in this article as a target of the real-world problem.

Specificity of the applied problem and assumptions made in this article. The model presented in this article is not limited to the case of motorway networks. It applies to any situation where the individuals in transit can be controlled on each section of their route through a network. A strong assumption of our model however is that we know the set of routed demands of the network, i.e. the number of individuals taking each possible route. In our model, the drivers do not have the choice of their route between their source and destination. We plan to search in this direction for future work. In particular, it might be relevant to consider that the drivers can take some sections of a trunk road to avoid the toll motorway.

We do not pretend that our model is representative of all the complexity of the drivers' reaction to inspectors' behavior, in particular because certain facts are particularly hard to model. For example, the perception of the penalty is not the same for all truck drivers. If an evader is caught with a second offense, he may get a higher fine in a trial.

In this article, we assume that the users of the network act on a selfish behaviour, and decide to pay or to evade so as to minimize the expected cost of their trip. This is obviously wrong, since there is certainly a large fraction of honest people who always pay the toll. However, we claim that our simplified model still leads to significant spatial distributions of the controls, because: (i) the number of evaders that we compute in this model corresponds to the number of network users for which it is more interesting to evade the toll; (ii) hence, the toll revenue in this model is a lower bound for the true revenue; (iii) if the fraction of honest drivers is the same on every route, we could solve the problem by considering the remaining fraction of *crafty* drivers only, which would lead to the same results.

Organization and contribution. We present our model in Section 2. We show that the optimal distribution of controls (with respect to the aforementioned goals) is the optimal strategy of the inspectors in a Stackelberg game, and can be found by mean of mathematical programming formulations. Then we exhibit in Section 2.3 a relation between the optimal solution of our model and the Nash equilibriums of a particular polymatrix game. Finally, experimental results from the application to the truck toll in Germany are presented in Section 3.

2 A Network Spot-Checking Game

We make use of the standard notation $[n] := \{1, \ldots, n\}$ and we denote vectors by boldface lower case letters. We model the transportation network by a directed graph $G = (V, E)$. We assume that the users of the network are distributed over a set of routes $\mathcal{R} = \{r_1, \ldots, r_m\}$, where each $r_i \subset E$. In addition, we are given the demand x_i of each route, that is, the number of users that take the route r_i per unity of time (typically one hour; we assume a constant demand, i.e., we do not take the diurnal variations of the traffic into account). We denote by $y_e := \sum_{\{i \in [m]:\ r_i \ni e\}} x_i$ the number of users passing through edge e per unity of time.

Every user of the route r_i has to pay a toll fee T_i, but he may also decide to evade the toll, with the risk to pay a penalty P_i if he gets controlled. We assume that the inspectors have a total capacity of control κ. This means that κ individuals can be controlled per unity of time. We consider two manners of spreading out the controls over the network in the next subsections. In the first one, we simply assume that the control force can be arbitrarily distributed over the network. The second one is a more theoretical approach, where we consider all possible allocations of a finite number of inspectors over the sections $e \in E$, and we search for the best mixed strategy combining these allocations.

2.1 Arbitrarily Splittable Controls

We denote by $\boldsymbol{q} \in \Delta_E$ the distribution of the controls, where Δ_E is the set of all probability distributions over E:

$$\Delta_E := \{\boldsymbol{q} \in [0,1]^{|E|} : \sum_{e \in E} q_e = 1\}.$$

Each coordinate of $\boldsymbol{q}$ represents the proportion of the total capacity of control κ that is allocated to the corresponding edge, i.e., κq_e individuals can be controlled on the section e per unity of time.

Strategy of the users. We denote by π_i the probability for a user of the route r_i to be controlled during its trip. We assume a stationary regime in which the users have learned the values of the probability π_i. Hence, a user of the route r_i will pay if π_i is above the threshold $\frac{T_i}{P_i}$, and evade if it is below. In other words, the proportion p_i of payers on the route r_i minimizes the average cost per user of this route:

$$\lambda_i := \min(T_i, P_i\pi_i) = \min_{p_i \in [0,1]} \Big(p_i T_i + (1-p_i) P_i \pi_i\Big).$$

A user passing on the section e has a probability $(\frac{\kappa q_e}{y_e} \wedge 1)$ to be controlled on this section, where we have used the notation $a \wedge b := \min(a,b)$. Hence, the probability π_i of being controlled during a trip on route r_i can be expressed as a function of the control distribution $\boldsymbol{q}$:

$$\pi_i = 1 - \prod_{e \in r_i} \big(1 - (\frac{\kappa q_e}{y_e} \wedge 1)\big).$$

In this section, we will use the classical approximation

$$\pi_i \simeq \pi_i' := \sum_{e \in r_i} (\frac{\kappa q_e}{y_e} \wedge 1), \tag{1}$$

which is valid when the right hand side of Equation (1) is small. In the experiments presented in Section 3, we obtain values of π_i' that never exceed 0.2. Note that this approximation is equivalent to assuming that a user pays twice the fine if he is caught twice.

Strategy of the inspectors. We think of the set of inspectors as a single player who splits the total control force κ according to a distribution $\boldsymbol{q} \in \Delta_E$, called the *mixed strategy of the controller*. Similarly, the users of the route $r_i \in \mathcal{R}$ are considered as a single player (called the i^{th} user), who pays the toll with a probability p_i and tries to evade with the complementary probability $1 - p_i$. We say that the i^{th} user plays with mixed strategy $\boldsymbol{p_i} = [p_i, 1-p_i]^T \in \Delta_2$. Our assumption that the users have the ability to *learn* the control distribution $\boldsymbol{q}$ can be described in the framework of a *Stackelberg game*, in which the controller is the *leader*, who makes the first *move*, while the users are *followers* who react to the leader's action. The controller knows that the users will adjust their strategies p_i depending on the control distribution $\boldsymbol{q}$, and plays accordingly. We can now formulate the problem of optimally distributing the controls over the networks, with respect to two distinct objectives.

Maximizing the profit. If the controller wants to maximize the total revenue generated by the toll, which is, by construction, equal to the total loss of the users, the problem to solve is:

$$\max_{\boldsymbol{q} \in \Delta_E} \quad \sum_{i \in [m]} x_i \lambda_i \quad = \quad \max_{\boldsymbol{q} \in \Delta_E} \quad \sum_i x_i \; \min(T_i, P_i \pi_i'), \tag{2}$$

where π_i' depends on $\boldsymbol{q}$ through Equation (1). If the costs of the controls must be taken into account, and the cost for a control on section e is c_e, then we can solve:

$$\max_{\boldsymbol{q} \in \Delta_E^-} \quad \sum_{i \in [m]} x_i \min(T_i, P_i \pi_i') - \sum_{e \in E} q_e \kappa c_e, \tag{3}$$

where $\Delta_E^- := \{\boldsymbol{q} \in [0,1]^{|E|} : \sum_{e \in E} q_e \leq 1\}$ (we do not necessarily use all the control capacity). It is not difficult to see that there must be an optimum such that $\forall e \in E$, $\frac{\kappa q_e}{y_e} \leq 1$, because the controller never has interest to place more capacity of control on a section than the number of users that pass through it. If we impose this constraint, the expression of π_i' simplifies to $\sum_{e \in r_i} \frac{\kappa q_e}{y_e}$, and Problem (3) becomes a linear program:

$$\begin{aligned} \max_{\substack{\boldsymbol{q} \in \Delta_E^- \\ \boldsymbol{\lambda} \in \mathbf{R}^m}} \quad & \sum_i x_i \lambda_i - \sum_{e \in E} q_e \kappa c_e \\ \text{s.t.} \quad & \forall i \in [m], \quad \lambda_i \leq P_i \sum_{e \in r_i} \frac{\kappa q_e}{y_e} \\ & \forall i \in [m], \quad \lambda_i \leq T_i \\ & \forall e \in E, \quad \kappa q_e \leq y_e. \end{aligned} \tag{4}$$

Minimizing the number of evaders. If the goal of the controller is to minimize the number of evaders, the problem to solve is:

$$\min_{\boldsymbol{q} \in \Delta_E} \sum_{\{i \in [m]:\ P_i \pi_i' < T_i\}} x_i.$$

Note that we have chosen to consider here that the i^{th} user is paying when the threshold $\pi_i' = \frac{T_i}{P_i}$ is reached but not exceeded. We can formulate this problem as a mixed integer program (MIP), by introducing a binary variable δ_i which is forced to take the value 1 when $\pi_i' < \frac{T_i}{P_i}$:

$$\begin{aligned} \min_{\substack{\boldsymbol{q} \in \Delta_E \\ \boldsymbol{\delta} \in \{0;1\}^m}} \quad & \sum_i x_i \delta_i \\ \text{s.t.} \quad & \forall i \in [m], \quad \frac{T_i}{P_i} \leq \sum_{e \in r_i} \frac{\kappa q_e}{y_e} + \delta_i \\ & \forall e \in E, \quad \kappa q_e \leq y_e. \end{aligned} \tag{5}$$

2.2 Coalition of a Finite Number of Controllers

In this section, we consider a more realistic setting, in which N inspectors, each having a capacity of control $\frac{\kappa}{N}$, play a cooperative game in order to maximize the revenue generated from the toll. A strategy of the coalition of controllers consists of a vector $\boldsymbol{n} \in \mathcal{S}_N := \{\boldsymbol{n} \in \mathbb{N}^{|E|} : \sum_{e \in E} n_e = N\}$ that indicates how many inspectors are allocated to each edge of the network. We assume that the inspectors play with a mixed strategy $\boldsymbol{q} \in \Delta_{\mathcal{S}_N}$, i.e. they choose the spatial distribution $\boldsymbol{n} \in \mathcal{S}_N$ with the probability $q_{\boldsymbol{n}}$. With this setting, the probability for a user of the route r_i to be controlled during its trip becomes

$$\bar{\pi}_i = \sum_{\boldsymbol{n} \in \mathcal{S}_N} q_{\boldsymbol{n}} \underbrace{\left(1 - \prod_{e \in r_i} \left(1 - \left(\frac{n_e \kappa}{N y_e} \wedge 1\right)\right)\right)}_{\alpha_{\boldsymbol{n},i}}.$$

As in Section 2.1, the problem to maximize the revenue generated from the toll can be formulated as an LP (we do not consider control costs for the sake of simplicity). Note that this time, we do not need to take a linear approximation of $\bar{\pi}_i$, because $\alpha_{\boldsymbol{n},i}$ is a fixed parameter:

$$\begin{aligned} \max_{\substack{\boldsymbol{q}\in\Delta_{\mathcal{S}_N}\\ \boldsymbol{\lambda}\in\mathbf{R}^m}} \quad & \sum_i x_i\lambda_i \\ \text{s.t.} \quad & \forall i \in [m], \quad \lambda_i \leq T_i && \text{(6a)}\\ & \forall i \in [m], \quad \lambda_i \leq \sum_{\boldsymbol{n}\in\mathcal{S}_n} q_{\boldsymbol{n}} P_i \alpha_{\boldsymbol{n},i}. && \text{(6b)} \end{aligned}$$

Although the number of strategies of the inspectors' coalition is exponential with respect to N, we will see that this problem can be solved efficiently by column generation. Let $\boldsymbol{v_n}$ denote the vector of $\mathbb{R}^m$ with coordinates $v_{\boldsymbol{n},i} = P_i\alpha_{\boldsymbol{n},i}$. From a geometrical point of view, the constraint (6b) restricts $\boldsymbol{\lambda}$ to the polyhedron $\mathcal{P}$ which is defined as the convex hull of the vertices $(\boldsymbol{v_n})_{\boldsymbol{n}\in\mathcal{S}_N}$, plus the cone of nonpositive vectors:

$$\mathcal{P} = \{\boldsymbol{v} + \boldsymbol{z} : \ \boldsymbol{v} \in \text{convex-hull}(\{\boldsymbol{v_n} : \ \boldsymbol{n} \in \mathcal{S}_N\}), \ \boldsymbol{z} \in \mathbb{R}^m_-\}. \tag{7}$$

The next proposition shows that if the capacity of control κ is smaller than the traffic on every edge, then $\mathcal{P}$ has no more than $|E|$ extreme points, so that we can impose $q_{\boldsymbol{n}} = 0$ for almost all $\boldsymbol{n} \in \mathcal{S}_N$.

Proposition 1. *Assume that $\forall e \in E, \kappa \leq y_e$, and denote by $\tilde{\boldsymbol{n}}(e)$ the allocation where all the inspectors are concentrated on edge e. Then, every extreme point of $\mathcal{P}$ is of the form $\boldsymbol{v}_{\tilde{\boldsymbol{n}}(e)}$ for an $e \in E$. Hence, Problem (6) has a solution in which $q_{\boldsymbol{n}} = 0$ for all $\boldsymbol{n} \in \mathcal{S}_N \setminus \{\tilde{\boldsymbol{n}}(e) : \ e \in E\}$.*

Proof. It is clear that the extreme points of convex-hull($\mathcal{S}_N$) are the vectors of the form $\tilde{\boldsymbol{n}}(e) := [0,\dots,0,N,0,\dots,0]^T$, with the nonzero in position e. The application $\boldsymbol{n} \mapsto \boldsymbol{u_n}$, which maps $\mathcal{S}_N$ onto $\mathbb{R}^m$, and where

$$u_{\boldsymbol{n},i} := P_i \sum_{e\in r_i} \frac{n_e\kappa}{Ny_e}$$

is linear, and hence the extreme points of the polyhedron with vertices $(\boldsymbol{u_n})_{\boldsymbol{n}\in\mathcal{S}_N}$ are among the images of the extreme points of convex-hull($\mathcal{S}_N$), that is, the vectors $\boldsymbol{u}_{\tilde{\boldsymbol{n}}(e)}$ $(e \in E)$. Let $\boldsymbol{n} \in \mathcal{S}_N$. Since $\kappa \leq y_e$ for all e, the expression of $v_{\boldsymbol{n},i}$ can be simplified to:

$$v_{\boldsymbol{n},i} = P_i \left(1 - \prod_{e\in r_i} (1 - \frac{n_e\kappa}{Ny_e})\right) \leq u_{\boldsymbol{n},i},$$

where the inequality follows from the log-concavity of $\boldsymbol{x} \mapsto \prod_i (1 - x_i)$. Moreover the equality is attained for the vectors of the type $\boldsymbol{v}_{\tilde{\boldsymbol{n}}(e)}$, because the product consists of only one factor (or even 0 factor if $e \notin r_i$), i.e., $\forall e \in E$ we have

$\boldsymbol{v}_{\tilde{\boldsymbol{n}}(e)} = \boldsymbol{u}_{\tilde{\boldsymbol{n}}(e)}$. This shows that $\boldsymbol{u}_{\boldsymbol{n}} \in \mathcal{P}$, because it can be written as a convex combination of the vectors $(\boldsymbol{v}_{\tilde{\boldsymbol{n}}(e)})_{e\in E}$. Finally, if $\boldsymbol{n}$ is not of the type $\tilde{\boldsymbol{n}}(e)$, i.e., $\max_{e\in E} n_e < N$, then we know that $\boldsymbol{u}_{\boldsymbol{n}}$ is not an extreme point of $\mathcal{P}$, and hence the vector $\boldsymbol{v}_{\boldsymbol{n}}$, which can be written as $\boldsymbol{u}_{\boldsymbol{n}} + \boldsymbol{z}$ for a vector $\boldsymbol{z} \in \mathbb{R}^m_-$ is not an extreme point of $\mathcal{P}$.

If $\kappa > y_e$ for some $e \in E$, then some other extreme points will appear. However, we expect the solution to be sparse and we can solve Problem (6) by column generation. In addition to the columns corresponding to the variable $\boldsymbol{\lambda}$, we start with the columns that correspond to the fully concentrated allocations $(q_{\tilde{\boldsymbol{n}}(e)})_{e\in E}$. After each iteration, the subproblem that we must solve to add a new column is the maximization of the reduced cost $\boldsymbol{\mu}^T \boldsymbol{v}_{\boldsymbol{n}} - \mu_0$, where $\boldsymbol{\mu} \geq \boldsymbol{0}$ is the current dual variable associated with the constraints (6b), and μ_0 is the dual of the constraint $\sum_{\boldsymbol{n}} q_{\boldsymbol{n}} \leq 1$:

$$\max_{\boldsymbol{n}} \left\{ \sum_{i\in[m]} \mu_i P_i \left(1 - \prod_{e\in r_i} \left(1 - \left(\frac{n_e \kappa}{N y_e} \wedge 1\right)\right)\right) - \mu_0 : \ \boldsymbol{n} \in \mathcal{S}_N \right\} \tag{8}$$

We use a greedy heuristic to find an approximate solution of Problem (8): we start from the configuration $\boldsymbol{n^{(0)}} = \boldsymbol{0}$ without any inspector, and for $k = 1, \ldots, N$ we add an inspector on the section which causes the largest possible increase of the reduced cost:

$$\forall e \in E, \ n_e^{(k)} = \begin{cases} n_e^{(k-1)} + 1 & \text{if } e = e_k \\ n_e^{(k-1)} & \text{otherwise,} \end{cases}$$

$$\text{where } e_k \in \operatorname*{argmax}_{e'\in E} \sum_{i\in[m]} \mu_i P_i \left(1 - \prod_{e\in r_i} \left(1 - \frac{(n_e^{(k-1)} + \delta_{e,e'})\kappa}{N y_e} \wedge 1 \right)\right).$$

In the above equation, δ stands for the Kronecker delta function. We use the vector $\boldsymbol{n^{(N)}}$ generated by this greedy procedure as an approximation for the solution of (8), and we add the column $\boldsymbol{v}_{\boldsymbol{n}^{(N)}}$ in the linear program. Finally, we solve this augemented linear program and repeat the above procedure.

An argument justifying this greedy method is that if we use the same approximation as in Equation (1), the objective of Problem (8) becomes separable and concave, and it is well known that the greedy procedure finds the optimum (see e.g. [5]). The column generation procedure can be stopped when the optimal value of Problem (8) is 0, which guarantees that no other column can increase the value of Problem (6). In practice, we stop the column generation as soon as the reduced cost of the strategy $\boldsymbol{n^{(N)}}$ returned by the greedy procedure is 0.

2.3 Relation with Polymatrix Games

A polymatrix game is a multiplayer game in which the payoff of player i is the sum of the partial payoffs received from a bimatrix game against each other player:

$$\text{Payoff}(i) = \sum_{j \neq i} \boldsymbol{p_i}^T \boldsymbol{A_{ij}}\ \boldsymbol{p_j}.$$

In this section, we establish a relation between the solutions of the model (3) presented above and the Nash equilibriums of a particular polymatrix game. For the model without costs (2), it is not difficult to write the payoff of the controller as the sum of partial payoffs from zero-sum bimatrix games played against each user (recall that $\boldsymbol{p_i} = [p_i, 1 - p_i]^T$):

$$\text{Payoff(controller)} = \sum_i x_i \lambda_i = \sum_i \text{Loss(user } i) = \sum_i \boldsymbol{p}_i^T \boldsymbol{A_i q},$$

where $\boldsymbol{A_i}$ is the $2 \times |E|-$matrix with elements

$$\forall e \in E, \quad (\boldsymbol{A_i})_{1,e} = x_i T_i; \quad (\boldsymbol{A_i})_{2,e} = \begin{cases} \frac{\kappa}{y_e} x_i P_i & \text{if } e \in r_i; \\ 0 & \text{otherwise.} \end{cases}$$

This particular polymatrix game has a special structure, since the interaction between the players can be modelled by a star graph with the controller in the central node, and each edge represents a zero-sum game between a user and the controller. Modulo the additional constraint $\kappa q_e \leq y_e$, which bounds from above the mixed strategy of the controller, any Nash equilibrium $(\boldsymbol{q}, \boldsymbol{p_1}, \ldots, \boldsymbol{p_m})$ of this polymatrix game gives a solution $\boldsymbol{q}$ to the Stackelberg competition problem studied in Section 2.1. The model with control costs (3) can also be formulated in this way, by adding a new player who has a single strategy. This player plays a zero-sum game against the controller, whose payoff is the sum of the control costs $\sum_e c_e q_e$.

Interestingly, the fact that Problem (3) is representable by a LP is strongly related to the fact that every partial game is zero-sum. We point out a recent paper of Daskalakis and Papadimitriou [3], who have generalized the Neumann's minmax theorem to the case of zero-sum polymatrix games. In the introduction of the latter article, the authors moreover notice that for any star network, we can find an equilibrium of a zero-sum polymatrix game by solving a LP.

3 Experimental Results

We have solved the models presented in this paper for several regions of the German motorways network, based on real traffic data (averaged over time). We present here a brief analysis of our results. On Figure 1, we have represented the mixed strategy of the controller that maximizes the revenue from the toll (without control costs, for $\kappa = 60$ controls per hour), for the regions of Berlin-Brandenburg and North Rhine-Westphalia (NRW). The graphs corresponding to these regions consist of 57 nodes (resp. 111) and 120 directed edges (resp. 264), and we have taken in consideration 1095 routes (resp. 4905). We have used a toll fee of 0.176 € per kilometer, and a penalty of 400 € that does not depend on the route.

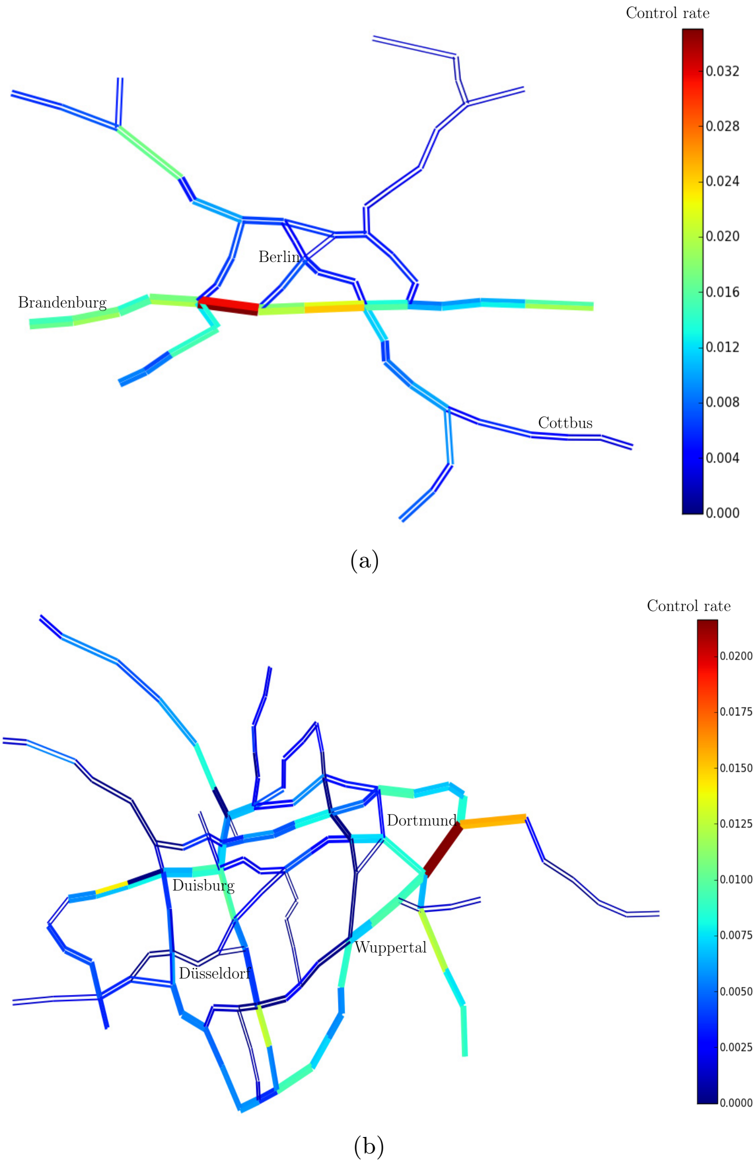
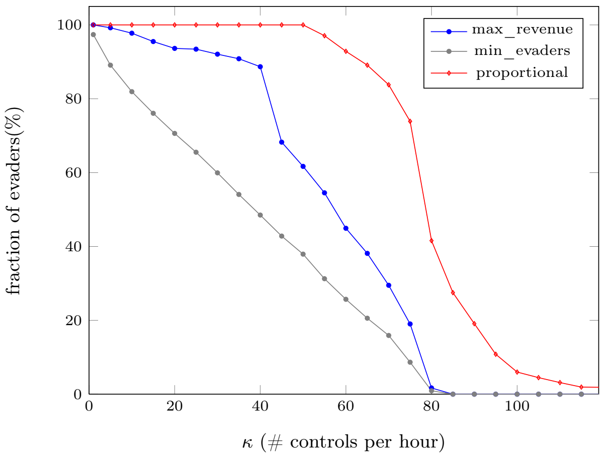
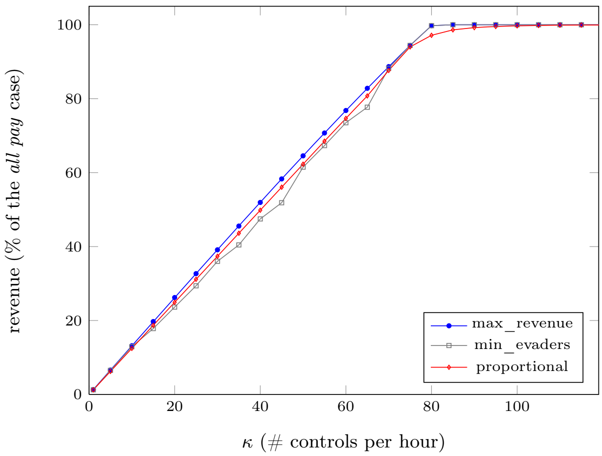


Fig. 1. Mixed strategy of the controller which maximizes the revenue (2), for the regions of Berlin-Brandenburg (a), and NRW (b). The widths of the sections indicate the traffic volumes.

For the region of Berlin-Brandenburg, we have plotted the evolution of the number of evaders and the revenue generated from the toll as a function of κ on Figure 2. Just to give an idea of the order of magnitudes, there is an average of 1620 trucks per hour in this instance. The strategies that maximize the revenue



(a)



(b)

Fig. 2. Evolution of the number of evaders (a) and of the toll revenue (b) with κ, for the region of Berlin-Brandenburg

and that minimize the number of evaders are compared to the case where the controls are proportional to the traffic. Several conclusions can be drawn from this Figure: first, the "proportional" strategy is not so bad in terms of revenue, however a difference of up to 4% with the max_revenue strategy is observed. Second, the number of evaders decreases much faster when the controls are distributed with respect to this goal. For $\kappa = 55$, the evasion rate achieved by the control distribution that is proportional to the traffic (resp. that maximizes the revenue) is of 97% (resp. 54%), while we can achieve an evasion rate of 31% with the min_evaders strategy. Third, both the max_revenue and the min_evaders strategies create a situation in which it is in the interest of no driver to evade for $\kappa \geq 80.3$. In contrast, there is still 2% of the drivers who had better evade with the proportional strategy for $\kappa = 115$.

We have also computed the optimal mixed strategy for a coalition of $N = 13$ inspectors, with the column generation procedure described in Section 2.2. For $\kappa = 60$, we found that the N inspectors should be simultaneously allocated to a common section 84% of the time. The column generation procedure, which allows to consider the strategies where the inspectors are spread over the network, yields an increase of revenue of only 1.84%. An intuitive explanation is that spreading out the inspectors leads to potentially controlling several times the same driver. Moreover, most of the traffic passes only through sections where $y_e \geq \kappa$, so that $\boldsymbol{v}_{\tilde{\boldsymbol{n}}(e)}$ is an extreme point of $\mathcal{P}$ (cf. Equation (7)).

4 Conclusion and Perspectives

We have presented a novel approach based on a Stackelberg game to spread out the controls over a transportation network, in order to enforce the payment of a transit toll. To the best of our knowledge, this is the first article which studies the distribution of controls while taking the topology of the network into account. The problem of distributing the controls so as to maximize the expected toll revenue (resp. minimize the number of evaders) was formulated as a linear program (resp. mixed integer program), and we have drawn a parallel with polymatrix games. Experimental results suggest that this approach can lead to significant improvements compared to the strategy which consists in controlling each section proportionally to the traffic volumes, especially when the goal is to minimize the number of toll evaders.

We have also shown that our model can be extended to deal with the problem of simultaneously deploying N controllers over the network. Despite the apparent complexity of this problem, we were able to find a solution by column generation in our experiments. The optimal strategy assigns most of the time the N controllers to the same section.

In future work, we want to improve the behavioral model of the users. A key point seems to be the perception of the probability to be controlled as a function of the control distributions, which can very different for several users [1]. We also want to introduce some time dynamics in the model, since the diurnal variations of the traffic can be very important.

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