

Numerical Investigation of the Primety of Real Numbers

Kristoffer Jensen

ad:mt, Aalborg University Esbjerg
Niels Bohrsvej 8, 6700 Esbjerg, Denmark
krist@create.aau.dk

Abstract. The Farey sequences can be used [1] to create the Eulers totient function $\phi(n)$, by identifying the fractions for number n that did not occur in all Farey sequences up to $n-1$. This function creates, when divided by $n-1$, what is here called the Primety measure, which is a measure of how close to being a prime number n is. $P(n)=\phi(n)/(n-1)$ has maximum 1 for all prime numbers and minimum that decreases non-uniformly with n . Thus $P(n)$ is the Primety function, which permits to designate a value of Primety of a number n . If $P(n)=1$, then n is a prime. If $P(n)<1$, n is not a prime, and the further $P(n)$ is from n , the less n is a prime. $\phi(n)$ and $P(n)$ is generalized to real numbers through the use of real numbered Farey sequences. The corresponding numerical sequences are shown to have interesting mathematical and artistic properties.

Keywords: Farey sequences, Totient function, Primety, Selfsimilarity, Fractals.

1 Introduction

The quest for (large) prime numbers is one of the important problems in mathematics, with repercussions into fundamental mathematics as well as contemporary society, with the use of prime numbers in coding and security systems. With the unpredictability of prime numbers, only brute force methods are guaranteed to render a result as to if a number is prime or not. Many mathematicians will find the mathematics involved in prime numbers beautiful, or even artistic in its own right. [2] states it *Mathematics, as I have been describing it, is an art form. The words ambiguity and metaphor are much more acceptable in the arts than they are in the sciences. But ambiguity and metaphor are the mechanisms through which that ultimate ambiguity, the one that divides the objective from the subjective, the natural world from the mind, is bridged.*

In addition to art in mathematical problem solving, the numbers can represent art in itself. Certainly, the Golden Ratio, and the accompanying Fibonacci numbers are found in many art works [3]. Another mathematic area of interest with artistic outcomes is fractals. Fractals are objects that are self-similar (i.e. details looks like the whole). Fractals are divided into exact or statistic self-similar objects, full or partial fractals, natural and mathematic fractals, etc. A common natural fractal is the coastline that was used as an example by the ‘father’ of fractal geometry, Benoit

Mandelbrot [5]. Common examples of mathematic fractals are the Cantor set, the Koch curve, and the Mandelbrot set. Fractals have two possible definitions [6], both proposed by Mandelbrot, the first saying that the Hausdorff dimension strictly exceeds the topological dimensions, and the second saying that the fractal is a shape made of parts similar to the whole in some way. Often, fractals are created using some kind of feedback mechanism, in which the input of the next step is a function including the output of the current step.

Larry Austin composed several music pieces based on natural fractal shapes [7], including Maroon Bells (1976), based in part on a mountain range, and Canadian Coastlines (1981), based on actual coastlines.

This work proposes to investigate the Farey sequence, in particular in the case of real numbers, and show some interesting and aesthetic outcomes of these sequences.

2 Mathematical Development

The Farey sequence [1] for a number n consists of $[0/n, 1/1, 1/2, 1/3, 2/3, \dots, 1/n, \dots, n-1/n]$. For $n=0$, it is the empty set $[\]$, for $n=1$, it is $[0, 1]$, for $n=2$, it is $[0, 1/2, 1]$, for $n=3$ it is $[0, 1/3, 1/2, 2/3, 1]$, etc. If all unique elements up to $n-1$ are retained, then for $n=1, 2$ there is one new element and for $n=3$ there are two new elements. For $n=4$, there are 2 new elements $[1/4, 3/4]$, etc. The number of new element is $[1, 1, 2, 2, 4, 2, 6, 4, 6, 4, \dots]$ and it corresponds [1] to the Euler totient function $\varphi(n)$. $\varphi(n)$ has the property [2] that $\varphi(p)=(p-1)$ for p primes, and that $\varphi(n) \leq (n-1)$, equal only in the case of n prime.

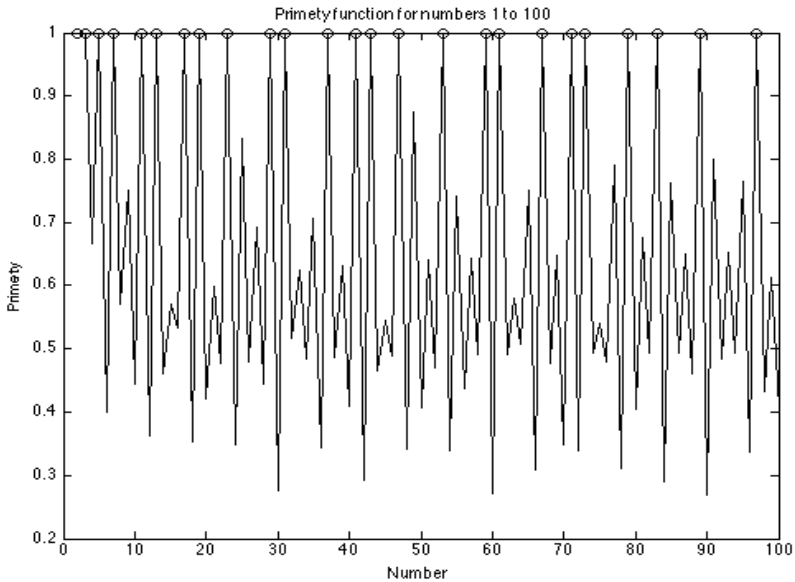


Fig. 1. The Primety for numbers 1 to 100

Therefore,

$$P(n) = \frac{\phi(n)}{n-1} \tag{1}$$

is here set as the Primety function, i.e. a numerical indication of how close to being prime a number is. The Primety values are shown in figure 1 for the numbers 1 to 100. The Prime numbers, for which $P(n)=1$, are denoted with a ring.

Now, if we use real numbers in the construction of the Farey sequence, $[0/r, 1/r, \dots, r/r, \dots, 1/(r+\delta), \dots, (r)/(r+\delta), \dots]$, and then find all new unique rations for each incremental δ , it is possible to show that the Primety function for real numbers,

$$P(r) = \frac{\phi(r)}{r-1} \tag{2}$$

where $\phi(r)$ is calculated as the new unique element in the Farey sequence for increasing real numbers (r). $P(r)$ follows a simple rule, as is shown in figure 2.

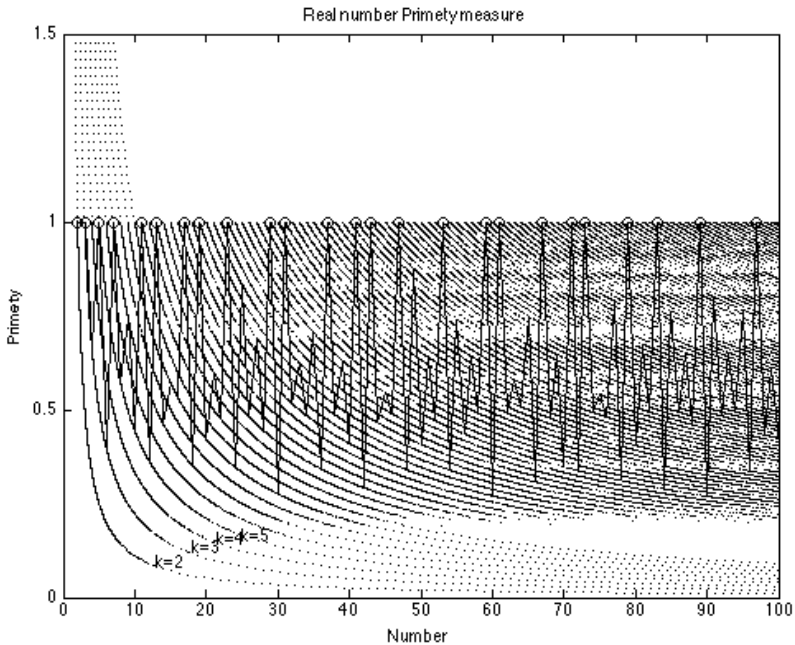


Fig. 2. The Primety values for real numbers between 1 and 100. The integer numbers are shown in the figure, and the primes are denoted with 'o'. The eq. (2) is shown, for low numbers ($2 \leq k \leq 10$), in dashed.

The Primety values are always found on one of the curves;

$$c_k = \frac{k}{x}, \tag{3}$$

where k is an integer, and $x > 0$ is an arbitrary value. $P(r)$ jumps between the different c_k , as r is increasing, so it is not possible to predict $P(r)$ from it. Nonetheless, this seems like a promising area of further research. For instance, the eq. (3) gives an absolute minimum for the value of $P(r)$. The maximum value seems to be equal to one for all values of x , with the exceptions of $r=1$. For $r < 1$, $O(r)=0$. All integer values approaches full Prime number Primety value; $P(r) \rightarrow 1$ as $r \rightarrow \infty$. It is not clear how this relates to the fact that many of these integer numbers has Primety values less than 1.

This is investigated further by transforming $P(r)$, so that the shape of it, according to eq. (3) is visible. By identifying the values of $P(r)$ that correspond to each c_k , and then multiply the found values by x a constant function is obtained. The result is shown in figure 3. It is clear that $P(r') \cdot r'$, where r' corresponds to the values where $P(r) = c_k$, is a constant function equal to k . The min and max index values shown in the right of figure 3 indicates that the minimum index is equal to k , and the maximum index is fluctuating slowly.

The first index corresponds to $P(r) \rightarrow 1$ as $r \rightarrow \infty$. The last index fluctuates slowly, in a non-predictable way, it seems. The Primety function for real numbers has the appearance to bring new findings in the quest for the understanding of prime numbers, but more work is necessary in order to assert this further. Hopefully this will allow more inspiring and artistically ambiguous mathematic development and further the art of mathematics and the mathematic art. Some initial investigations of the latter are made in the next section.

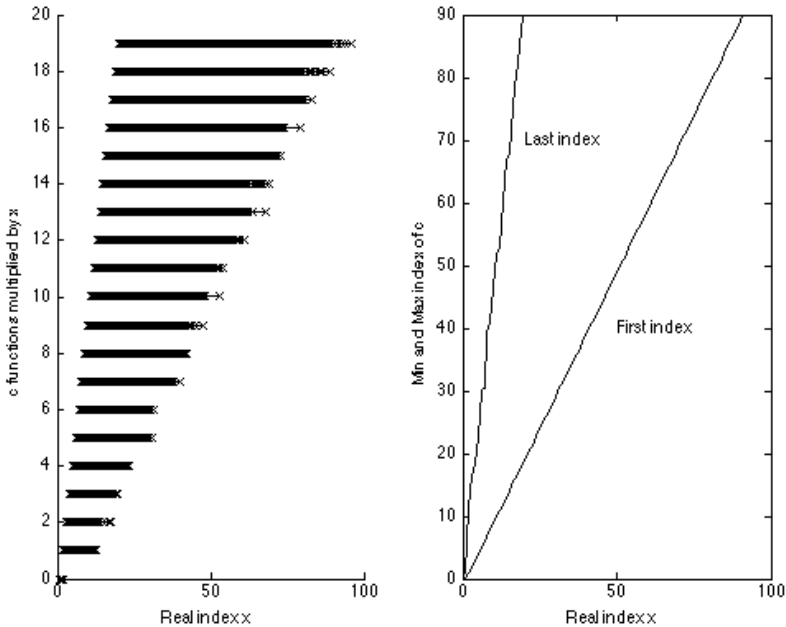


Fig. 3. Transformed real value Primety function (left) with corresponding min and max index values (right)

3 Fractal Behaviour

This section will show some of the fractal behaviours of the Primety function, and the related Farey sequence. First, we calculate the Primety for all numbers up to a large number, and show the scatterplot, i.e. the following Primety value as a function of the current value, as seen in figure 4.

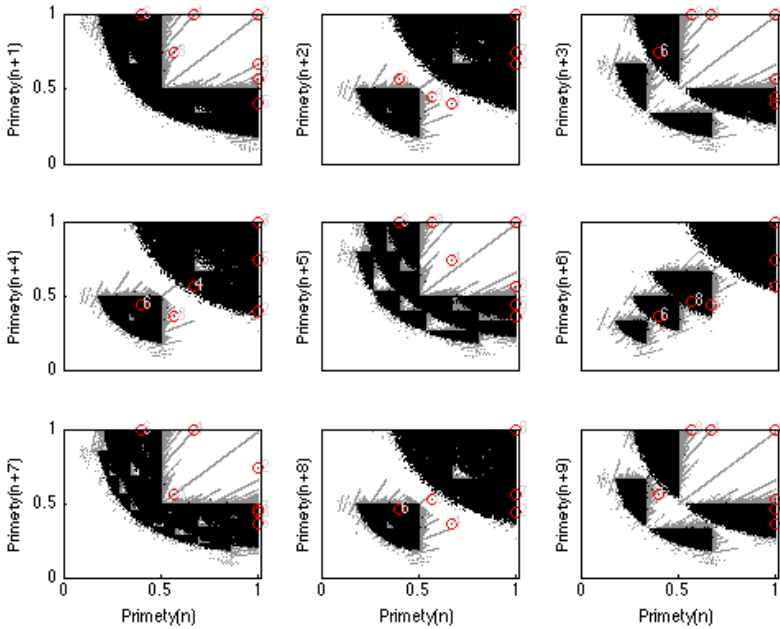


Fig. 4. Next Primety values as function of current Primety value for integers (black) and real numbers (grey). Context going from next Primety (upper left corner) to nine values after (lower right corner). The first 8 integer values are denoted with a ring, and the corresponding value of n .

Some initial conclusions can be made from figure 4. First, except for very low numbers, all integer Primety values are found above the half-circle - $P(n)-1)^2+(P(n+1)-1)^2=z$. Furthermore, except for a few low numbers, all high Primety values - $P(n)>0.5$ gives $P(n+1)<0.5$ and vice-versa. There is a systematic shift from high to low values. Thus, all even numbered Primety values are found below 0.5 and all odd numbered values are found above 0.5. Furthermore, a majority of $P(n)$ are found at or close to small integer fractions. Inside, the hole visible above $P(n)=1/3$, $P(n+1)=2/3$, a shape very similar to the full plot is visible. If it is a fractal shape, it is a very slow one, i.e. the fractals only show up in detail after many numbers. Each scatterplot reveals interesting and informative aspects of the Primety values.

For the real numbered Primety values, $P(r)$, even though the scatterplots place the real Primety values in the vicinity of the integer values, the situation is different, as it

seems the scatterplot reveals lines; $P(n+k)=(l/m)*P(n)$, where l and m are low integers. For instance, (for the next value scatterplot), for the line leading to the '2', $l=m=1$, for '3' $l=2, m=3$, for '4', $l=3, m=2$, etc.

More fractal behaviour is coming from the Farey sequences. The difference signal of the integer Farey sequence has a characteristic shape, which is independent on the size of the Farey sequence. Furthermore, this same shape is exactly replicated inside the larger shape (assuming n is large). These findings are shown in figure 5 for three different sizes, $n=100$ (left), $n=1000$, and $n=10000$ (right). All three sizes have the same shape, albeit the small sequence (left) is coarser. The peaks of the differences are found on small integer ratios, ($1/2, 1/3, 3/8, 2/5, 5/12$, etc)

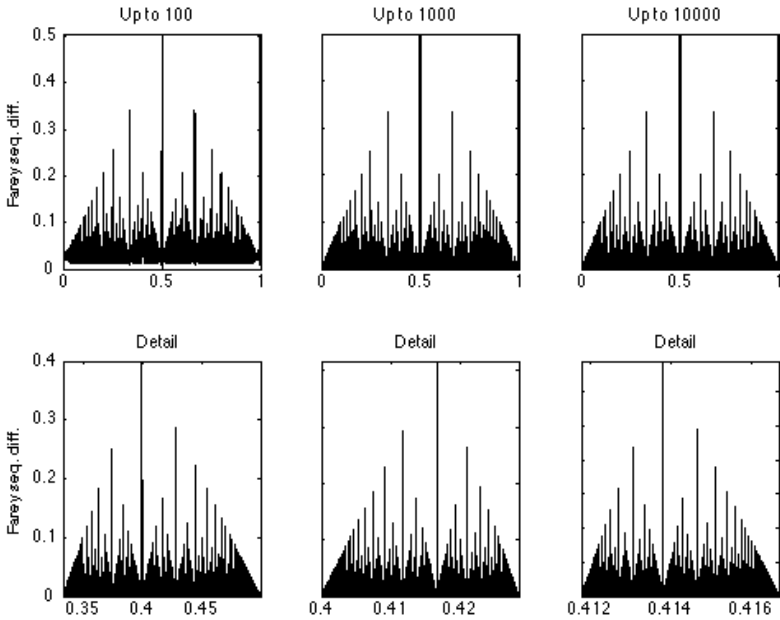


Fig. 5. Difference of Farey sequence for three sizes (top), and three different details of the large size (bottom). The different sizes show how the Farey difference signal always retains the same shape, and the different details show how the outer shape is copied in the details.

Figure 5 show the fractal behaviour of the Farey difference signal, both for different sizes (top), and for different details (bottom), and for different details (bottom).

In addition to this, the real signal Primety function $P(r)$ exposes an interesting behaviour. In figure 6, the real values Primety function is show for $0 < r < 10$, along with the first to third difference signals (top) and spectrograms (bottom). It is interesting to observe how the difference signals look similar to the Primety signal itself, although with larger values for each difference, and also clearly changing shape at each integer value for higher difference signals. Treated as an audio signal, this function exposes a characteristic toned and rhythmic sound that eventually blends into a noisy signal. The sound is very similar across the Primety difference signals.

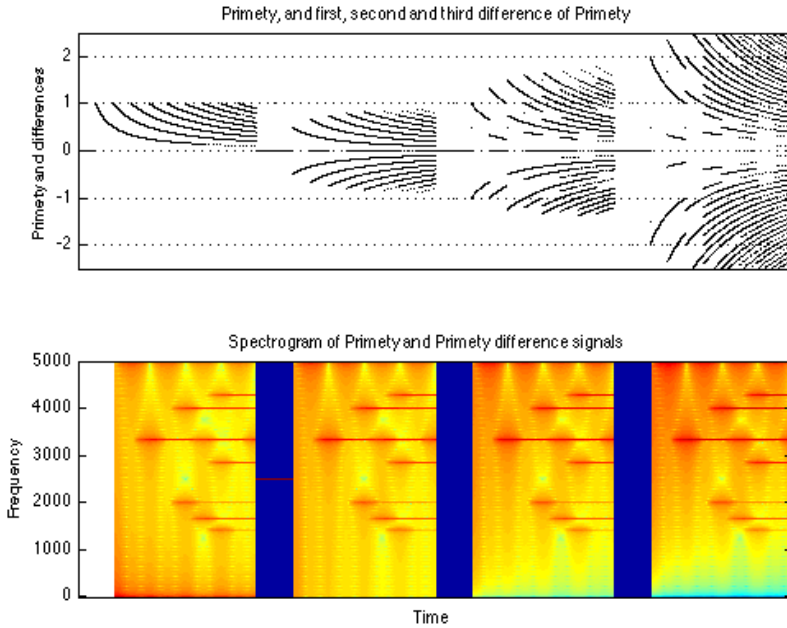


Fig. 6. Real value Primety function and first three difference signals (top), and spectrograms (bottom)

It is certain that the nature contains vast amount of at least statistic self-similarities fractals [5]. Many artists have, consciously or not, integrated fractal shapes in their art. Among audio signals to potentially be used in music, as shown above, fractals have a long tradition for quasi-artistic qualities. While the figures shown here are not necessary polished enough, or do not contain the aesthetic or otherwise qualities to be used in artwork, it is still possibly to envision such a use.

4 Conclusions

The Farey sequence of real numbers, along with the introduction of the Primety value permits to expose a certain number of laws that governs prime numbers. Among these are that the Primety values are always found on functions of the shape $c_k = k/x$, which enables to assert a (low) lowest possible value of the Primety function. Furthermore, all even Primety values are less than one half, and all odd are more than one half.

Three interesting visual results from this research are; 1) the k/x shape of the Primety function, 2) the almost fractal scatterplot of the Primety function, and the complete fractal shape of the Farey difference signal. Some interesting visual and sonic results are obtained from these signals.

Fractals and related mathematic results, as presented here, are placed in a double art context, first, because the quest for the mathematic results leads to ambiguity, *one of art's most potent aesthetic functions* [2], secondly, because the end result, graphics, plots, music, represents art in itself.

References

1. Sylvester, J.J.: On the number of fractions contained in any 'Farey series' of which the limiting number is given. *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science* 15(5), 251–257 (1883)
2. Byers, W.: *How mathematicians think: using ambiguity, contradiction, and paradox to create mathematics*. Princeton University Press (2007)
3. Posamentier, A.S., Lehmann, I.: *The fabulous Fibonacci numbers*. Prometheus Books (2007)
4. Graham, R.L., Knuth, D.E., Patashnik, O.: *Concrete Mathematics*, p. 147. Addison-Wesley (1989)
5. Mandelbrot, B.B.: How long is the coast of Britain? Statistical self-similarity and fractional dimension. *Science* 156, 636–638 (1967)
6. Addison, P.S.: *Fractals and chaos: an illustrated course*. The Institute of Physics, London (1997)
7. Dodge, C., Jerse, T.A.: *Computer Music: Synthesis, Composition and Performance*, 2nd edn. Macmillan Library Reference (1997)