Optimal Contract Design for an Efficient Secondary Spectrum Market*

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Abstract. In this paper we formulate a contract design problem where a primary license holder wishes to profit from its excess spectrum capacity by selling it to potential secondary users/buyers, but needs to determine how to optimally price it to maximize its profit, knowing that this excess capacity is stochastic in nature and cannot provide deterministic service guarantees to a buyer. We address this problem by adopting as a reference a traditional spectrum market where the buyer can purchase exclusive access with fixed/deterministic guarantees. We consider two cases; in one the seller has full information on the buyer, including its service requirement and quality constraint, and in the other the seller only knows possible types and their distribution. In the first case we fully characterize the nature of the optimal contract design. In the second case, we find the optimal contract design when there are two possible types and determine a design procedure and show that it is optimal when the nature of the stochastic channel is common to all possible types.

Keywords: contract design, incentives, quality of service constraint, secondary spectrum market.

1 Introduction

The scarcity of spectrum resources and the desire to improve spectrum efficiency have led to extensive research and development in recent years in such concepts as dynamic spectrum access/sharing, open access, and secondary (spot or short-term) spectrum market, see e.g., [1, 2].

One of the fundamental premises behind a secondary (and short-term) spectrum market is the existence of excess capacity due to the primary license holder's own spectrum under-utilization. However, this excess capacity is typically uncontrolled and random, both spatially and temporally, and strongly dependent on the behavior of the primary users. One may be able to collect statistics and make predictions, as has been done in numerous spectrum usage studies [3–5], but it is fundamentally stochastic in nature. The primary license holder can of course choose to eliminate the randomness by setting aside resources (e.g., bandwidth) exclusively for secondary users. This will however likely impinge on its current users and may not be in the interest of its primary business model.

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The alternative is to simply give non-exclusive access to secondary users for a fee, which allows the secondary users to *share* a certain amount of bandwidth simultaneously with its existing licensed users, but only under certain conditions on the primary traffic/spectrum usage. For instance, a secondary user is given access but can only use the bandwidth if the current activity by the licensed users is below a certain level, e.g., as measured by received SNR. This is a typical scenario under the spectrum overlay and underlay models [6]; many spectrum sharing schemes proposed in the literature fall under this scenario, see e.g., [7–10].

In this case a secondary user pays (either in the form of money or services in return) for gaining spectrum access but not for guaranteed use of the spectrum. This presents a challenge to both the primary and the secondary users: On one hand, the secondary user must assess its needs and determine whether the uncertainty in spectrum quality is worth the price asked for and what level of uncertainty can be tolerated. On the other hand, the primary must decide how stochastic service quality should be priced so as to remain competitive against guaranteed (or deterministic) services which the secondary user may be able to purchase from a traditional market or a different primary license holder.

In this paper we formulate this as a contract problem for the primary user and seek to address the question of what type of contracts should the primary design so as to maximize its profit. Within this framework we adopt a reference point in the form of a traditional spectrum market from where a secondary user can purchase deterministic or guaranteed service, i.e., exclusive access rights to certain bandwidth, at a fixed price per unit. This gives the secondary user a choice to reject the offer from the primary user if it is risk-averse or if the primary user's offer is not attractive. This also implies that the price per unit of bandwidth offered by the primary user must reflect its stochastic quality.

Work most relevant to the study presented in this paper includes [11], which considers a contract problem where the secondary users help relay primary user's data and in return are allowed to send their own data, as well as [12], which considers the convexity of an optimal portfolio of different stochastic purchases, under two types of bandwidth shortage constraints. The work in [12] however considers only the perspective of the buyer but not the seller.

Our main results are as follows. We formally present the contract design problem in Section 2, and consider two cases. In the first case the seller is assumed to have full information on the buyer, including its service requirement and quality constraint. For in this case we fully characterize the optimal contract design (Section 3). In the second case the buyer belongs to a set of types and the seller knows only the set and its distribution but not the buyer's exact type. We again fully characterize the nature of the optimal contract design when the number of types is limited to two. In the case of having more than two possible types of buyer, we assume that the channel condition is common among the buyers. Under this assumption, we determine the optimal contract when the seller can design as many contract as the it wants. When the number of contracts is limited, we describe a design procedure and prove the optimality (Section 4).

2 Model and Assumptions

The basic idea underlying our model is to capture the value of secondary spectrum service, which is random and non-guaranteed in nature, by using guaranteed service as a reference.

2.1 The Contract Setup

The contract is setup to be distributed from the seller to the buyer in this model. The *seller*, who is also referred to as the owner or the primary license holder, uses the spectrum to provide business and service to its *primary users*, and carry *primary traffic*. The seller is willing to sell whatever underutilized bandwidth it has as long as it generates positive profit and does not impact negatively its primary business. It knows that the bandwidth it is selling is stochastic and cannot provide hard guarantees. We will assume that the seller pre-designs up to M contracts and announce them to a potential buyer. If the buyer accepts one of the contracts, they come to an agreement and they have to follow the contract up to a predetermined period of time. It is up to the seller to design the contracts, but up to the buyer to decide whether or not to accept it.

Each contract is in the form of a pair of real numbers (x, p), where $x \in R^+$ and $p \in R^+$:

- x is the amount of bandwidth they agree to trade on (given from the seller to buyer).
- p is the price per unit of x (total of xp paid to the seller).

When a contract (x, p) is signed, the seller's profit or utility is defined by

$$U(x, p) = x(p - c),$$

where $c(> 0)$ is a predetermined constant cost which takes into account the operating cost of the seller. If none of the contract is accepted by the buyer, the reserved utility of the owner is defined by $U(0, 0) = 0$.

2.2 A Reference Market of Fixed/Deterministic Service or Exclusive Use

We next consider what a contract specified by the pair (x, p) means to a potential buyer. To see this, we will assume that there exists a traditional (as opposed to this emerging, secondary) market from where the buyer can purchase services with fixed or deterministic guarantees. What this means is that the buyer can purchase *exclusive* use of certain amount of bandwidth, which does not have to be shared with other (primary) users. This serves as an alternative to the buyer, and will be used in our model as a point of reference. We will leave it unspecified how the price of exclusive use is set, and will thus normalize it to be unit price per unit of bandwidth (or per unit of transmission rate). The idea is that given this alternative, the owner cannot arbitrarily set its price because the buyer can always walk away and purchase from this traditional market. This traditional market will also be referred to as the *reference* market, and the service it sells as the *fixed* or *deterministic* service/channel. Our model does allow a buyer to purchase from both markets should there be a benefit.

2.3 The Buyer's Consideration

When the set of M contracts are presented to a buyer, its choices are (1) to choose one of the contracts and abide by its terms, (2) to reject all contracts and go to the traditional market, and (3) to purchase a certain combination from both market. The buyer's goal is to minimize its purchasing cost as long as a certain quality constraint is satisfied. The framework presented here applies to any meaningful quality constraint; to make our discussion concrete, below we will focus on a loss constraint for concreteness.

Suppose the buyer chooses to purchase y unit of fixed service from the reference market together with a contract (x, p). Then its constraint on expected loss of transmission can be expressed as:

$$E[(q - y - xB)^{+}] \leq \epsilon \ ,$$

where

- q: the amount of data/traffic the buyer wishes to transmit.
- $B \in \{0, 1\}$: a binary random variable denoting the quality of the channel for this buyer. We will denote $b := P(B = 1)$.
- ϵ: a threshold on expected loss that's acceptable to the buyer.

Here we have adopted a simplifying assumption that the purchased channel (in the amount of x) is either available in the full amount or completely unavailable. More sophisticated models can be adopted here, by replacing xB with another random variable $X(x)$ denoting the random amount of data transmission the buyer can actually realize. This will not affect the framework presented here, but will alter the technical details that follow.

With this purchase $(y, (x, p))$, the buyer's cost is given by $y + xp$. The cost of the contract (x, p) to this buyer is given by the value of the following minimization problem:

$$C(x, p) = \underset{y}{\text{minimize}} \ \ y + px \tag{1}$$

$$\text{subject to} \ \ E[(q - y - xB)^{+}] \leq \epsilon \tag{2}$$

That is, to assess how much this contract actually costs him, the buyer has to consider how much additional fixed service he needs to purchase to fulfill his needs.

The buyer can always choose to not enter into any of the presented contracts and only purchase from the traditional market. In this case, its cost is given by the value of the following minimization problem:

$$C(0, 0) = \underset{y}{\text{minimize}} \ \ y$$

$$\text{subject to} \ \ E[(q - y)^{+}] \leq \epsilon$$

Since every term is deterministic, we immediately conclude that $C(0,0) = q - \epsilon$, which will be referred to as the *reserve price* of the buyer. Obviously if a contract's cost is higher than this price then there is no incentive for the buyer to enter into that contract.

2.4 Informational Constraints

We investigate the following two possible scenarios.

1. Information is symmetric:
   Under this assumption, the seller knows exactly the values q, b, ϵ of the buyer. The seller can thus extract all of the buyer's surplus (over the reserve price), resulting in $C(x, p) = C(0, 0)$ at the optimal contract point.
2. Information is asymmetric:
   Under this assumption, the seller can no longer exploit all of the buyer's surplus, resulting in a more complicated contract design process. We assume there are possibly K types of buyers, each having a different triple (q, b, ϵ). We further assume that the seller has a prior belief of the distribution of the buyer types; a buyer is of type i with probability r_i and has the triple (q_i, b_i, ϵ_i) as its private information. We will also assume that at most M different contracts are announced to the buyer.

3 Optimal Contract under Symmetric Information

In the symmetric information case, the seller can custom-design a contract for the buyer, subject to the constraint that it offers an incentive for the buyer to accept, referred to as the *individual rationality* (IR) constraint. In other words, the buyer (by accepting the contract) has to be able to achieve a cost no higher than the reserve price: $C(x, p) \leq C(0, 0) = q - \epsilon$. Knowing this, the seller can exactly determined the region where the buyer would accept a contract (x, p) since it knows the values q, ϵ, b.

Theorem 1. *When $q(1 - b) \leq \epsilon$, the buyer accepts the contract (x, p) if and only if*

$$p \leq \begin{cases} b & \text{if } x \leq \frac{q-\epsilon}{b} \\ \frac{q-\epsilon}{x} & \text{if } x > \frac{q-\epsilon}{b} \end{cases} \tag{3}$$

When $q(1 - b) > \epsilon$, the buyer accepts the contract if and only if

$$p \leq \begin{cases} b & \text{if } x \leq \frac{\epsilon}{1-b} \\ \frac{b\epsilon}{x(1-b)} & \text{if } x > \frac{\epsilon}{1-b} \end{cases} \tag{4}$$

The above result is illustrated in Fig. 1. The meaning of the two different types of regions are as follows. (i) When $q(1 - b) \leq \epsilon$, or $b \geq \frac{q-\epsilon}{q}$, the quality of the

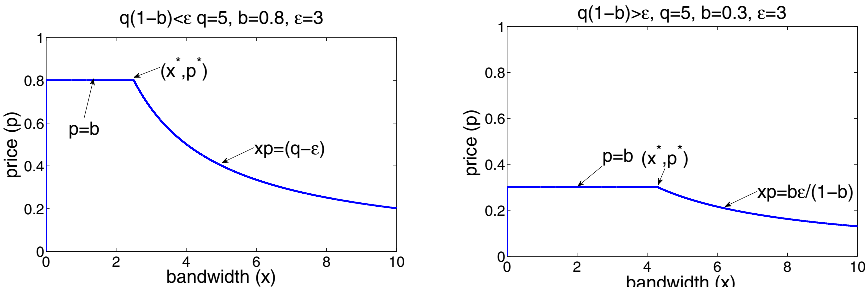

Fig. 1. Examples of $q(1-b) \le \epsilon$ (left) and $q(1-b) > \epsilon$ (right)

stochastic channel is sufficiently good such that, when x is large enough, the constraint Eqn. (2) can be achieved without any purchase of the deterministic channel (fixed service y). Thus, the buyer is willing to spend up to $C(0,0) = q-\epsilon$. (ii) When $q(1-b) > \epsilon$, or $b < \frac{q-\epsilon}{q}$, the quality of the stochastic channel is not so good that no matter how much is purchased, some deterministic channel (y) has to be purchased to satisfy the loss constraint. Thus, the buyer is not willing to spend all of $q-\epsilon$ on the contract. Below we prove the sufficient condition of the acceptable region when $q(1-b) \le \epsilon$; other parts of the above theorem can be done using similar arguments.

1. The buyer accepts the contract (x,p) if $x \le \frac{q-\epsilon}{b}$ and $p \le b$.

   Proof. We start by letting $y = q-\epsilon-xp$ and show that the IR constraint is satisfied:
   $$y + xp = q - \epsilon - xp + xp = q - \epsilon \le U(0,0) \ .$$
   The loss constraint is satisfied because,
   $$\begin{aligned} E[(q-y-xB)^+] &= (q-y-x)^+ b + (q-y)^+(1-b) \\ &= (\epsilon + xp - x)^+ b + (\epsilon + xp)(1-b) \\ &= \begin{cases} (\epsilon+xp)(1-b) \le (\epsilon + b\frac{q-\epsilon}{b})(1-b) \le \epsilon & \text{if } \epsilon + xp - x \le 0 \\ \epsilon + x(p-b) \le \epsilon & \text{if } \epsilon + xp - x > 0 \end{cases} \end{aligned}$$
   □

2. The buyer is willing to accept the contract (x,p) if $x \ge \frac{q-\epsilon}{b}$ and $xp \le U(0,0) = q-\epsilon$.

   Proof. The IR constraint is satisfied when the buyer does not purchase any y. We next examine whether the quality constraint is satisfied with $y=0$.
   $$\begin{aligned} E[(q-x)^+] &= (q-x)^+ b + q(1-b) \le (q - \frac{q-\epsilon}{b})^+ b + q(1-b) \\ &= (qb - (q-\epsilon))^+ + q(1-b) = (\epsilon - q(1-b))^+ + q(1-b) \\ &= (\epsilon - q(1-b)) + q(1-b)) = \epsilon \ , \end{aligned}$$
   where the second to last equality follows from the fact that $q(1-b) \le \epsilon$. □

After determining the feasible region of contracts for a given type (q, ϵ, b), the seller can choose any point in this region to maximize its utility. We next show that the optimal contract for the seller is determined by the intersection of the two boundary lines derived above, which we will denote as (x^*, p^*) throughout the rest of the paper. Here we assume that there exists a contract with $p > c$ such that the buyer will accept, for otherwise the seller has no incentive to sell the stochastic channel.

Theorem 2. *The optimal contract is the intersection point of the two lines:*

$$p^* = b \tag{5}$$

$$x^* p^* = \begin{cases} q - \epsilon \ \textit{if}\ q(1-b) \leq \epsilon \\ \frac{b\epsilon}{1-b} \ \textit{if}\ q(1-b) > \epsilon \end{cases} \tag{6}$$

Proof. From the form of the seller's utility $(U(x,p) = x(p-c))$, it can be easily verified that the profit is increasing in p. Using this property and the fact that we already determined the feasible contracts in Theorem 1, we can show that the contract pair (x, p) that generates the highest profit for the seller is the intersection point (x^*, p^*) (as illustrated in Figure 1). □

Once the seller determines the optimal contract and presents it to the buyer, the buyer chooses to accept because it satisfies the loss constraint and the IR constraint. It can be shown that the buyer's utility is exactly $C(0,0)$, as we expected.

The optimal contract for buyer of type (q, ϵ, b) defined in Theorem 2 can be written in a compact form in the following theorem.

Theorem 3. *The optimal contract* (x^*, p^*) *of a buyer type* (q, ϵ, b) *is given by* $(x^*, p^*) = (min(\frac{\epsilon}{1-b}, \frac{q-\epsilon}{b}), b)$.

Proof. By Theorem 2, when $q(1-b) \leq \epsilon$,

$$\frac{q-\epsilon}{b} \leq \frac{\frac{\epsilon}{1-b} - \epsilon}{b} = \frac{\epsilon}{1-b}$$
$$\Rightarrow \quad x^* = \frac{q-\epsilon}{b} = min(\frac{\epsilon}{1-b}, \frac{q-\epsilon}{b})$$

Similarly, when $q(1-b) > \epsilon$,

$$\frac{q-\epsilon}{b} > \frac{\frac{\epsilon}{1-b} - \epsilon}{b} = \frac{\epsilon}{1-b}$$
$$\Rightarrow \quad x^* = \frac{\epsilon}{1-b} = min(\frac{\epsilon}{1-b}, \frac{q-\epsilon}{b})$$

□

We now introduce the concept of an *equal-cost line* of a buyer, this concept will be used to find the optimal contract when there are more than one possible type of buyer. Consider a contract (x', p'). Denote by $P(x', p', x)$ a price such that the contract $(x, P(x', p'\ , x))$ has the same cost as contract (x', p') to a buyer. This will be referred to as an *equivalent price*. Obviously $P(x', p', x)$ is a function of x, x', and p'.

Definition 1. *The equal-cost line E of a buyer of type (q, ϵ, b) is the set of contracts within the buyer's acceptance region T that are of equal cost to the buyer. Thus $(x, p) \in E$ if and only if $p = P(x', p', x)$ for some other $(x', p') \in E$. The cost of this line is given by $C(x', p')$, $\forall (x', p') \in E$.*

It should be clear that there are many equal-cost lines, each with a different cost. Figure 2 shows an example of a set of equal-cost lines. We will therefore also write an equal-cost line as $E_{x',p'}$ for some (x', p') on the line to distinguish it from other equal-cost lines. The next theorem gives a precise expression for the equivalent price that characterizes an equal-cost line.

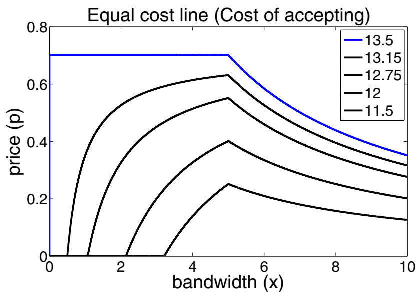

Fig. 2. Example of equal cost lines

Theorem 4. *For a buyer of type (q, ϵ, b) with an intersection point (x^*, p^*) on its acceptance region boundary, and given a contract (x', p'), an equal-cost line $E_{x',p'}$ consists of all contracts $(x, P(x', p', x))$ such that*

$$P(x', p', x) = \begin{cases} b - \frac{x'}{x}(b - p') & \text{if } x, x' \leq x^* \\ x'p'/x & \text{if } x, x' \geq x^* \\ (b(x^* - x') + x'p')/x & \text{if } x' < x^* < x \\ b - (x^* b - x'p')/x & \text{if } x < x^* < x' \end{cases}$$

Proof. We will prove this for the case $q(1 - b) \leq \epsilon$; the other case can be shown with similar arguments and is thus omitted for brevity. In this case $x^* = \frac{q-\epsilon}{b}$. When $x, x' \leq x^*$, without buying deterministic service the loss is given by

$$\begin{aligned} E[(q - xB)^+] &= (q - x)^+ b + q(1 - b) \\ &= (q - x)b + q(1 - b) = q - xb \geq \epsilon, \end{aligned}$$

where the second equality is due to the fact that $q(1-b) \le \epsilon \Rightarrow \frac{q-\epsilon}{b} \le q \Rightarrow x \le \frac{q-\epsilon}{b} \le q$. The incentive for the buyer is to purchase y such that the loss is just equal to ϵ.

$$\begin{aligned} E[(q-y-xB)^+] &= (q-y-x)b + (q-y)(1-b) \\ &= q-y-xb = \epsilon \ . \end{aligned}$$

The first equality follows from the fact that $q(1-b) \le \epsilon$, which implies both $(q-y-x) \ge 0$ and $(q-y) \ge 0$. This is true for both (x,p) and (x',p'). Since (x,p) is on the equal cost line $E_{x',p'}$, we know that $C(x,p) = C(x',p')$. We also know that $C(x,p) = y + xp$ and $C(x',p') = y' + x'p'$,

$$C(x,p) = q - \epsilon - xb + xp = q - \epsilon - x'b + x'p' = C(x',p') \ .$$

Rearranging the second equality such that p is a function of x, x', p' immediately gives the result. When $x, x' > x^*$, x (x') alone is sufficient to achieve the loss constraint. For $C(x,p) = C(x',p')$ we must have $x'p' = xp$, resulting in the second branch. The third and fourth branch can be directly derived from the first two branches. When $x > x^* > x'$ $(x' > x^* < x)$, we first find the equivalent price at x^* by the first branch (second branch), and then use the second branch (first branch) to find $P(x',p',x)$. This gives the third branch (fourth branch) □

Note that every contract below an equal-cost line is strictly preferable to a contract on the line for the buyer.

4 Contract under Asymmetric Information

We now turn to the case where parameters (q,b,ϵ) are private information of the buyer. The seller no longer knows the exact type of the buyer but only what types are out there and their distribution; consequently it has to guess the buyer's type and design the contract in a way that maximizes its expected payoff. In order to do so, the seller can design a specific contract for each type so that the buyers will reveal their true types. Specifically, when the buyer distributes a set of contracts $\mathbb{C} = \{(x_1,p_1),(x_2,p_2)......(x_K,p_K)\}$ specially designed for each of the K types, a buyer of type i will select (x_i,p_i) only if the following set of equations is satisfied:

$$C_i(x_i,p_i) \le C_i(x_j,p_j) \ \forall j \ne i \ ,$$

where C_i denotes the cost of a type i buyer. In other words, the contract designed for one specific type of buyer, must be as good as any other contract from the buyer's point of view. Let $R_i(\mathbb{C})$ denote the contract that a type i buyer will select given a set of contract $\mathbb{C}$. Then,

$$R_i(\mathbb{C}) = \underset{(x,p)\in\mathbb{C}}{\operatorname{argmin}} \ C_i(x,p) \ .$$

Given a set of contracts $\mathbb{C}$, we can now express the seller's expected utility as

$$E[U(\mathbb{C})] := \sum_i U(R_i(\mathbb{C}))r_i$$

where r_i is the *a priori* probability that the buyer is of type i. We further denote the set $T_i = \{(x,p) : C_i(x,p) \le C_i(0,0)\}$ as the set of all feasible contracts for type i buyer (feasible region in Theorem 1). The optimal contract (Theorem 2) designed for the type-i buyer, will also be called max_i:

$$\begin{aligned} max_i &:= (x_i^*, p_i^*) \\ &:= \underset{(x,p)\in T_i}{\operatorname{argmax}} \; U(x,p) \end{aligned}$$

4.1 Two Types of Buyer, K=2

We first consider the case when there are only two possible types of buyer $(q_i, \epsilon_i, b_i), i \in \{1,2\}$, with probability r_i, $r_1 + r_2 = 1$.

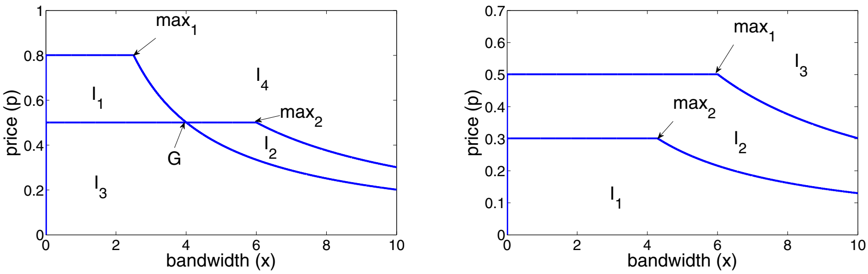

Fig. 3. Example when $max_1 \notin T_2$ and $max_2 \notin T_1$ (left), $max_1 \in T_2$ or $max_2 \in T_1$ (right)

***M* = 1.** We first consider the case when the seller hands out only one contract.

Theorem 5. *The optimal contract is as follows,*

– *if* $max_1 \notin T_2$ *and* $max_2 \notin T_1$,

$$optimal = \begin{cases} max_1 \text{ if } & r_1U(max_1) \ge r_2U(max_2) \text{ and } r_1U(max_1) \ge U(G) \\ max_2 \text{ if } & r_2U(max_2) \ge r_1U(max_1) \text{ and } r_2U(max_2) \ge U(G) \\ \quad G \quad \text{ if } & U(G) \ge r_2U(max_2) \text{ and } U(G) \ge r_1U(max_1) \end{cases}$$

– *if* $max_1 \in T_2$.

$$optimal = \begin{cases} max_1 \text{ if } & U(max_1) \ge r_2U(max_2) \\ max_2 \text{ if } & r_2U(max_2) \ge U(max_1) \end{cases}$$

– *if* $max_2 \in T_1$.

$$optimal = \begin{cases} max_2 \ \textit{if}\ U(max_2) \geq r_1 U(max_1) \\ max_1 \ \textit{if}\ r_1 U(max_1) \geq U(max_2) \end{cases}$$

When $max_1 \notin T_2$ and $max_2 \notin T_1$, we denote the intersecting point of the two boundaries (of the accepting region of the two types) as G (see Figure 3 (left)). Theorem 5 can be proved by showing that the payoffs of contracts in a particular region are no greater than special points such as G. For example, in the case of $max_1 \notin T_2$ and $max_2 \notin T_1$ any point in I_3 is suboptimal to point G because they are both acceptable by both types of buyers and G has a strictly higher profit than any other point in I_3.

$M = 2$, $max_1 \notin T_2$ and $max_2 \notin T_1$. The seller can hand out at most two contracts for the buyer to choose from. We will see that providing multiple contracts can help the seller obtain higher profits.

Theorem 6. *The set* $\{max_1, max_2\}$ *is the optimal set of contracts.*

Proof. The set $\mathbb{C} = \{max_1, max_2\}$ gives an expected payoff of

$$E[U(\mathbb{C})] = r_1 U(R_1(\mathbb{C})) + r_2 U(R_2(\mathbb{C}))) = r_1 U(R_1(max_1)) + r_2 U(R_2(max_2))$$

The last equality holds because $max_1 \notin T_2$ and $max_2 \notin T_1$ and both types choose the max_i intended for them. If $\mathbb{C}$ is not the optimal set, then there must exist some contract set $\mathbb{C}' = \{(x_1, p_1), (x_2, p_2)\}$ such that

$$\begin{aligned} E[U((\mathbb{C}'))] &= r_1 U(R_1(x_1, p_1)) + r_2 U(R_2(x_2, p_2)) \\ &> E[U(\mathbb{C})] = r_1 U(R_1(max_1)) + r_2 U(R_2(max_2)) \end{aligned}$$

This has to mean either $U(R_1(x_1, p_1)) > U(R_1(max_1))$, or $U(R_2(x_2, p_2)) > U(R_2(max_2))$, or both, all of which contradict the definition of max_i. Thus, $\{max_1, max_2\}$ is the optimal contract set. □

$M = 2$, $max_1 \in T_2$ or $max_2 \in T_1$. The seller can hand out at most two contracts.

Obviously, the seller cannot hand out the same contract $\mathbb{C} = \{max_1, max_2\}$ as in the previous section and claim that it is optimal. Without loss of genrality, we will assume that the type-1 buyer has a smaller b_1 ($b_1 \leq b_2$), thus, we are considering the $max_1 \in T_2$ case. We will first determine the optimal contract when $x_1^* \leq x_2^*$, the optimal contract when $x_1^* > x_2^*$ can be determined based on the results of the first case. To find the optimal contract set, we consider only the contract pairs $\{(x_1, p_1), (x_2, p_2)\}$ where each type-i buyer pick (x_i, p_i) instead of the other one. It is quite simple to show that we do not lose optimality by restricting to this type of contract sets.

To find the optimal contract, we will 1) first show that for each (x_1, p_1) we can express the optimal (x_2, p_2) in terms of x_1 and p_1; 2) then we will show that (x_1, p_1) must be on the boundary of T_1 with $x_1 \leq x_1^*$; 3) using 1) and 2) we can calculate the expected profit by a simpler optimization problem.

Lemma 1. *In the $K = 2$ case, if $max_1 \in T_2$ and $x_1^* \leq x_2^*$. Given a contract for type-1 (x_1, p_1), the optimal contract for type-2 must be $(x_2^*, P_2(x_1, p_1, x_2^*))$.*

Proof. Given a contract (x_1, p_1), the feasible region for the contract of type-2 buyer is the area below $P_2(x_1, p_1, x)$ as defined in Theorem 4 (see Figure 4). By noticing that the form of the seller's profit is increasing in both p and x $(U(x, p) = x(p - c))$, the contract that generates the highest profit will be such that $x_2 = x_2^*$ and $p_2 =, P_2(x_1, p_1, x_2^*)$. □

Lemma 2. *In the $K = 2$ case, if $max_1 \in T_2$ and $x_1^* \leq x_2^*$. An optimal contract for type-1 must be $p_1 = b_1$ and $x_1 \leq x_1^*$.*

Proof. Lemma 2 can be proved in two steps. First we assume the optimal contract has $(x_1, p_1) \in T_1$, where we can increase p_1 by some positive $\delta > 0$ but still have $(x_1, p_1 + \delta) \in T_1$. By noticing that both $U(x, p)$ and $P(x, p, x')$ are increasing in p. We know that both $U(x_1, p_1 + \delta)$ and $U(x_2^*, P_2(x_1, p_1 + \delta, x_2^*)))$ are strictly larger than $U(x_1, p_1)$ and $U(x_2^*, P_2(x_1, p_1, x_2^*)))$. This contradicts the assumption that it was optimal before, thus, we know that the optimal contract for (x_1, p_1) must be on the two lines (the upper boundary of T_1) defined in Theorem 2. Then we can exclude the possibility of having (x_1, p_1) on the boundary of T_1 with $x_1 > x_1^*$ by comparing the contract (x_1^*, b_1) with such a contract. □

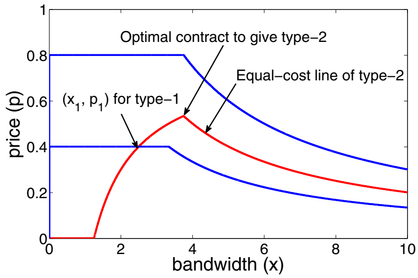

Fig. 4. The regions to distinguish type-2 given (x_1, p_1)

By putting the constraints from Lemmas 1, 2 and using Theorem 4, the expected profit can be expressed as follows.

$$\begin{aligned}
E[U(\mathbb{C})] &= r_1 U(x_1, p_1) + r_2 U(x_2, p_2) \\
&= r_1 U(x_1, p_1) + r_2 U(x_2, P_2(x_1, p_1, x_2^*)) \\
&= r_1 U(x_1, b_1) + r_2 U(x_2^*, b_2 - \frac{x_1}{x_2^*}(b_2 - b_1)) \\
&= r_1 x_1 (b_1 - c) + r_2 x_2^* (b_2 - \frac{x_1}{x_2^*}(b_2 - b_1) - c) \\
\frac{\partial E[U(\mathbb{C})]}{\partial x_1} &= r_1(b_1 - c) - r_2(b_2 - b_1)
\end{aligned}$$

The x_1 acheiving the optimal contract $\mathbb{C}$ is given by,

$$x_1 = \begin{cases} 0 & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) < 0 \\ x_1^* & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) > 0 \end{cases}$$

$$\mathbb{C} = \begin{cases} max_2 & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) < 0 \\ max_1,\ (x_2^*, b_2 - \frac{x_1^*}{x_2^*}(b_2 - b_1)) & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) > 0 \end{cases}$$

This result shows two different conditions: 1) When $\frac{r_1}{r_2} < \frac{b_2 - b_1}{b_1 - c}$, type-2 is more profitable and the seller will distribute max_2. If the seller chooses to distribute max_2, there is no way to distribute another contract for type-1 without affecting the behavior of type-2. Consequently, the seller only distributes one contract. 2) When $\frac{r_1}{r_2} > \frac{b_2 - b_1}{b_1 - c}$, type-1 is more profitable and the seller will distribute max_1. After choosing max_1, the seller can also choose $(x_2^*, b_2 - \frac{x_1^*}{x_2^*}(b_2 - b_1))$ for the type-2 buyer without affecting the type-1 buyer's choice. As a result, the seller distributes a pair of contracts to get the most profit.

With a very similar argument, the optimal contract for $x_1^* > x_2^*$ can be determined. Again, we can prove that the optimal contract must have $p_1 = b_1$ and $x_1 \leq x_1^*$. The difference is that when $x_1^* > x_2^*$, the expression for $(x_2^*, P_2(x_1, p_1, x_2^*))$ has two cases depending on whether $x_1 > x_2^*$ or $x_1 \leq x_2^*$.

$$E[U(\mathbb{C})] = \begin{cases} r_1 U(x_1, b_1) + r_2 U(x_2^*, b_2 - \frac{x_1}{x_2^*}(b_2 - b_1)) & \text{if } x_1 \leq x_2^* \\ r_1 U(x_1, b_1) + r_2 U(x_2^*, \frac{x_1 b_1}{x_2^*}) & \text{if } x_1 > x_2^* \end{cases}$$

$$\frac{\partial E[U(\mathbb{C})]}{\partial x_1} = \begin{cases} r_1(b_1 - c) - r_2(b_2 - b_1) & \text{if } x_1 \leq x_2^* \\ r_1(b_1 - c) + r_2 b_1 & \text{if } x_1 > x_2^* \end{cases}$$

To summarize, when $r_1(b_1 - c) - r_2(b_2 - b_1) > 0$, $E[R(\mathbb{C})]$ is strictly increasing in x_1 and we know that $x_1 = x_1^*$ maximizes the expected profit. When $r_1(b_1 - c) - r_2(b_2 - b_1) < 0$, $E[R(\mathbb{C})]$ is decreasing in x_1 if $x_1 \in [0, x_2^*]$ and increasing in x_1 if $x_1 \in [x_2^*, x_1^*]$. We can only conclude that either $x_1 = 0$ or $x_1 = x_1^*$ maximizes the expected profit.

$$x_1 = \begin{cases} 0 \text{ or } x_1^* & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) < 0 \\ x_1^* & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) > 0 \end{cases}$$

$$\mathbb{C} = \begin{cases} max_2 \text{ or } \{max_1, (x_2^*, \frac{x_1^* b_1}{x_2^*})\} & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) < 0 \\ \{max_1, (x_2^*, \frac{x_1^* b_1}{x_2^*})\} & \text{if } r_1(b_1 - c) - r_2(b_2 - b_1) > 0 \end{cases}$$

In the first condition, we can calculate the expected profit of the two contract sets and pick the one with the higher profit.

4.2 K Types of Buyer, $K \geq 2$, Common b_i

In this section we consider the case when different types share the same channel condition $b_i = b, \forall i = 1, \cdots, K$, which is also known to the seller. This models the case where the condition is largely determined by the seller's primary user traffic. An example of the acceptance regions of three buyer types are shown in Figure 5. We will assume that the indexing of the buyer is in the increasing order of x_i^*; this can always be done by relabeling the buyer indices. There are two possible cases: (1) the seller can announce as many contracts as it likes, i.e., $M = K$ (note that there is no point in designing more contracts than there are types); (2) the seller is limited to at most $M < K$ contracts. In the results presented below we fully characterize the optimal contract set in both cases.

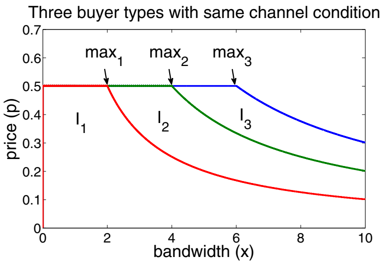

Fig. 5. Three buyer types with common b

Theorem 7. *When $M = K$ and $\forall b_i = b$, the contract set that maximizes the seller's profit is $(max_1, max_2, ..., max_K)$.*

This result holds for the following reason. As shown in Figure 5, with a constant b, the intersection points (max_i) of all acceptance regions are on the same line $p = b$. For a buyer of type i, all points to the left of max_i on this line cost the same as max_i, and all points to its right are outside the buyer's acceptance region. Therefore the type-i buyer will select the contract max_i given this contract set. Since this is the best the seller can do with a type-i buyer (see Theorem 4) this set is optimal for the seller. (see proof of Theorem 6)

Lemma 3. *When $M < K$ and $\forall b_i = b$, the optimal contract set is a subset of $(max_1, ..., max_K)$.*

Proof. Assume the optimal contract $\mathbb{C}$ is not a subset of $(max_1, ..., max_K)$. Then it must consists of some contract points from at least one of the I_i regions as demonstrated in Figure 5. Let these contracts be $A_i \subset I_i$ and $\bigcup_i A_i = \mathbb{C}$. For each non-empty A_i, we replace it by the contract max_i and call this new contract set $\mathbb{C}'$. The proof is to show that this contract set generates profit at least as large as the original one. For each type-i buyer that picked some contract $(x, p) \in A_j$ from the optimal contract $\mathbb{C}$, it must had a type greater than or equal to j otherwise (x, p) is not in its acceptance region. In the contract set $\mathbb{C}'$, type-i will now pick max_j or max_l with $l > j$. The choice of each possible type of buyer picks from $\mathbb{C}'$ is at least as profitable as the one they picked from $\mathbb{C}$. Thus, the expected profit of $\mathbb{C}'$ is at least as good as $\mathbb{C}$. □

The above lemma suggests the following iterative way of finding the optimal contract set.

Definition 2. *Define function* $g(m,i)$ *as the the maximum expected profit for the seller by picking contract* max_i *and selecting optimally* $m-1$ *contracts from the set* $(max_{i+1},...,max_K)$.

Note that if we include max_i and max_j $(i<j)$ in the contract set but nothing else in between i and j, then a buyer of type l $(i \le l < j)$ will pick contract max_i. These types contribute to an expected profit of $x_i^*(b-c)\sum_{l=i}^{j-1} r_l$. At the same time, no types below i will select max_i (as it is outside their acceptance regions), and no types at or above j will select max_i (as for them max_j is preferable).

Thus the function $g(m,i)$ can be recursively obtained as follows:

$$g(m,i) = \max_{j:i<j\le K-m+2} g(m-1,j) + x_i^*(b-c)\sum_{l=i}^{j-1} r_l,$$

with the boundary condition $g(1,i) = x_i^*(b-c)\sum_{l=i}^{K} r_l$.

Finally, it should be clear that the maximum expected profit for the seller is given by $\max_{1\le i\le K} g(M,i)$, and the optimal contract set can be determined by going backwards: first determine $i_M^* = \arg\max_{1\le i\le K} g(M,i)$, then $i_{M-1}^* = \arg\max_{1\le i\le K-1} g(M-1,i)$, and so on.

Theorem 8. *The set of contracts* $\{max_{i_1^*}, max_{i_2^*}, \cdots, max_{i_M^*}\}$ *obtained using the above procedure is optimal and its expected profit is given by* $g(M, i_M^*)$.

5 Conclusion

In this paper we considered a contract design problem where a primary license holder wishes to profit from its excess spectrum capacity by selling it to potential secondary users/buyers via designing a set of profitable contracts. We considered two cases. Under symmetric information, we found the optimal contract that achieves maximum profit for the primary user. Under asymmetric information, we found the optimal contract if the buyer belongs to one of two types. When there are more than two types we restricted our attention to the case where the channel condition is common to all types, and presented an optimal procedure to design the contracts.

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