Additively Coupled Sum Constrained Games

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Abstract. We propose and analyze a broad family of games played by resource-constrained players, which are characterized by the following central features: 1) each user has a multi-dimensional action space, subject to a single sum resource constraint; 2) each user's utility in a particular dimension depends on an additive coupling between the user's action in the same dimension and the actions of the other users; and 3) each user's total utility is the sum of the utilities obtained in each dimension. Familiar examples of such multi-user environments in communication systems include power control over frequency-selective Gaussian interference channels and flow control in Jackson networks. In settings where users cannot exchange messages in real-time, we study how users can adjust their actions based on their local observations. We derive sufficient conditions under which a unique Nash equilibrium exists and the best-response algorithm converges globally and linearly to the Nash equilibrium. In settings where users can exchange messages in real-time, we focus on user choices that optimize the overall utility. We provide the convergence conditions of two distributed action update mechanisms, gradient play and Jacobi update.

1 Introduction

Game theory provides a formal framework for describing and analyzing the interactions of multiple decision-makers. Recently, there has been a surge in research activities that adopt game theoretic tools to investigate a wide range of modern communications and networking problems. In resource-constrained communication networks, a user's utility is usually not only affected by its own action but also by the actions taken by all the other users sharing the same resources. Due to the mutual coupling among users, the performance optimization of multi-user communication systems is challenging. Depending on the characteristics of different applications, numerous game-theoretical models and solution concepts have been proposed to characterize the multi-user interactions and optimize the users' decisions in communication networks. A variety of game theoretic solutions have been developed to characterize the resulting performance of the multi-user interactions, including Nash equilibrium (NE) and Pareto optimality [1]. The purpose of this paper is to introduce and analyze a general framework that abstracts the common characteristics of this family of multi-user interaction scenarios, which includes, but is not limited to, the power control scenario. In particular, the main contributions of this paper are as follows.

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First of all, we define the class of *Additively Coupled Sum Constrained Games* (ACSCG), which captures and characterizes the key features of several communication and networking applications. In particular, the central features of ACSCG are: 1) each user has a multi-dimensional strategy that is subject to a single sum resource constraint; 2) each user's payoff in each dimension is impacted by an additive combination of its own action in the same dimension and a function of the other users' actions; 3) users' utilities are separable across different dimensions and each user's total utility is the sum of the utilities obtained within each dimension.

Second, based on the feasibility of real-time information exchange, we provide the convergence conditions of various generic distributed algorithms in different scenarios. When no message exchanges between users are possible and every user maximizes its own utility, it is essential to determine whether a NE exist and if yes, how to achieve such an equilibrium. In ACSCG, a pure NE exists because ACSCG belongs to concave games [1] [2]. Our key contribution in this context is that we investigate the uniqueness of pure NE and consider the best response dynamics to compute the NE. We explore the properties of the additive coupling among users given the sum constraint and provide several sufficient conditions under which best response dynamics converges linearly[1] to the unique NE, for any set of feasible initialization with either sequential or parallel updates. When users can collaboratively exchange messages with each other in real-time, we present the sufficient convergence conditions of two alternative distributed pricing algorithms, including gradient play and Jacobi update, to coordinate users' action and improve the overall system efficiency. The proposed convergence conditions generalize the results that have been previously obtained in [8]- [13] for the multi-user power control problem and they are immediately applicable to other multi-user applications in communication networks that fulfill the requirements of ACSCG.

The rest of this paper is organized as follows. Section 2 defines the model of ACSCG. For ACSCG models, Sections 3 and 4 present several distributed algorithms without and with real-time information exchanges, respectively, and provide sufficient conditions that guarantee the convergence of the proposed algorithms. Conclusions are drawn in Section 5. Due to space limitations, the formal proofs and numerical examples are omitted; for these proofs and examples, the reader is referred to [19].

2 Game Model

2.1 Strategic Games, Nash Equilibrium, and Pareto Optimality

A strategic game is a suitable model for the analysis of a game where all users act independently and simultaneously according to their own self-interests and with no or limited a priori knowledge of the other users' strategies. This can be

[1] A sequence $x^{(k)}$ with limit x^* is linearly convergent if there exists a constant $c \in (0,1)$ such that $|x^{(k)} - x^*| \leq c|x^{(k-1)} - x^*|$ for k sufficiently large.

formally defined as a tuple $\Gamma = \langle \mathcal{N}, \mathcal{A}, u \rangle$. In particular, $\mathcal{N} = \{1, 2, \ldots, N\}$ is the set of decision makers. Define $\mathcal{A}$ to be the joint action set $\mathcal{A} = \times_{n \in \mathcal{N}} \mathcal{A}_n$, with $\mathcal{A}_n \subseteq \mathcal{R}^K$ being the action set available for user n. The vector utility function $u = \times_{n \in \mathcal{N}} u_n$ is a mapping from the individual users' joint action set to real numbers, i.e. $u : \mathcal{A} \rightarrow \mathcal{R}^N$. In particular, $u_n(\mathbf{a}) : \mathcal{A} \rightarrow \mathcal{R}$ is the utility of the nth user that generally depends on the strategies $\mathbf{a} = (\mathbf{a}_n, \mathbf{a}_{-n})$ of all users, where $\mathbf{a}_n \in \mathcal{A}_n$ denotes a feasible action of user n, and $\mathbf{a}_{-n} = \times_{m \neq n} \mathbf{a}_m$ is a vector of the actions of all users except n. We also denote by $\mathcal{A}_{-n} = \times_{m \neq n} \mathcal{A}_m$ the joint action set of all users except n. To capture the multi-user performance tradeoff, the utility region is defined as $\mathcal{U} = \{(u_1(\mathbf{a}), \ldots, u_N(\mathbf{a})) | \ \exists \ \mathbf{a} \in \mathcal{A}\}$. Various solutions, such as NE and Pareto optimality, were developed in the literature [1]. Significant research efforts have been devoted in the literature to constructing operational algorithms in order to achieve NE and Pareto optimality in various games with special structures of action set $\mathcal{A}_n$ and utility function u_n.

Nash Equilibrium: Definition, Existence, and Convergence. To avoid the overhead associated with exchanging information in real-time, network designers may prefer fully decentralized solutions in which the participating users simply compete against other users by choosing actions $\mathbf{a}_n \in \mathcal{A}_n$ to selfishly maximize their individual utility functions $u_n(\mathbf{a}_n, \mathbf{a}_{-n})$, given the actions $\mathbf{a}_{-n} \in \mathcal{A}_{-n}$. Most of these approaches focus on investigating the existence and properties of NE. NE is defined to be an action profile $(\mathbf{a}_1^*, \mathbf{a}_2^*, \ldots, \mathbf{a}_N^*)$ with the property that for every player, it satisfies $u_n(\mathbf{a}_n^*, \mathbf{a}_{-n}^*) \geq u_n(\mathbf{a}_n, \mathbf{a}_{-n}^*)$ for all $\mathbf{a}_n \in \mathcal{A}_n$, i.e. given the other users' actions, no user can increase its utility alone by changing its action. Many of the well-known results on NE rely on specific structural properties of action set $\mathcal{A}$ and utility function u in the investigated multi-user interactions. For an extensive discussion of the methodologies studying the existence, uniqueness, and convergence of various equilibria in communication networks, we refer the readers to [14].

Pareto Optimality and Network Utility Maximization. A profile of actions is Pareto optimal if there is no other profile of actions that makes every user at least as well off and at least one user strictly better off. It is important to note that operating at a NE will generally limit the performance of the user itself as well as that of the entire network, because the available network resources are not always effectively exploited due to the conflicts of interest occurring among users. As opposed to the NE-based approaches, there exists a large body of literature that focuses on studying how to compute Pareto optimal solutions in large-scale networks where centralized solutions are infeasible by optimizing a certain common objective function $f(u_1(\mathbf{a}), u_2(\mathbf{a}), \ldots, u_N(\mathbf{a}))$. This function represents the fairness rule based on which the system-wide resource allocation is performed. Different objective functions, e.g. sum utility maximization in which $f(u_1(\mathbf{a}), u_2(\mathbf{a}), \ldots, u_N(\mathbf{a})) = \sum_{n=1}^{N} u_n(\mathbf{a})$, can provide reasonable allocation outcomes by jointly considering fairness and efficiency. An important example is the NUM framework that develops distributed algorithms to solve network resource allocation problems [6]. The majority of the results in the existing NUM literature are based on convex optimization theory. It is well-known that, for convex optimization problems, users can collaboratively exchange price signals that reflect the "cost" for consuming the constrained resources and the Pareto optimal allocation that maximizes the network utility can be determined in a fully distributed manner [7].

Summarizing, these general structural results without and with real-time message exchange turn out to be very useful when analyzing various multi-user interactions in communication networks. In the remaining part of this paper, we will derive several structural results for a particular type of multi-user interaction scenario.

2.2 Additively Coupled Sum Constrained Games

Definition 1. *A multi-user interaction $\Gamma = \langle \mathcal{N}, \mathcal{A}, u \rangle$ is a ACSCG if it satisfies the following assumptions:*

A1: *$\forall n \in \mathcal{N}$, action set $\mathcal{A}_n \subseteq \mathcal{R}^K$ is defined as $\mathcal{A}_n =$* [2]

$$\Big\{(a_n^1, a_n^2, \cdots, a_n^K) \mid a_n^k \in [a_{n,k}^{\min}, a_{n,k}^{\max}] \text{ and } \sum_{k=1}^{K} a_n^k \le M_n \Big\}. \tag{1}$$

A2: *There exist $h_n^k : \mathcal{R} \to \mathcal{R}$, $f_n^k : \mathcal{A}_{-n} \to \mathcal{R}$, and $g_n^k : \mathcal{A}_{-n} \to \mathcal{R}$, $k = 1, \ldots, K$, such that*

$$u_n(\mathbf{a}) = \sum_{k=1}^{K} \Big[h_n^k\big(a_n^k + f_n^k(\mathbf{a}_{-n})\big) - g_n^k(\mathbf{a}_{-n}) \Big], \tag{2}$$

for all $\mathbf{a} \in \mathcal{A}$ and $n \in \mathcal{N}$. $h_n^k(\cdot)$ is an increasing, twice differentiable, and strictly concave function and $f_n^k(\cdot)$ and $g_n^k(\cdot)$ are both twice differentiable.

The ACSCG model defined by assumptions A1 and A2 covers a broad class of multi-user interactions. Assumption A1 indicates that each player's action set is a K-dimensional vector set and its action vector is sum-constrained. This represents the communication scenarios in which each user needs to determine its multi-dimensional action in various channels or networks while the total amount of resources it can consume is constrained. Assumption A2 implies that each user's utility is separable and can be represented by the summation of concave functions h_n^k minus "penalty" functions g_n^k across the K dimensions. In particular, within each dimension, the input of h_n^k is an additive combination of user n's action a_n^k and function $f_n^k(\mathbf{a}_{-n})$ that depends on the remaining users' joint action $\mathbf{a}_{-n}$. Since a_n^k only appears in the concave function h_n^k, it implies that each user's utility is concave in its own action, i.e. diminishing returns per unit

[2] We consider a sum constraint throughout the paper rather than a weighted-sum constraint, because a weighted-sum constraint can be easily converted to a sum constraint by rescaling $\mathcal{A}_n$. Besides, we nontrivially assume that $\sum_{k=1}^{K} a_{n,k}^{\max} \ge M_n$.

of user n's invested action $\mathbf{a}_n$, which is common for many application scenarios in communication networks.

Summarizing, the key features of the game model defined by A1 and A2 include: each user's action is subject to a *sum constraint*; users' utilities are impacted by *additive combinations* of a_n^k and $f_n^k(\mathbf{a}_{-n})$ through concave functions h_n^k. Therefore, we term the game Γ that satisfies assumptions A1 and A2 as ACSCG. In [19], we present several illustrative multi-user interaction examples that belong to ACSCG, including power control in frequency-selective Gaussian interference channel, delay minimization in Jackson networks, and asynchronous transmission in digital subscriber lines network.

2.3 Issues Related to ACSCG

Since ACSCG represents a good abstraction of numerous multi-user resource allocation problems, we aim to investigate the convergence properties of various distributed algorithms in ACSCG without and with real-time message passing.

ACSCG is a concave game [1] [2] and therefore, it admits at least one pure NE. In practice, we want to provide the sufficient conditions under which best response dynamics provably and globally converges to a pure NE. However, the existing literature, e.g. the diagonal strict concavity (DSC) conditions in [2] and the supermodular game theory [3]- [5], does not provide such convergence conditions for the general ACSCG model. On the other hand, if we want to maximize the sum utility by enabling real-time message passing among users, we also note that, the utility u_n is not necessarily jointly concave in $\mathbf{a}$ because of the existence of $g_n^k(\cdot)$. Therefore, the existing algorithms developed for the convex NUM are not immediately applicable either.

In the following sections, we will fully explore structures of ACSCG and address the convergence properties of various distributed algorithms in two different scenarios. Specifically, Section 3 investigates the scenarios in which each user n can only observe $\{f_n^k(\mathbf{a}_{-n})\}_{k=1}^K$ and cannot exchange any information with any other user. Section 4 focuses on the scenarios in which each user n is able to announce and receive information in real-time to and from the remaining users about $\frac{\partial u_n(\mathbf{a})}{\partial a_m^k}$ and $\frac{\partial u_m(\mathbf{a})}{\partial a_n^k}$, $\forall m \neq n, k = 1, \ldots, K$.

3 Scenario I: No Message Exchange among Users

In communication scenarios where users cannot exchange messages to achieve coordination, the participating users can simply choose actions to selfishly maximize their individual utility functions $u_n(\mathbf{a})$ by solveing the following optimization program:

$$\max_{\mathbf{a}_n \in \mathcal{A}_n} u_n(\mathbf{a}). \tag{3}$$

The steady state outcome of such a multi-user interaction is usually characterized as a NE.

3.1 Properties of Best Response Dynamics in ACSCG

In this subsection, we first focus on the scenarios in which $f_n^k(\mathbf{a}_{-n})$ is the linear combination of the remaining users' action in the same dimension k, i.e.

$$f_n^k(\mathbf{a}_{-n}) = \sum_{m \neq n} F_{mn}^k a_m^k \tag{4}$$

and $F_{mn}^k \in \mathcal{R}$, $\forall m, n, k$. In Section 3.2, we will extend the results derived for the functions $f_n^k(\mathbf{a}_{-n})$ defined in (4) to general $f_n^k(\mathbf{a}_{-n})$.

Since $h_n^k(\cdot)$ is concave, the objective in (3) is a concave function in a_n^k when the other users' actions $\mathbf{a}_{-n}$ are fixed. To find the globally optimal solution of the problem in (3), we can first form its Lagrangian

$$L_n(\mathbf{a}_n, \lambda) = u_n(\mathbf{a}) + \lambda(M_n - \sum_{k=1}^{K} a_n^k), \tag{5}$$

in which $a_n^k \in [a_{n,k}^{\min}, a_{n,k}^{\max}]$. By taking the first derivatives of (5), we have

$$\frac{\partial L_n(\mathbf{a}_n, \lambda)}{\partial a_n^k} = \frac{\partial h_n^k(a_n^k + \sum_{m \neq n} F_{mn}^k a_m^k)}{\partial a_n^k} - \lambda = 0. \tag{6}$$

Denote

$$l_n^k(\mathbf{a}_{-n}, \lambda) \triangleq \left[\left\{ \frac{\partial h_n^k}{\partial x} \right\}^{-1} (\lambda) - \sum_{m \neq n} F_{mn}^k a_m^k \right]_{a_{n,k}^{\min}}^{a_{n,k}^{\max}}, \tag{7}$$

in which $\left\{ \frac{\partial h_n^k}{\partial x} \right\}^{-1}$ is the inverse function[3] of $\frac{\partial h_n^k}{\partial x}$ and $[x]_b^a = \max\{\min\{x, a\}, b\}$. The optimal solution of (3) is given by $a_n^{*k} = l_n^k(\mathbf{a}_{-n}, \lambda^*)$, where the Lagrange multiplier λ^* is chosen to satisfy the sum constraint $\sum_{k=1}^{K} a_n^{*k} = M_n$.

We define the best response operator $B_n^k(\cdot)$ as

$$B_n^k(\mathbf{a}_{-n}) = l_n^k(\mathbf{a}_{-n}, \lambda^*). \tag{8}$$

We consider the best response algorithm in which each user updates its action using the best response strategy that maximizes its utility function in (2). We consider two types of update orders, including sequential update and parallel update. Specifically, in sequential update, individual players iteratively optimize in a circular fashion with respect to their own actions while keeping the actions of their opponents fixed. At stage t, user n chooses its action according to

$$a_n^{k,t} = B_n^k([\mathbf{a}_1^t, \ldots, \mathbf{a}_{n-1}^t, \mathbf{a}_{n+1}^{t-1}, \ldots, \mathbf{a}_N^{t-1}]). \tag{9}$$

On the other hand, players adopting the parallel update revise their actions at stage t according to

$$a_n^{k,t} = B_n^k(\mathbf{a}_{-n}^{t-1}). \tag{10}$$

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[3] If $\nexists\, x = x^*$ such that $\frac{\partial h_n^k}{\partial x}|_{x=x^*} = \lambda$, we let $\left\{ \frac{\partial h_n^k}{\partial x} \right\}^{-1}(\lambda) = -\infty$.

We obtain several sufficient conditions under which best response dynamics converges. Similar convergence conditions are proved in [9]- [11] in which $h_n^k(x) = \log_2(\sigma_n^k + H_{nn}^k x)$. We consider more general functions $h_n^k(\cdot)$ and further extend the convergence conditions in [9]- [11]. The key differences among all the sufficient conditions which will be provided in this section are summarized in Table 1.

Table 1. Comparison among conditions (C1)-(C6)

Conditions	Assumptions about $f_n^k(\mathbf{a}_{-n})$	$h_n^k(x)$	Measure of residual error $\mathbf{a}_n^{t+1} - \mathbf{a}_n^t$	Contraction factor
(C1)	(4)	A2	1-norm	$2\rho(\mathbf{T}^{\max})$
(C2)	(4) and F_{mn}^k have the same sign for $\forall k, m \neq n$	A2	1-norm	$\rho(\mathbf{T}^{\max})$
(C3)	(4)	(13)	weighted Euclidean norm	$\rho(\mathbf{S}^{\max})$
(C4)	general	A2	1-norm	$2\rho(\bar{\mathbf{T}}^{\max})$
(C5)	$\frac{\partial f_n^k(\mathbf{a}_{-n})}{\partial a_m^{k'}}$ have the same sign for $\forall \mathbf{a} \in \mathcal{A}, k, k', m \neq n$	A2	1-norm	$\rho(\bar{\mathbf{T}}^{\max})$
(C6)	general	(13)	weighted Euclidean norm	$\rho(\bar{\mathbf{S}}^{\max})$

General $h_n^k(\cdot)$. The first sufficient condition is developed for the general cases in which the functions $h_n^k(\cdot)$ in the utilities $u_n(\cdot)$ are specified in assumption A2. Define

$$[\mathbf{T}^{\max}]_{mn} \triangleq \begin{cases} \max_k |F_{mn}^k|, & \text{if } m \neq n \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

and let $\rho(\mathbf{T}^{\max})$ denote the spectral radius of the matrix $\mathbf{T}^{\max}$.

Theorem 1. *If*

$$\rho(\boldsymbol{T}^{\max}) < \frac{1}{2}, \tag{C1}$$

then there exists a unique NE in game Γ and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to $\mathcal{A}$ with either sequential or parallel updates.

Proof: This theorem is proved by showing that the best response dynamics defined in (9) and (10) is a contraction mapping under (C1). See Appendix A in [19] for details. ■

In multi-user communication applications, it is common to have games of *strategic complements* (or *strategic substitutes*), i.e. the marginal returns to any one component of the player's action rise with increases (or decreases) in the components of the competitors' actions [15]. For instance, in power control applications, increasing user n's transmitted power creates stronger interference to the other users and decreases their marginal achievable rates. Mathematically, if u_n

is twice differentiable, strategic complementarities (or strategic substitutes) can be described as

$$\frac{\partial^2 u_n(\mathbf{a}_n, \mathbf{a}_{-n})}{\partial a_n^j \partial a_m^k} \geq 0, \ \forall m \neq n, j, k, (\text{or } \frac{\partial^2 u_n(\mathbf{a}_n, \mathbf{a}_{-n})}{\partial a_n^j \partial a_m^k} \leq 0, \ \forall m \neq n, j, k). \quad (12)$$

For the ACSCG models that exhibit strategic complementarities (or strategic substitutes), the following theorem further relaxes condition (C1).

Theorem 2. *Let Γ be an ACSCG with strategic complementarities (or strategic substitutes), i.e. $F_{mn}^k \leq 0$, $\forall k, m \neq n$, (or $F_{mn}^k \geq 0$, $\forall k, m \neq n$). If*

$$\rho(\boldsymbol{T}^{\max}) < 1, \quad (\text{C2})$$

then there exists a unique NE in game Γ and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to $\mathcal{A}$ with either sequential or parallel updates.

Proof: This theorem is proved by adapting the proof of Theorem 1. See Appendix B in [19]. ■

Remark 1. (Implications of conditions (C1) and (C2)) Theorem 1 and Theorem 2 give sufficient conditions for best response dynamics to globally converge to a unique fixed point. Specifically, $\max_k |F_{mn}^k|$ can be regarded as a measure of the strength of the mutual coupling between user m and n. The intuition behind (C1) and (C2) is that, the weaker the coupling among different users is, the more likely that best response dynamics converges. Consider the extreme case in which $F_{mn}^k = 0, \forall k, m \neq n$. Since each user's best response is not impacted by the remaining users' action $\mathbf{a}_{-n}$, the convergence is immediately achieved after a single best-response iteration. If no restriction is imposed on F_{mn}^k, Theorem 1 specifies a mutual coupling threshold under which best response dynamics provably converge. The proof of Theorem 1 can be intuitively interpreted as follows. We regard every best response update as the users' joint attempt to approach the NE. Due to the linear coupling structure in (4), user n's best response in (7) contains a term $\sum_{m \neq n} F_{mn}^k a_m^k$ that is a linear combination of $\mathbf{a}_{-n}$. As a result, the residual error $\left|\mathbf{a}_n^{t+1} - \mathbf{a}_n^t\right|_1$, which is the 1-norm distance between the updated action profile $\mathbf{a}_n^{t+1}$ and the current action profile $\mathbf{a}_n^t$, can be upper-bounded using linear combinations of $\left|\mathbf{a}_m^t - \mathbf{a}_m^{t-1}\right|_1$ in which $m \neq n$. Recall that F_{mn}^k can be either positive or negative. We also note that, if $\mathbf{a}_m^t \neq \mathbf{a}_m^{t-1}$, $\mathbf{a}_m^t - \mathbf{a}_m^{t-1}$ contains both positive and negative terms due to the sum-constraint. In the worst case, the distance $\left|\mathbf{a}_n^{t+1} - \mathbf{a}_n^t\right|_1$ is maximized if $\left\{F_{mn}^k\right\}$ and $\left\{a_m^{k,t} - a_m^{k,t-1}\right\}$ are co-phase multiplied and additively summed, i.e. $F_{mn}^k\left(a_m^{k,t} - a_m^{k,t-1}\right) \geq 0$, for $\forall k = 1, \ldots, K, m \neq n$. After an iteration, all users except n contributes to user n's residual error at stage $t+1$ up to $\sum_{m \neq n} 2 \max_k \left|F_{mn}^k\right| \left|\mathbf{a}_m^t - \mathbf{a}_m^{t-1}\right|_1$. Under condition (C1), it is guaranteed that the residual error contracts. Theorem 2 focuses on the situations in which the signs of F_{mn}^k are the same, $\forall m \neq n, k$. In this case, $\left\{F_{mn}^k\right\}$ and $\left\{a_m^{k,t} - a_m^{k,t-1}\right\}$ cannot be co-phase multiplied. Therefore, the region of convergence enlarges and hence, condition (C2) stated in Theorem 2 is weaker than condition (C1) in Theorem 1.

Remark 2. (Relation to the results in references [9]- [11]) Similar to [9] [10], our proofs choose 1-norm as the distance measure for the residual errors $\mathbf{a}_n^{t+1} - \mathbf{a}_n^t$ after each best-response iteration. However, by manipulating the inequalities in a different way, condition (C2) is more general than the results in [9] [10], where they require $\max_k F_{mn}^k < \frac{1}{N-1}$. Interestingly, condition (C2) recovers the result obtained in [11] where it is proved by choosing the Euclidean norm as the distance measure for the residual errors $\mathbf{a}_n^{t+1} - \mathbf{a}_n^t$ after each best-response iteration. However, the approach in [11] using the Euclidean norm only applies to the scenarios in which $h_n^k(\cdot)$ is a logarithmic function. We prove that condition (C2) applies to any $h_n^k(\cdot)$ that is increasing and strictly concave.

A Special Class of $h_n^k(\cdot)$. In addition to conditions (C1) and (C2), we also develop a sufficient convergence condition for a family of utility functions parameterized by a negative number θ. In particular, $h_n^k(\cdot)$ satisfies[4]

$$h_n^k(x) = \begin{cases} \log(\alpha_n^k + F_{nn}^k x), \text{ if } \theta = -1, \\ \frac{(\alpha_n^k + F_{nn}^k x)^{\theta+1}}{\theta+1}, \quad \text{if } -1 < \theta < 0 \text{ or } \theta < -1. \end{cases} \tag{13}$$

and $\alpha_n^k \in \mathcal{R}$ and $F_{nn}^k > 0$. The interpretation of this type of utilities has been addressed in [16]. It is shown that varying the parameter θ leads to different types of fairness across $\alpha_n^k + F_{nn}^k(a_n^k + \sum_{m \neq n} F_{mn}^k a_m^k)$ for all k. In particular, $\theta = -1$ corresponds to the proportional fairness; if $\theta = -2$, then harmonic mean fairness; and if $\theta = -\infty$, then max-min fairness. In these cases, best response dynamics in equation (7) is reduced to

$$l_n^k(\mathbf{a}_{-n}, \lambda) = \left[\left(\frac{1}{F_{nn}^k}\right)^{1+\frac{1}{\theta}} \lambda^{\frac{1}{\theta}} - \frac{\alpha_n^k}{F_{nn}^k} - \sum_{m \neq n} F_{mn}^k a_m^k \right]_{a_{n,k}^{\min}}^{a_{n,k}^{\max}}, \tag{14}$$

Define $[\mathbf{S}^{\max}]_{mn} \triangleq$

$$\begin{cases} \frac{\sum_{k=1}^K (F_{mm}^k)^{1+\frac{1}{\theta}}}{\sum_{k=1}^K (F_{nn}^k)^{1+\frac{1}{\theta}}} \max_k \left\{ |F_{mn}^k| \left(\frac{F_{nn}^k}{F_{mm}^k}\right)^{1+\frac{1}{\theta}} \right\}, \text{ if } m \neq n \\ 0, \qquad \text{otherwise.} \end{cases} \tag{15}$$

For the class of utility functions in (13), Theorem 3 gives a sufficient condition that guarantees the convergence of the best response dynamics defined in (14).

Theorem 3. *For $h_n^k(\cdot)$ defined in (13), if*

$$\rho(\boldsymbol{S}^{\max}) < 1, \tag{C3}$$

then there exists a unique NE in game Γ and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to $\mathcal{A}$ and with either sequential or parallel updates.

[4] If $\alpha_n^k + F_{nn}^k x \leq 0$, we let $h_n^k(x) = -\infty$. We assume for this class of $h_n^k(\cdot)$ that for $\forall \mathbf{a}_{-n} \in \mathcal{A}_{-n}$, there exists $\mathbf{a}_n \in \mathcal{A}_n$ such that $\alpha_n^k + F_{nn}^k x > 0$ for $\forall n, k$.

Proof: It can be proved by showing that the best response dynamics defined in (14) is a contraction mapping with respect to the weighted Euclidean norm. See Appendix C in [19] for details. ■

Remark 3. (Relation between conditions (C3) and the results in reference [11]) For power control in frequency-selective Gaussian interference channel, Scutari et al. established in [11] a sufficient condition under which the iterative water-filling algorithm converges. The iterative water-filling algorithm essentially belongs to best response dynamics. Specifically, in [11], Shannon's formula leads to $\theta = -1$ and cross channel coefficients satisfy $F_{mn}^k \geq 0, \forall k, m \neq n$. Equation (14) reduces to the water-filling formula

$$l_n^k(\mathbf{a}_{-n}, \lambda) = \left[\frac{1}{\lambda} - \frac{\alpha_n^k}{F_{nn}^k} - \sum_{m \neq n} F_{mn}^k a_m^k\right]_{a_{n,k}^{\min}}^{a_{n,k}^{\max}}, \tag{16}$$

and $[\mathbf{S}^{\max}]_{mn} = \max_k F_{mn}^k$. By choosing the weighted Euclidean norm as the distance measure for the residual errors $\mathbf{a}_n^{t+1} - \mathbf{a}_n^t$ after each best-response iteration, Theorem 3 generalizes the results in [11] for the family of utility functions defined in (13).

Remark 4. (Relation between conditions (C1), (C2) and (C3)) The connections and differences between conditions (C1), (C2) and (C3) are summarized in Table 1. We have addressed the implications of (C1) and (C2) in Remark 1. Now we discuss their relation with (C3). First of all, condition (C1) is proposed for general $h_n^k(\cdot)$ and condition (C3) is proposed for the class of utility functions defined in (13). However, Theorem 1 and Theorem 3 individually establish the fact that best response dynamics is a contraction map by selecting different vector and matrix norms. Therefore, in general, (C1) and (C3) do not immediately imply each other. Note that $[\mathbf{S}^{\max}]_{mn} \leq \zeta_{mn} \cdot \max_k |F_{mn}^k|$ in which ζ_{mn} satisfies

$$\zeta_{mn} = \frac{\sum_{k=1}^K (F_{mm}^k)^{1+\frac{1}{\theta}}}{\sum_{k=1}^K (F_{nn}^k)^{1+\frac{1}{\theta}}} \cdot \max_k \frac{(F_{nn}^k)^{1+\frac{1}{\theta}}}{(F_{mm}^k)^{1+\frac{1}{\theta}}} \in \left[1, \frac{\max_k (F_{nn}^k/F_{mm}^k)^{1+\frac{1}{\theta}}}{\min_k (F_{nn}^k/F_{mm}^k)^{1+\frac{1}{\theta}}}\right]. \tag{17}$$

The physical interpretation of ζ_{mn} is the similarity between the preferences of user m and n across the total K dimensions of their action spaces. Recall that both $\mathbf{S}^{\max}$ and $\mathbf{T}^{\max}$ are non-negative matrices and $\mathbf{S}^{\max}$ is element-wise less than or equal to $\max_{m \neq n} \zeta_{mn} \mathbf{T}^{\max}$. By the property of non-negative matrix and condition (C1), we can conclude $\rho(\mathbf{S}^{\max}) \leq \rho(\max_{m \neq n} \zeta_{mn} \mathbf{T}^{\max}) < \max_{m \neq n} \frac{\zeta_{mn}}{2}$. If users have similar preference in their available actions and the upper bound of ζ_{mn} that measures the difference of their preferences is below the following threshold:

$$\frac{\max_{k, m \neq n} (F_{nn}^k/F_{mm}^k)^{1+\frac{1}{\theta}}}{\min_{k, m \neq n} (F_{nn}^k/F_{mm}^k)^{1+\frac{1}{\theta}}} < 2, \tag{18}$$

we know that (C1) implies (C3) in this situation because $\rho(\mathbf{S}^{\max}) < \max_{m,n} \zeta_{mn} \cdot \rho(\mathbf{T}^{\max}) < 2 \cdot \frac{1}{2} = 1$. We also would like to point out that, the LHS of (18) is a function of θ and the LHS $\equiv 1$ if $\theta = -1$. When $\theta = -1$, $\mathbf{T}^{\max}$ coincides with $\mathbf{S}^{\max}$. Mathematically, in this case, (C3) is actually more general than (C2), because it still holds even if coefficients F_{mn}^k have different signs.

3.2 Extensions to General $f_n^k(\cdot)$

As a matter of fact, the results above can be extended to the more general situations in which $f_n^k(\cdot)$ is a nonlinear differentiable function, $\forall n, k$ and its input $\mathbf{a}_{-n}$ consists of the remaining users' action from all the dimensions. Accordingly, equation (7) becomes

$$l_n^k(\mathbf{a}_{-n}, \lambda) \triangleq \left[\left\{ \frac{\partial h_n^k}{\partial x} \right\}^{-1} (\lambda) - f_n^k(\mathbf{a}_{-n}) \right]_{a_{n,k}^{\min}}^{a_{n,k}^{\max}}. \tag{19}$$

The conclusions in Theorem 1, 2, and 3 can be further extended as Theorem 4, and 5, 6 that are listed below.

For general $f_n^k(\cdot)$, we denote

$$[\bar{\mathbf{T}}^{\max}]_{mn} \triangleq \begin{cases} \max_{\mathbf{a} \in \mathcal{A}, k'} \sum_{k=1}^K \left| \frac{\partial f_n^k(\mathbf{a}_{-n})}{\partial a_m^{k'}} \right|, & \text{if } m \neq n \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Besides, for $h_n^k(\cdot)$ defined in (13), we define $[\bar{\mathbf{S}}^{\max}]_{mn} \triangleq$

$$\begin{cases} \frac{\sum\limits_{k=1}^K (F_{mm}^k)^{1+\frac{1}{\theta}}}{\sum\limits_{k=1}^K (F_{nn}^k)^{1+\frac{1}{\theta}}} \max\limits_{\mathbf{a} \in \mathcal{A}, k'} \left\{ \sum\limits_{k=1}^K \left| \frac{\partial f_n^k(\mathbf{a}_{-n})}{\partial a_m^{k'}} \right| \left(\frac{F_{nn}^{k'}}{F_{mm}^{k'}} \right)^{1+\frac{1}{\theta}} \right\}, & \text{if } m \neq n \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

Theorem 4. *If*

$$\rho(\bar{\boldsymbol{T}}^{\max}) < \frac{1}{2}, \tag{C4}$$

then there exists a unique NE in game Γ and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to $\mathcal{A}$ with either sequential or parallel updates.

Proof: This theorem can be proved by combining the proof of Theorem 1 and the mean value theorem for vector-valued functions. See Appendix D in [19] for details. ∎

Similarly as in Theorem 2, for the general ACSCG models that exhibit strategic complementarities (or strategic substitutes), we can relax condition (C4).

Theorem 5. *For Γ with strategic complementarities (or strategic substitutes), i.e. $\frac{\partial f_n^k(\boldsymbol{a}_{-n})}{\partial a_m^{k'}} \geq 0, \forall m \neq n, k, k', \boldsymbol{a} \in \mathcal{A}$, (or $\frac{\partial f_n^k(\boldsymbol{a}_{-n})}{\partial a_m^{k'}} \leq 0, \forall m \neq n, k, k', \boldsymbol{a} \in \mathcal{A}$), if*

$$\rho(\bar{\boldsymbol{T}}^{\max}) < 1, \tag{C5}$$

then there exists a unique NE in game Γ and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to $\mathcal{A}$ with either sequential or parallel updates.

Theorem 6. *For $h_n^k(\cdot)$ defined in (13), if*

$$\rho(\bar{\boldsymbol{S}}^{\max}) < 1, \tag{C6}$$

then there exists a unique NE in game Γ and best response dynamics converges linearly to the NE, for any set of initial conditions belonging to $\mathcal{A}$ with either sequential or parallel updates.

Remark 5. (Implications of conditions (C4), (C5), and (C6)) Based on the mean value theorem, we know that the upper bound of the additive sum of first derivatives $\sum_{k=1}^{K} \left| \frac{\partial f_n^k(\mathbf{a}_{-n})}{\partial a_m^{k'}} \right|$ governs the maximum impact that user m's action can make over user n's utility. As a result, Theorem 4, Theorem 5, and Theorem 6 indicate that $\sum_{k=1}^{K} \left| \frac{\partial f_n^k(\mathbf{a}_{-n})}{\partial a_m^{k'}} \right|$ can be used to develop similar sufficient conditions for the global convergence of best response dynamics. Table 1 summarizes the connections and differences among all the aforementioned conditions from (C1) to (C6).

Remark 6. (Impact of sum constraints) An interesting phenomenon that can be observed from the analysis above is that, the convergence condition may depend on the maximum constraints $\{M_n\}_{n=1}^N$. This differs from the observation in [11] that the presence of the transmit power and spectral mask constraints does not affect the convergence capability of the iterative water-filling algorithm. This is because when functions $f_n^k(\mathbf{a}_{-n})$ are affine, the elements in $\bar{\mathbf{T}}^{\max}$ and $\bar{\mathbf{S}}^{\max}$ are independent of the values of $\{M_n\}_{n=1}^N$. Therefore, (C1)-(C6) are independent of M_n for affine $f_n^k(\mathbf{a}_{-n})$. However, for non-linear $f_n^k(\mathbf{a}_{-n})$, the values of $\{M_n\}_{n=1}^N$ specify the range of users' joint feasible action set $\mathcal{A}$, and this will affect $\bar{\mathbf{T}}^{\max}$ and $\bar{\mathbf{S}}^{\max}$ accordingly. In other words, in the presence of non-linearly coupled $f_n^k(\mathbf{a}_{-n})$, convergence may depend on the players' maximum sum constraints $\{M_n\}_{n=1}^N$.

4 Scenario II: Message Exchange among Users

In this section, our objective is to coordinate the users' actions in ACSCG to maximize the overall performance of the system, measured in terms of their total utilities, in a distributed fashion. Specifically, the optimization problem we want to solve is

$$\max_{\mathbf{a} \in \mathcal{A}} \sum_{n=1}^{N} u_n(\mathbf{a}). \tag{22}$$

We will study two distributed algorithms in which the participating users exchange price signals that indicate the "cost" or "benefit" that its action causes to the other users. Allocating network resources via pricing has been well-investigated for convex NUM problems [6], where the original NUM problem can be decomposed into distributedly solvable subproblems by setting price for each constraint resource, and each subproblem has to decide the amount of resources to be used depending on the charged price. However, unlike in the conventional convex NUM, pricing mechanisms may not be immediately applicable in ACSCG if the objective in (22) is not jointly concave in $\mathbf{a}$. Therefore, we are interested in characterizing the convergence condition of different pricing algorithms in ACSCG.

We know that for any local maximum $\mathbf{a}^*$ of problem (22), there exist Lagrange multipliers $\lambda_n, \nu_n^1, \cdots, \nu_n^N$ and $\nu_n'^1, \cdots, \nu_n'^N$ such that the following Karush-Kuhn-Tucker (KKT) conditions hold for all $n \in \mathcal{N}$:

$$\frac{\partial u_n(\mathbf{a}^*)}{\partial a_n^k} + \sum_{m \neq n} \frac{\partial u_m(\mathbf{a}^*)}{\partial a_n^k} = \lambda_n + \nu_n^k - \nu_n'^k, \ \forall n \tag{23}$$

$$\lambda_n \Big(\sum_{k=1}^{K} a_n^{k*} - M_n \Big) = 0, \ \lambda_n \geq 0 \tag{24}$$

$$\nu_n^k (a_n^{k*} - a_{n,k}^{\max}) = 0, \ \nu_n'^k (a_{n,k}^{\min} - a_n^{k*}) = 0, \ \nu_n^k, \nu_n'^k \geq 0. \tag{25}$$

Denote π_{mn}^k user m's marginal fluctuation in utility per unit decrease in user n's action a_n^k within the kth dimension

$$\pi_{mn}^k (a_m^k, \mathbf{a}_{-m}^k) = -\frac{\partial u_m(\mathbf{a})}{\partial a_n^k}, \tag{26}$$

which is announced by user m to user n and can be viewed as the cost charged (or compensation paid) to user n for changing user m's utility. Using (26), equation (23) can be rewritten as

$$\frac{\partial u_n(\mathbf{a}^*)}{\partial a_n^k} - \sum_{m \neq n} \pi_{mn}^k (a_m^{k*}, \mathbf{a}_{-m}^{k*}) = \lambda_n + \nu_n^k - \nu_n'^k. \tag{27}$$

If we assume fixed prices $\{\pi_{mn}^k\}$ and action profile $\mathbf{a}_{-n}^k$, condition (27) gives the necessary and sufficient KKT condition of the following problem:

$$\max_{\mathbf{a}_n \in \mathcal{A}_n} \ u_n(\mathbf{a}) - \sum_{k=1}^{K} a_n^k \cdot \Big(\sum_{m \neq n} \pi_{mn}^k \Big). \tag{28}$$

At an optimum, a user behaves as if it maximizes the differences between its utility minus its payment to the other users in the network due to its impact over the other users' utilities. Different distributed pricing mechanisms can be developed based on the individual objective function in (28) and the convergence conditions may also vary based on the specific action update equation.

We will investigate two distributed pricing mechanisms for non-convex ACSCG and provide two sufficient conditions that guarantee their convergence. Specifically, under these sufficient conditions, both algorithms guarantee that the total utility is monotonically increasing until it converges to a feasible operating point that satisfies the KKT conditions. Similarly as in Section 3.1, we first assume $f_n^k(\mathbf{a}_{-n})$ takes the form in (4) and users update their actions in parallel.

4.1 Gradient Play

The first distributed pricing algorithm that we consider is gradient play. The update iterations of gradient play need to be properly redefined in presence of real-time information exchange. Specifically, at stage t, users adopting this algorithm exchange price signals $\{\pi_{mn}^{k,t-1}\}$ using the gradient information at stage $t-1$. Within each iteration, each user first determines the gradient of the objective in (28) based on the price vectors $\{\pi_{mn}^{k,t-1}\}$ and its own utility function $u_n(\mathbf{a}_n, \mathbf{a}_{-n}^{t-1})$. Then each user updates its action a_n^t using gradient projection algorithm according to

$$a_n^{\prime k,t} = a_n^{k,t-1} + \kappa\Big(\frac{\partial u_n(\mathbf{a}_n, \mathbf{a}_{-n}^{t-1})}{\partial a_n^k} - \sum_{m\neq n} \pi_{mn}^{k,t-1}\Big). \tag{29}$$

and

$$\mathbf{a}_n^t = [a_n^{1,t} a_n^{2,t} \cdots a_n^{K,t}] = \Big[a_n^{\prime 1,t} a_n^{\prime 2,t} \cdots a_n^{\prime K,t}\Big]_{\mathcal{A}_n}^{\|\cdot\|_2}. \tag{30}$$

in which the stepsize $\kappa > 0$. The following theorem provides a sufficient condition under which gradient play will converge monotonically provided that we choose small enough constant stepsize κ.

Theorem 7. *If* $\forall n, k, \mathbf{x}, \mathbf{y} \in \mathcal{A}_{-n}$,

$$\inf_x \frac{\partial^2 h_n^k(x)}{\partial^2 x} > -\infty, \ \textit{and} \ \Big\| \bigtriangledown g_n^k(\mathbf{x}) - \bigtriangledown g_n^k(\mathbf{y})\Big\| \le L' \big\|\mathbf{x} - \mathbf{y}\big\|, \tag{C7}$$

gradient play converges for a small enough stepsize κ.

Proof: This theorem can be proved by showing the gradient of the objective function in (22) is Lipschitz continuous and applying Proposition 3.4 in [17]. See Appendix E in [19] for details. ■

Remark 7. (Application of condition (C7)) A sufficient condition that guarantees the convergence of distributed gradient projection algorithm is the Lipschitz continuity of the gradient of the objective function in (22). For example, in the power control problem in multi-channel networks [12], we have $h_n^k(x) = \log_2(\alpha_n^k + H_{nn}^k x)$ and $g_n^k(\mathbf{P}_{-n}) = \log_2(\sigma_n^k + \sum_{m\neq n} H_{mn}^k P_m^k)$. For this configuration, we can immediately verify that condition (C7) is satisfied. Therefore, gradient play can be applied. Moreover, as in [12], if we can further ensure that the problem in (22) is convex for some particular utility functions, gradient play converges to the unique optimal solution of (22) at which achieving KKT conditions implies global optimality.

4.2 Jacobi Update

We consider another alternative strategy update mechanism called Jacobi update [18]. In Jacobi update, every user adjusts its action gradually towards the best response strategy. Specifically, the maximizer of problem (28) takes the following form

$$B_n^{'k}(\mathbf{a}_{-n}) = \left\{\frac{\partial h_n^k}{\partial x}\right\}^{-1}\left(\lambda_n + \nu_n^k - \nu_n^{'k} + \sum_{m\neq n}\pi_{mn}^k\right) - \sum_{m\neq n}F_{mn}^k a_m^k, \tag{31}$$

in which λ_n, ν_n^k, and $\nu_n^{'k}$ are the Lagrange multipliers that satisfy complementary slackness in (24) and (25), and π_{mn}^k is defined in (26). In Jacobi update, at stage t, user n chooses its action according to

$$a_n^{k,t} = a_n^{k,t-1} + \kappa\left[B_n^{'k}(\mathbf{a}_{-n}^{t-1}) - a_n^{k,t-1}\right], \tag{32}$$

in which the stepsize $\kappa \in (0,1]$. The following theorem establishes a sufficient convergence condition for Jacobi update.

Theorem 8. *If* $\forall n, k, \mathbf{x}, \mathbf{y} \in \mathcal{A}_{-n}$,

$$\begin{array}{c}\inf_x \frac{\partial^2 h_n^k(x)}{\partial^2 x} > -\infty,\ \sup_x \frac{\partial^2 h_n^k(x)}{\partial^2 x} < 0,\ \textit{and} \\ \left\| \nabla g_n^k(\mathbf{x}) - \nabla g_n^k(\mathbf{y})\right\| \leq L'\left\|\mathbf{x}-\mathbf{y}\right\|,\end{array} \tag{C8}$$

Jacobi update converges if the stepsize κ *is sufficiently small.*

Proof: This can be proved using the descent lemma and the mean value theorem. The details of the proof are provided in Appendix F in [19]. ■

Remark 8. (Relation between condition (C8) and the result in [13]) Shi et al. considered the power allocation for multi-carrier wireless networks with non-separable utilities. Specifically, $u_n(\cdot)$ takes the form

$$u_n(\mathbf{P}) = r_i\left(\sum_{k=1}^{K}\log_2\left(1 + \frac{H_{nn}^k P_n^k}{\sigma_n^k + \sum_{m\neq n} H_{mn}^k P_m^k}\right)\right), \tag{33}$$

in which $r_i(\cdot)$ is an increasing and strictly concave function. Since the utilities are non-separable, the distributed pricing algorithm proposed in [13], which in fact belongs to Jacobi update, requires only one user to update its action profile at each stage while keeping the remaining users' action fixed. The condition in (C8) gives the convergence condition of the same algorithm in ACSCG. We prove in Theorem 7 that, if the utilities are separable, convergence can still be achieved even if these users update their actions at the same time. Therefore, we do not need an arbitrator to select the single user that updates its action at each stage.

5 Conclusion

In this paper, we propose and investigate a new game model, which we refer to as additively coupled sum constrained games, in which each player is subject to a sum constraint and its utility is additively impacted by the remaining users' actions. The convergence properties of various generic distributed adjustment algorithms, including best response, gradient play, and Jacobi update, have been investigated. The sufficient conditions obtained in this paper generalize the existing results developed in the multi-channel power control problem and can be extended to other applications that belong to ACSCG.

References

1. Fudenberg, D., Tirole, J.: Game Theory. MIT Press, Cambridge (1991)
2. Rosen, J.: Existence and uniqueness of equilibrium points for concave n-person games. Econometrica 33, 520–534 (1965)
3. Topkis, D.M.: Supermodularity and Complementarity. Princeton Univ. Press, Princeton (1998)
4. Yao, D.: S-modular games with queueing applications. Queueing Syst. 21, 449–475 (1995)
5. Altman, E., Altman, Z.: S-modular games and power control in wireless networks. IEEE Trans. Automatic Control 48(5), 839–842 (2003)
6. Chiang, M., Low, S.H., Calderbank, A.R., Doyle, J.C.: Layering as optimization decomposition. Proceedings of the IEEE 95, 255–312 (2007)
7. Palomar, D.P., Chiang, M.: A tutorial on decomposition methods for network utility maximization. IEEE JSAC 24(8), 1439–1451 (2006)
8. Yu, W., Ginis, G., Cioffi, J.: Distributed multiuser power control for digital subscriber lines. IEEE JSAC 20(5), 1105–1115 (2002)
9. Chung, S.T., Seung, J.L., Kim, J., Cioffi, J.: A game-theoretic approach to power allocation in frequency-selective Gaussian interference channels. In: Proc. IEEE Int. Symp. on Inform. Theory, p. 316 (June 2003)
10. Cendrillon, R., Huang, J., Chiang, M., Moonen, M.: Autonomous spectrum balancing for digital subscriber lines. IEEE Trans. on Signal Process. 55(8), 4241–4257 (2007)
11. Scutari, G., Palomar, D.P., Barbarossa, S.: Optimal linear precoding strategies for wideband noncooperative systems based on game theory - Part II: Algorithms. IEEE Trans. Signal Process. 56(3), 1250–1267 (2008)
12. Huang, J., Berry, R., Honig, M.: Distributed interference compensation for wireless networks. IEEE JSAC 24(5), 1074–1084 (2006)
13. Shi, C., Berry, R., Honig, M.: Distributed interference pricing for OFDM wireless networks with non-seperable utilities. In: Proc. of Conference on Information Sciences and Systems (CISS), March 19-21, pp. 755–760 (2008)
14. Lasaulce, S., Debbah, M., Altman, E.: Methodologies for analyzing equilibria in wireless games. IEEE Signal Process. Magazine 26, 41–52 (2009)
15. Bulow, J., Geanakoplos, J., Klemperer, P.: Multimarket oligopoly: strategic substitutes and strategic complements. Journal of Political Economy 93, 488–511 (1985)
16. Mo, J., Walrand, J.: Fair end-to-end window-based congestion control. IEEE Trans. on Networking 8(5), 556–567 (2000)
17. Bertsekas, D.P., Tsitsiklis, J.N.: Parallel and Distributed Computation. Prentice Hall, Englewood Cliffs (1997)
18. La, R., Anantharam, V.: Utility based rate control in the internet for elastic traffic. IEEE/ACM Trans. Networking 10(2), 271–286 (2002)
19. Su, Y., van der Schaar, M.: Structural Solutions For Additively Coupled Sum Constrained Games, UCLA Tech. Rep. (2010), http://arxiv.org/abs/1005.0880