Stability and Efficiency in the General-Priority-Based Assignment
(Extended Abstract)

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Abstract. We introduce a general class of priority orders over sets, which captures both indifferences and substitutability. Our notion of substitutability ensures the existence of stable assignment. The characterization of efficient priority structures implies that there is usually a conflict between efficiency and stability. Thus we turn to the problem of finding a constrained efficient assignment, and give an algorithm which solves the problem for any priority structure that falls into our class. As an important application, gender equality or racial equality in school choice can be captured by our model, but not previous models in the literature.

Keywords: Stability, Efficiency, Generalized deferred acceptance algorithm, Substitutability, Indifferences, Stable agent improving cycle, Equal treatment of equal agents, Acyclicity.

Overview

We develop a general-priority-based assignment problem by introducing a general notion of *substitutability* on priority orders which captures indifferences and substitutability in the previous literature. Indifferences or ties are fairly common in practice, but only recently has received attention in the two-sided matching literature. See for example, Abdulkadiroğlu, Pathak and Roth (2009), Ehlers and Erdil (2010), and Erdil and Ergin (2006, 2008). On the other hand, *substitutability*, originally introduced by Kelso and Crawford (1982), is the most general class of the strict priority orders for the existence of a stable assignment. Our notion of substitutability combines these two notions in a unified framework, and we analyze *stability* and *efficiency* properties in this more general environment.

Since our priority structure involves indifferences, and is not necessarily responsive, a strict resolution of the priority orders may fail to preserve substitutability, and thus we do not straightforwardly guarantee a stable assignment. However, thanks to the monotonic property of our substitutability condition,

the deferred acceptance algorithm, first introduced by Gale and Shapley (1962), can be appropriately extended to our model to get a generalized deferred acceptance algorithm (GDA). Provided our notion of substitutability holds, the GDA algorithm always returns a stable assignment. As is the case for responsive priorities with ties, the set of constrained efficient allocations is not necessarily a singleton, and the arbitrary choices within the implementation of the GDA algorithm may lead to a constrained inefficient outcome.

Thus the question of finding a constrained efficient assignment remains. In the case of responsive priorities, the stable improvement cycles algorithm by Erdil and Ergin (2008) reaches a constrained efficient matching. We point out that their 'cycles of schools' are not sufficient to recover efficiency in the more general model. We introduce the *stable agent improving cycles* (SAIC), and by further assuming *equal treatment of equal agents*, we show that a stable assignment is constrained efficient if and only if it does not admit SAIC. Hence, we guarantee a constrained efficient assignment through the GDA and SAIC algorithms.

Further we ask when a constrained efficient outcome is indeed efficient. We formulate a general *acyclicity* condition, and show that the constrained efficient stable correspondence is efficient if and only if the priority structure is *acyclic*, generalizing Ergin (2002), and Ehlers and Erdil (2010). Acyclity is typically interpreted as the similarity of priorities of objects, however, this interpretation is no longer valid in our class. We observe that a priority structure may be cyclic even when every object is endowed with the same priority order.

As an application, our notion of substitutability can incorporate some notion of 'gender equality' or 'racial equality' in school choice. These notions are increasingly considered in many school choice districts, and our approach helps capture these notions in a novel way.