Paris Metro Pricing for Internet Service Differentiation

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Abstract. This paper analyzes the Paris Metro Pricing (PMP) strategy for differentiating Internet service. PMP has several advantages over other pricing schemes that guarantee quality of service (QoS) such as simplicity and less bandwidth overhead. In this paper, we develop a simple analytical model for PMP. We first assume that there is only one network service provider (a monopolist) serving N users and model the user behavior and the provider's profit. Then we derive the optimal ratio of dividing a given network capacity in order to maximize the profit of the service provider. Our results show that, by maximizing providers profit, the subscription is also maximized which can be interpreted as a higher satisfaction of users compared to that of not using PMP. In addition, by taking into account various network types, we show that in a monopoly environment, it is always better to implement PMP regardless of user populations we considered. We then further extend our model to a duopoly setting. We found that there exist no Nash equilibrium even when both providers do not differentiate the network service.

Keywords: Internet services, price discrimination, Paris Metro Pricing, revenue maximization, user subscription, Nash equilibrium.

1 Introduction

The usage of the *Internet* was dominated by traditionally 'typical' data services (e.g., e-mail, web browsing, file transfer, etc.). These services do not require severe bandwidth overhead on the network since the traffic generated by those applications usually tolerate relatively large packet delays. However, as we clearly witness these days, new internet applications such as VoIP, IPTV, and many smart phone applications can be charaterized as delay-constrained (or delay-sensitive), and thus require higher requirements. In addition, by 2014, mobile data traffic will double every year through 2014, increasing 39 times between 2009 and 2014, according to Cisco forecast [1]. This increase of diverse quality of service (QoS) requirements gives the rationale to develop new methods capable of treating the delay-constrained and non-delay-constrained traffic differently. A

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possible solution is to give priority to the delay-constrained traffic in the queues of the network [2] but without an appropriate *pricing* scheme, any prioritization is useless [3].

In this paper, we study the use of pricing mechanism called Paris Metro Pricing (PMP) proposed by Odlyzko [4]. Under the PMP scheme, the network is split into subnetworks. The tariff for each subnetwork is different, expecting a lower congestion for highly priced networks. This method does not offer any QoS guarantees, so that it is somewhat weak compared to several other approaches. However, due to its simplicity, it is, indeed, very attractive to many practitioners.

In this regard, a few papers have examined the PMP scheme for charging packet networks. In [5], the authors present a mathematical model in which all packets are generated by the same kind of application and all users have the same valuation of QoS. Even though they have showed the existence and uniqueness of the stability using queueing theory, the assumptions seem too strong to have practical implications. Paper [6] is the closest work to ours in that, based on their proposed model, they tried to show whether a network service provider (a single-constrained monopolist) can be profitable using PMP strategy. However, there are several differences with our study. First, although they modeled the user's satisfaction as the utility function they only showed the surplus from the providers perspective. In addition, the results they showed were merely for uniform distribution. Most of the papers in which PMP is analyzed take the uniform distribution of users as one of the assumptions for their model. However, in [7] the authors pointed out that it is required to assess the importance of those assumptions. In this spirit, we adopt a basic model from [8] and further extend it so as to analyze the economic aspects of this pricing scheme and to address the limitations of the existing papers.

The rest of the paper is organized as follows. In Section 2 we develop a simplified model to reflect both the provider revenue and user behavior. In section 3 we illustrate the impact of PMP on revenue and user subscription with several user distribution. In the subsequent section we analyze the PMP when two firms compete for the same set of users (duopoly). Section 5 concludes the paper.

2 System Model

2.1 Single Pricing

In this section, we develop a system model based on [8] with one network which will be used throughout the paper. Consider a communication system with a large population of N users each charaterized by a type θ that is an independent random variable distributed in $[0, 1]$ with $f(\theta)$ and $F(\theta)$ as its probability density function and cumulative distribution function respectively. A user of type θ finds the network connection acceptable if the number of users X using the network and the price p are such that

$$\frac{X}{N} \leq 1 - \theta \quad \text{and} \quad p \leq \theta$$

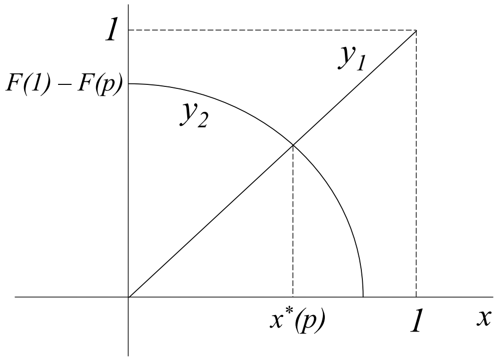


Fig. 1. The two functions y_1 and y_2 for the illustration of solution $x^*(p)$ of (2)

In this expression, N is the capacity of the network. Due to our analysis purpose, note that the capacity in the model is different from the conventional one that uses *bps*. Also, we will later divide the network into two subnetworks, each with capacity $N/2$. The expression interprets that a user with a large value of θ is willing to pay quite a lot for the connection but he expects a low utilization (or congestion) for a high quality of service. Conversely, a user with a small value of θ does not want to pay much for his connection but is willing to tolertate high delays. For example, we can view the users with large θ as users of VoIP and those with small θ as web browsers.

Assume that the network connection price is $p \in (0, 1)$. If the number of users in the network at a certain time period is X, then a user of type θ connects if the inequalities above are satisfied, i.e., if $\theta \in [p, 1 - X/N]$. Since θ is distributed in $[0, 1]$, the probability that a random user connects is $[F(1 - X/N) - F(p)]^+$. Accordingly, the number X of users that connect is binomial with mean $N \times [F(1 - X/N) - F(p)]^+$, so that

$$\frac{X}{N} \approx \left[F(1 - \frac{X}{N}) - F(p)\right]^+ \tag{1}$$

by the law of large numbers, since N is large. By letting $x = X/N$, (1) can be expressed as

$$x = [F(1 - x) - F(p)]^+ \tag{2}$$

Let the left-hand side and the right-hand side of (2) be $y_1(x)$ and $y_2(x)$ respectively. Then, for a fixed p, the solution of (2) can be illustrated as in Figure 1.

Since y_1 is increasing and y_2 is non-increasing, with both functions sharing the same domain of $[0, 1]$, a unique solution $x(p)$ exists. By taking the derivative of (2) with respect to p we have

$$\frac{dx}{dp} = -\frac{f(p)}{1 + f(1 - x)} \; < \; 0$$

which shows that x is a decreasing function of p.

Now, we solve the provider revenue ($R(p)$) maximization problem. In other words, we find the price p that maximizes the product of the number of users of the network and the price p, that is

$$p^* = \arg\max_p R(p) = p \cdot x(p) \tag{3}$$

Since $R(p)$ is a continuous function with $R(0) = 0$ and $R(1) = 0$, there exists at least one solution to the above maximization problem. Generally, it is nontrivial to find a closed form solution for $x(p)$, hence, for solving (3) a numerical method is performed. Nevertheless, to illustrate, we show a simple example when θ is uniformly distributed in $[0, 1]$. Equation (1), then, reduces to

$$\frac{X}{N} \approx \left(1 - \frac{X}{N} - p\right)^+$$

Solving this expression we find that $x := X/N = (1 - p)/2$. The operator can maximize his revenues by choosing the value of p that maximizes $px = p(1-p)/2$. The maximizing price is $p = 1/2$ and the corresponding value of px is $1/8$, which measures the revenue divided by N[1].

2.2 Differentiated Pricing

Consider now a situation where Paris Metro Pricing is applied. The network service provider divides the network into two subnetworks, each with different capacity. That is, there exist two networks: network 1 with price p_1 and capacity $N\alpha_1$ and network 2 with price p_2 and capacity $N\alpha_2$ (where $\alpha_1 + \alpha_2 = 1$). Without loss of generality, we assume $p_1 > p_2$. The users will select one of the two networks, based on the prices and utilizations. A user joins if there is an acceptable network and he chooses the cheapest network if both are acceptable. Moreover, if both networks are acceptable and, by any chance, have the same price, a user will join the one with the smallest utilization because it offers a marginally better QoS.

If the number of users in the two networks are X_1 and X_2 respectively, then a user of type θ chooses network 2 if $X_2/(N\alpha_2) \leq 1-\theta$ and $p_2 \leq \theta$. The probability that θ falls between p_2 and $1 - X_2/(N\alpha_2)$ is then

$$\left[F(1 - \frac{X_2}{N\alpha_2}) - F(p_2)\right]^+$$

Then, $x_2 := X_2/N$ is given by

$$x_2 = \left[F(1 - \frac{x_2}{\alpha_2}) - F(p_2)\right]^+$$

[1] For merely flat pricing, due to its simplicity, we were able to get results from curvilinear distributions (beta distribution). The pdf of the beta distribution is defined as $f(x, \alpha, \beta) = x^{\alpha-1}(1 - x)^{\beta-1} / \int_0^1 u^{\alpha-1}(1 - u)^{\beta-1}\, du$ and the results (maximum revenue) for some instances are as follows: 0.1045 ($\alpha = 0.5$, $\beta = 0.5$), 0.1176 ($\alpha = 2$, $\beta = 5$), 0.1620 ($\alpha = 5$, $\beta = 2$), 0.1488 ($\alpha = 2$, $\beta = 2$), and 0.1610 ($\alpha = 3$, $\beta = 3$).

A user will opt for network 1 if $X_1/(N\alpha_1) \leq 1-\theta$, $p_1 \leq \theta$ and $X_2/(N\alpha_2) > 1-\theta$. Thus, we find that $x_1 := X_1/N$ is such that

$$x_1 = \left[F(1 - \frac{x_1}{\alpha_1}) - F(\max\{p_1, 1 - \frac{x_2}{\alpha_2}\})\right]^+$$

To determine the prices p_1 and p_2 that maximize the revenue of the operator, one needs to maximize the total revenue R obtained by both two subnetworks over p_1 and p_2, mathematically it is equivalent to solving the following problem

$$R = \max_{p_1,p_2} \; p_1x_1 + p_2x_2$$

By solving the problem when the network capacity is divided exactly half (again with uniform distribution of θ), one can show that the maximum occurs for $p_1 = 8/13$ and $p_2 = 11/26$ and that the maximum revenue equals to $25/156$. Eventually, we may conclude that the service differentiation with Paris Metro Pricing increases the revenue from $1/8$ to $25/156$, or by 28.2%. This can be one good rationale to use PMP instead of not using it. In this regard, we show, from another perspective, that it is recommended to use PMP. The provider who is willing to use PMP will want to know how much additional capacity is needed (without using PMP) in order to have as much as revenue obtained from using PMP (In this analysis we assume that the cost of increasing the network capacity is considerable).

Proposition 1. *PMP gives an increase in revenue by 28.2% which is equivalent to the amount when using only one network (and not using PMP) and increasing the network capacity by 78.6% given the original capacity N*

Proof. Let the increased capacity be $N(1+\beta)$. Considering the new increased capacity gives

$$\frac{X}{N} \approx \left(1 - \frac{X}{N(1+\beta)} - p\right)^+$$

solving for x, we have

$$x = \frac{1+\beta}{2+\beta}(1-p)$$

then solving the optimization problem (3), we have the optimal revenue with extended capacity as R_{ext}. By equating this with the optimal revenue obtained by PMP, R_{pmp} gives corresponding β as

$$R_{ext} = \frac{1}{4}\left(\frac{1+\beta}{2+\beta}\right) = \frac{25}{156} = R_{pmp}$$

$$\beta = \frac{44}{56}$$

3 Impact of PMP on Revenue and User Subscription

Since we have seen the price that maximizes the provider's revenue and its corresponding revenue with θ following uniform distribution, we will now see how these values as well as the fraction of users joining the network change with more general cdfs of θ. The rationale behind this is that uniform distribution alone cannot represent various type of user population. For instance, there might be a community of users in which the majorities use real-time services. On the other hand, one user group might consist of majority of people requiring non-delay-constrained services (e.g., email). Therefore, we need to represent these typical user distributions and incorporate them in the study of analyzing the impact of PMP.

In this regard, we define three additional probability distributions (f_1, f_3 and f_4) that capture three typical user distributions as follows (figure 2):

$$f_i(\theta) = \begin{cases} 2 - 2\theta & \text{if} \quad i = 1 \\ 1 & \text{if} \quad i = 2 \\ 2\theta & \text{if} \quad i = 3 \end{cases}$$

where $\theta \in [0, 1]$ for all three distributions. Also,

$$f_4(\theta) = \begin{cases} 4\theta & \text{if} \quad 0 \leq \theta \leq \frac{1}{2} \\ 4 - 4\theta & \text{if} \quad \frac{1}{2} < \theta \leq 1 \end{cases}$$

The first pdf (f_1) comes from a general concave cdf of θ and stands for a network consisting of high population of users generating non-delay-constrained traffic . The second function comes from a general linear cdf of θ and exactly corresponds to the uniform distribution and represents a well balanced network of users. The third one (f_3) is a pdf that represents a convex cumulative distribution function indicating the networks where the majority of users require high QoS levels. The last one (f_4) is a pdf of a 'S-shaped' cdf and designates a network with majority of users requiring intermediate QoS levels. We will denote the network environment for each user population type as NT1, NT2, NT3, and NT4 henceforth.

To see whether PMP actually gives benefits to both providers and users we, first, have to check whether it is always better irrespective of network environment. Secondly, even though this is true, we also have to check whether the average surplus of using PMP over every network environment is better than the value when it is not used. This comes from the fact that the type of user population will change over time since users consisting the network will require different service types at different time segment of interest. In light of this, we derive the maximum revenue and the corresponding user participation for each of the four user distributions. We will denote $\widehat{R}^1_{max}$, $\widehat{R}^2_{max}$, $\widehat{R}^3_{max}$ and $\widehat{R}^4_{max}$, and $\widehat{\alpha}^1_2$, $\widehat{\alpha}^2_2$, $\widehat{\alpha}^3_2$ and $\widehat{\alpha}^4_2$ as the maximum revenues and their corresponding α_2 values achieved by using PMP scheme for the distributions f_1, f_2, f_3 and f_4

3.1 Uniform Distribution

For the uniform distribution, we could express $\widehat{R}^2_{max}$, p_1, p_2, x_1, and x_2 with respect to just one variable α_2 after some manipulation. Each of these values

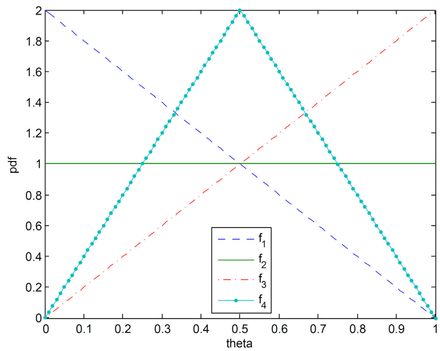


Fig. 2. The pdfs of θ representing each different network environment

are illustrated in Figure 3 and we find that the maximum revenue as well as the corresponding α_2 is as follows:

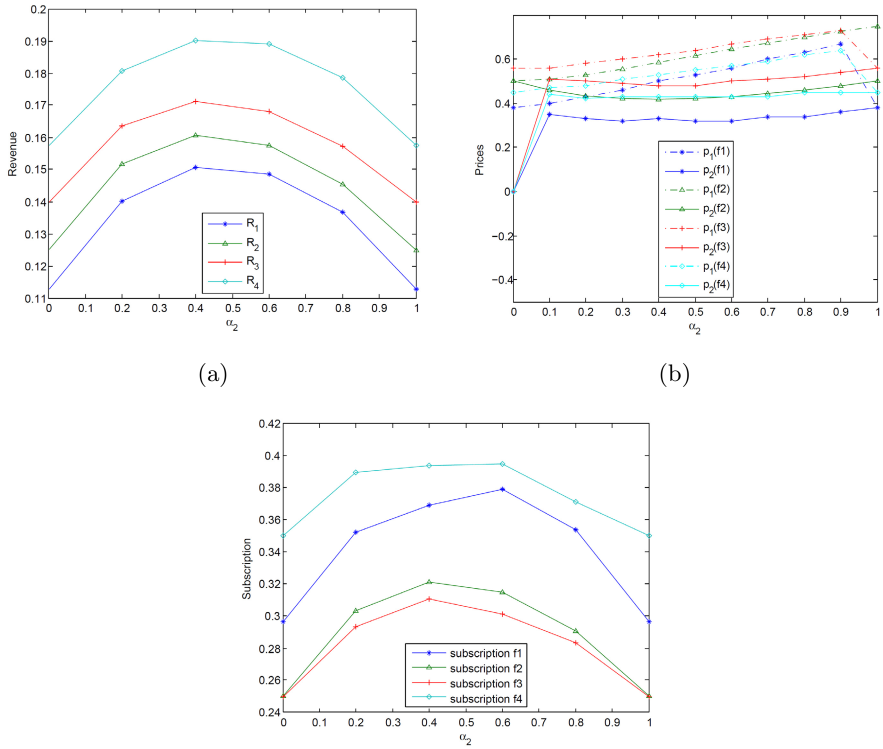
$$\widehat{R}^2_{max} = 0.1608 \ , \quad \widehat{\alpha}^2_2 = 0.43$$

In this example, the revenue increases by 28.6% compared to flat pricing. We could notice, from this analysis, that α_2 which maximizes revenue is in fact slightly less than 0.5. This tells us that when using two subnetworks to apply PMP, dividing it exactly half may not be the best choice for the network service provider. Also, since the range of α_2 that results in the revenue within 90% of the maximum value covers about 68% ($\geq$ 50%) of its whole interval, we could say that the revenue is not that sensitive to the ratio of dividing a network.

Figure 3c shows how the network utilization changes as α_2 increases. As you see from the figure, the total network utilization (denoted as subscription f2 in Fig. 3c) is at maximum when $\alpha_2 = 0.43$. This is the exact α_2 value that maximizes the total revenue. This implies that, when the network service provider maximizes his own profit, the network utilization is also maximized which is naturally true since the more users join the network the more revenue can be achieved (however, this does not imply that those two values have to be exactly the same, instead it means they will tend to have similar values). Moreover, it is easy to see from Figure 1c that whatever the value of α_2 is, PMP strategy always results in higher utilization than not performing it.

Non-uniform Distributions. In effect, if the user population is not represented by a uniform distribution, it is difficult to derive the maximum revenue of the service provider when PMP is used. This is due to the fact that it involves a maximization problem over three variables of α_2, p_1, and p_2. The results for the three non-uniform user distributions is illustrated in figure 3 and we find the maximum revenues and corresponding α_2 values as follows (for the derivation of all the values for uniform and non-uniform distributions see [9]):

$$\widehat{R}^1_{max} = 0.1512, \quad \widehat{\alpha}^1_2 = 0.45$$



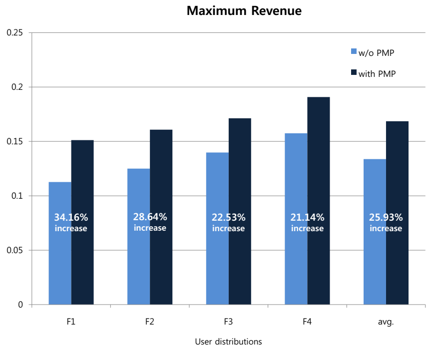
(a) (b)

(c)

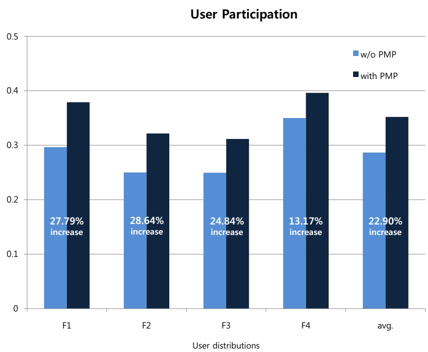
Fig. 3. Numerical results for the three non-uniform distribution cases (including that of uniform distribution) : (a) Revenues with respect to α_2 for each user distribution, (b) Prices with respect to α_2 for each user distribution, (c) User participation with respect to α_2 for each user distribution

$$\widehat{R}^3_{max} = 0.1713, \qquad \widehat{\alpha}^3_2 = 0.45$$
$$\widehat{R}^4_{max} = 0.1908, \qquad \widehat{\alpha}^4_2 = 0.50$$

From the previous section we have seen that when the network consists of a well-balanced users, the capacity ratio between two subnetworks is not that sensitive as long as it is close enough to the optimal ratio. We, now, try to see whether this is also true for the rest of the three network types. The sensitivity of the total revenue with respect to α_2 can be derived from the R-α_2 plots for each of the three non-uniform distributions (Figure 3a). That is, we would like to see how the revenue changes when the ratio of the capacities between two subnetwork is slightly altered from the optimal value (α_2^*). From Figure 3 we could see that the range of α_2 that produces the revenue within 90% of maximum value spans 65%, 74%, and 77% of its overall interval for NT1, NT3 and NT4 respectively. Thus, with the sensitivity of NT2 we could say that the revenue achieved in the



(a) Maximum revenue



(b) User subscription

Fig. 4. Maximum revenues and user subscription for each user distribution

environment NT1 is relatively sensitive to the ratios of its subnetwork capacity which should be considered for a provider when its servicing environment is of NT1 most of the time. However, in general (and on average), determining the proportion of the subnetworks is not that critical.

Comparing with the maximum revenues when PMP is not used, we eventually see that no matter what type of user population may be, using PMP always produces a higher revenue for the network service providers (Figure 4a). Similarly, PMP scheme is always favored by network users, regardless of the characteristics of user groups as shown in Figure 4b. The users' satisfaction is measured by the number of users who join the network (user participation) because, based on our system model, only the satisfied users receive the network services. In summary, the average values of 4 network environment between using PMP and not using PMP clearly show that PMP is, indeed, a superior pricing scheme in terms of both the revenue and network utilization (user participation).

Observation 1. *In a network where only one network service provider (a monopolist) provides internet service, it is always better for the provider to use Paris Metro Pricing scheme than to just provide a general service with flat pricing. In addition, PMP scheme is always favored by the users consisting the network.*

Observation 2. *For a network service provider to achieve the maximum revenue, determining the fraction of the capacity of the subnetworks is less critical for all of the network environment.*

4 Competition: Duopoly

In previous sections, we have noticed that a single network service provider (a monopolist) will have an advantage for using PMP scheme. Then one might ask, "what happens when there are two network service providers? Does the Nash equilibrium even exist for the duopoly case?" In this section, we try to answer this question by modeling the situation using game theory.

4.1 Three Prices

We first analyze the situation where there are two service providers with only one using PMP. That is, one ISP (firm A) provides the service with a single price p_a and the other ISP (firm B) uses PMP with the lower price p_{b1} and the higher price p_{b2}. Both the competing firms have equal capacity where firm B splits the two subnetworks exactly half. Then, with this situation, there are 5 possible cases as follows:

1. $p_a < p_{b1} < p_{b2}$
2. $p_{b1} < p_a < p_{b2}$
3. $p_{b1} < p_{b2} < p_a$
4. $p_a = p_{b1} < p_{b2}$
5. $p_{b1} < p_a = p_{b2}$

Let R_A and R_B be the revenue (function) of the two firms respectively. From the similar derivation as in section 2 we can obtain the revenue functions of both firms for each of the 5 cases as follows:

1. $R_A = \frac{2}{3}(p_a - p_a^2)$,
   $R_B = p_{b1}\min(\frac{1-p_{b1}}{2}, \frac{1-p_a}{6}) + p_{b2}\min(\frac{1-p_{b2}}{2}, \frac{1}{2}\min(\frac{1-p_{b1}}{2}, \frac{1-p_a}{6}))$
2. $R_A = p_a\frac{2}{3}\min(1-p_a, \frac{1-p_{b1}}{2})$,
   $R_B = \frac{p_{b1}(1-p_{b1})}{2} + p_{b2}\frac{1}{2}\min(1-p_{b2}, \frac{2}{3}\min(1-p_a, \frac{1-p_{b1}}{2}))$
3. $R_A = p_a\frac{2}{3}\min(1-p_a, \frac{1}{2}\min(1-p_{b2}, \frac{1-p_{b1}}{2}))$,
   $R_B = \frac{p_{b1}(1-p_{b1})}{2} + p_{b2}\frac{1}{2}\min(1-p_{b2}, \frac{1-p_{b1}}{2})$
4. $R_A = \frac{1}{2}(p_a - p_a^2)$,
   $R_B = \frac{p_{b1}(1-p_{b1})}{4} + p_{b2}\frac{1}{2}\min(1-p_{b2}, \frac{3(1-p_{b1})}{4})$

5. $R_A = p_a \frac{1}{2}\min(1-p_a, \frac{1-p_{b1}}{2})$,
   $R_B = \frac{p_{b1}(1-p_{b1})}{2} + p_{b2}\frac{1}{4}\min(1-p_{b2}, \frac{1-p_{b1}}{2})$

Let R_A^* and R_B^* be the optimal revenue for each firm and p_a^*, p_{b1}^*, and p_{b2}^* be the corresponding prices. Then the values for each of the 5 cases are summarized in table 1.

Based on the results, we try to find the Nash equilibrium price vector $(\bar{p}_a, \bar{p}_{b1}, \bar{p}_{b2})$ such that

$$R_A(\bar{p}_a, \bar{p}_{b1}, \bar{p}_{b2}) \geq R_A(p_a, \bar{p}_{b1}, \bar{p}_{b2}) \quad \forall p_a$$

and

$$R_B(\bar{p}_a, \bar{p}_{b1}, \bar{p}_{b2}) \geq R_B(\bar{p}_a, p_{b1}, p_{b2}) \quad \forall p_{b1}, p_{b2}$$

However, one can verify that there are no such vector and conclude that there exist no Nash equilibrium with three prices case.

Table 1. Optimal revenues and prices for firms A and B (three prices)

Case	p_a^*	p_{b1}^*	p_{b2}^*	R_A^*	R_B^*
1)	1/2	5/6	11/12	0.167	0.108
2)	3/4	1/2	5/6	0.125	0.194
3)	17/20	2/5	7/10	0.085	0.225
4)	1/2	1/2	5/8	0.125	0.180
5)	3/4	1/2	3/4	0.090	0.172

Table 2. User subscription for firms A and B (three prices)

Case	x_a^*	x_{b1}^*	x_{b2}^*
1)	1/3	1/12	1/24
2)	1/6	1/4	1/12
3)	1/10	3/10	3/20
4)	3/8	3/8	3/16
5)	3/16	1/4	3/16

4.2 Four Prices

We now consider the situation where the two firms both go for PMP scheme. In other words, firm A provides a differentiated service with two prices p_{a1} and $p_{a2} (> p_{a1})$ and, similarly, firm B divides the network into two subnetworks with prices p_{b1} and $p_{b2} (> p_{b1})$. Now, due to the symmetry, we need only consider 6 cases which are

1. $p_{b1} < p_{b2} < p_{a1} < p_{a2}$
2. $p_{b1} < p_{a1} < p_{b2} < p_{a2}$

3. $p_{a1} < p_{b1} < p_{b2} < p_{a2}$
4. $p_{b1} = p_{a1} < p_{b2} < p_{a2}$
5. $p_{b1} < p_{a1} < p_{b2} = p_{a2}$
6. $p_{b1} = p_{a1} < p_{b2} = p_{a2}$

And one can find the revenue functions of both firms for the three cases as follows:

1. $R_A = p_{a2}\frac{1}{2}\min(1-p_{a2}, x_{a1}) + p_{a1}\min(\frac{1-p_{a1}}{2}, \frac{1-p_{b2}}{4})$
   $R_B = p_{b2}\frac{1-p_{b2}}{2} + p_{b1}\frac{1-p_{b1}}{2}$
2. $R_A = p_{a2}\frac{1}{2}\min(1-p_{a2}, x_{b2}) + p_{a1}\min(\frac{1-p_{a1}}{2}, \frac{x_4}{2})$
   $R_B = p_{b2}\frac{1}{2}\min(1-p_{b2}, x_2) + p_{b1}\frac{1-p_{b1}}{2}$
3. $R_A = p_{a2}\frac{1}{2}\min(1-p_{a2}, x_{b2}) + p_{a1}\frac{1-p_{a1}}{2}$
   $R_B = p_{b2}\frac{1}{2}\min(1-p_{b2}, x_{b1}) + p_{b1}\frac{1}{2}\min(1-p_{b1}, x_{a1})$
4. $R_A = p_{a2}\frac{1}{2}\min(1-p_{a2}, x_{b2}) + p_{a1}(\frac{1-p_{a1}}{3})$
   $R_B = p_{b2}\frac{1}{2}\min(1-p_{b2}, x_{b1}) + p_{b1}\frac{1-p_{b1}}{3}$
5. $R_A = p_{a2}\frac{1}{3}\min(1-p_{a2}, x_{a1}) + p_{a1}\min(\frac{1-p_{a1}}{2}, \frac{x_4}{2})$
   $R_B = p_{b2}\frac{1}{3}\min(1-p_{b2}, x_2) + p_{b1}\frac{1-p_{b1}}{2}$
6. $R_A = p_{a2}\frac{1}{3}\min(1-p_{a2}, x_{a1}) + p_{a1}(\frac{1-p_{a1}}{3})$
   $R_B = p_{b2}\frac{1}{3}\min(1-p_{b2}, x_{b1}) + p_{b1}\frac{1-p_{b1}}{3}$

where x_{a2}, x_{a1}, x_{b2}, and x_{b1} are the number of users in the networks in which the prices are p_{a2}, p_{a1}, p_{b2}, and p_{b1} divided by the capacity respectively and are as follows:

$$x_{b1} = \frac{1}{2}(1-p_{b1}),\ x_{b2} = \min(\frac{1-p_{b2}}{2}, \frac{x_{b1}}{2})$$

$$x_{a1} = \min(\frac{1-p_{a1}}{2}, \frac{x_{b2}}{2}),\ x_{a2} = \min(\frac{1-p_{a2}}{2}, \frac{x_{a1}}{2})$$

For each of the three cases, the optimal revenues (R_A^*, R_B^*) and corresponding prices (p_{a2}^*, p_{a1}^*, p_{b2}^*, p_{b1}^*) are shown in table 3.

Table 3. Optimal revenues and prices for firms A and B (four prices)

Case	p_{a2}^*	p_{a1}^*	p_{b2}^*	p_{b1}^*	R_A^*	R_B^*
1)	0.93	0.86	0.7	0.4	0.0927	0.225
2)	0.93	0.75	0.88	0.5	0.1217	0.1778
3)	0.94	0.5	0.88	0.76	0.1532	0.144
4)	0.84	0.5	0.67	0.5	0.1505	0.1943
5)	0.86	0.7	0.86	0.4	0.1453	0.2007
6)	0.67	0.5	0.67	0.5	0.1578	0.1578

Similar to the three prices case, we observe that there are no Nash equilibrium price vector for this case. Thus, in the duopoly model, we conclude that there

Table 4. User subscription for firms A and B (four prices)

Case	x^*_{a2}	x^*_{a1}	x^*_{b2}	x^*_{b1}
1)	7/200	7/100	3/20	3/10
2)	3/100	1/8	3/50	1/4
3)	3/100	1/4	3/50	3/25
4)	1/12	1/3	1/6	1/3
5)	1/10	3/20	1/10	3/10
6)	2/9	1/3	2/9	1/3

are no equilibria based on the solution concept of Nash equilibrium when both firms use PMP[2]. Moreover, it is interesting to notice that even when both firms decide to provide services with single pricing there are no Nash equilibria [8].

4.3 Discussion

It is also interesting to notice that the result is model-dependent. In a related literature, Gibbens et al. [7] showed that if there are more than one firm (in the market) then they do not differentiate their networks in equilibrium. That is, in a duopoly setting, the result which states that there is no equilibrium with PMP is same as ours but the fact that there is indeed an equilibrium with single prices is different.

The nonexistence of equilibria does not mean that there is no incentive to differentiate the prices in the real-world duopoly setting[3]. For instance, assume there are two providers each dividing the market share equally with single prices. If firm A's price (p_A) is lower than firm B's price (p_B), clearly, firm A will increase its profit by introducing another price, say p_{A2}, around p_B to take away some of firm B's market share. On the other hand, firm B will introduce a lower price, say p_{B2}, around p_A to maximize its profit. Therefore, in reality, we might observe the situation where there are 2 providers each with 2 prices.

5 Conclusion

In this paper, we developed a model for Paris Metro Pricing strategy and demonstrated the profit incentive for a NSP to use PMP in a variety of scenarios. In particular, we analyzed the consumer behavior under PMP by allowing the model to define each users' condition which, when satisfied, they opt for joining the network. We evaluated the revenue of using PMP when there is a single provider (a monopolist) and determined the optimal fraction of the two subnetworks to be divided in order to maximize the profit. We have seen that using PMP, indeed,

[2] We have analyzed the cases with the concept of ϵ-equilibrium as well but the result did not change.

[3] Here we assume that, in the real world, ISPs cannot change its level of service price freely over time due to political and social reasons.

increases both the revenue and subscription. Also, we have looked at a competition setting where two NSPs provide PMP. As it turned out, we noticed that in a duopoly case there exist no (pure) Nash equilibrium even when the duopolists go for single pricing. As directions for future research, we wish to investigate more deeply the duopoly case where providers are capable to differentiate their services other than just PMP scheme.

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