Coalition Stability under QoS Based-Market Segmentation

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Abstract. N independent sources choose their provider depending on the perceived costs associated with each provider. The perceived cost is the sum of the price and quality of service proposed by the provider coefficiented by the source sensitivity to the quality of service. The source chooses the smallest cost provider or refuses to subscribe if all the perceived costs are above her maximum admissible opportunity cost. First, we detail the market segmentation between the providers as function of the quality of service sensitivity. Then, we prove that in case where coalitions emerged and under defensive equilibria, the game characteristic function would be submodular meaning that the Shapley value would be a fair and stable way to share the grand coalition revenue.

Keywords: cooperative game, Shapley value.

1 Introduction

The inter-carrier network of Internet that interconnects different operator domains is a wide and diversified network in constant growth in terms of number of domain operators. These operators have to manage complex technical and economic interactions. In this context, the different operators first provide (and sell) network services needed to ensure the level of performance required by the end-user applications. The inter-carrier network is thus today a technico-economic system in which competition and inter-dependencies prevail. The study presented in this paper is a contribution to the FP7 project ETICS that aims at creating a new ecosystem of innovative QoS-enabled interconnection models between Network Service Providers allowing for a fair distribution of revenue shares among all the actors of the service delivery value-chain.

The objective of this paper is to evaluate the benefit of network providers acting on a same market when they have together a privileged partnership in terms of economic alliance. As an example of network scenario in this context, we briefly describe here the Game as a Service scenario studied in the context of ETICS (see Figure 1). We consider a set of network providers, each one proposing a catalogue of on-line games to final users (such providers act as service retailers). The games they propose are developed and managed by cloud providers and require network transport services. In

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the context of the present article, cloud providers and network services are considered as cost impacts for service retailers, and these costs can be shared by some of these retailers if they choose to be member of a same alliance.

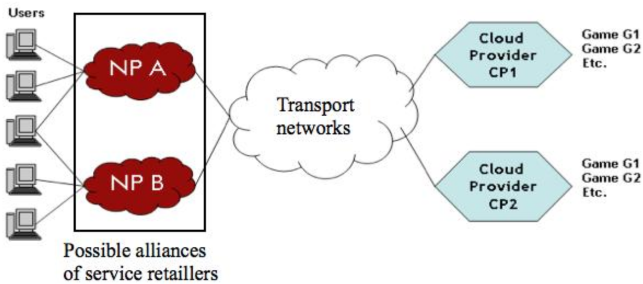

Fig. 1. "Game as a service" scenario

Most articles in economics assume that the providers have a fixed consumer basis i.e., that no churn is possible, or that demand is simply a linear function of price. In the rapid growing literature on revenue management, one of the most important issues is how to model provider demand forecasting. Demand is usually represented as an explicit/implicit function of price and the root tactic upon which revenue management is based is to change prices dynamically to maximize immediate or short-run revenue [9]. Kwon et al. propose a novel approach in [5] where non-cooperative providers learn dynamically their demand which is governed by a continuous time differential equation based on an evolutionary game theory perspective. However, the learning process requires data to efficiently update its forecastings. The problem is that in most real systems, data are missing or even altered by noise or measurement errors. The idea of our article is to take into account the source perception of both the prices and quality of service of the providers, and also their capacity to churn from one provider to the others or even, to refuse to subscribe. Moreover, uncertainty on the providers' knowledge about the sources' preferences is incorporated by assuming that their sensitivity to the quality of service is distributed according to a density function.

Taking into account the individual source preferences, the article shows that providers have always interest to cooperate through a grand coalition, provided the coalition revenue is shared according to the Shapley value, and that this is a stable organization of the market. Besides, most articles in the economic literature are restricted to monopolistic or duopolistic cases of competition [4]; on the contrary, our article is extended to an arbitrary large number of interacting providers using specific game properties.

The paper is organized as follow. The game is described in Section 2: first the relations between two interacting providers are analyzed in Section 2.1, then a two provider game is solved in Section 2.2 and generalized to the case of three providers in Section 2.3. Finally, the game is extended to the n provider case in Section 3: the market segmentation is determined in Section 3.1 and we prove that the characteristic function associated with the cooperative game is submodular, implying that the grand coalition should remain stable with the Shapley value as revenue sharing mechanism, in Section 3.2. We conclude in Section 4.

2 Game Description

In this work, n providers in competition want to interconnect with N sources. On one hand, each provider i's quality of service (QoS) level q_i, is fixed and known publicly. It proposes an access price p_i according to its QoS level q_i. On the other hand, the sources have the choice either to connect or not. Besides, they select their provider depending on the QoS they perceive and also on the proposed access prices. The provider selection process of a generic source k depends its opportunity cost. *Opportunity cost* is the cost related to the best choice available to someone who has picked among several mutually exclusive choices.

In our article, the sources have indeed the choice between buying an access to one of the n providers or refusing to subscribe. Opportunity cost is a key concept in economics [1]; it has already been applied to the telecommunications bundle offer market [12] and to the pricing of virtual mobile network operators' services [6]. In our model, each source k ($k = 1, 2, ..., N$) has an opportunity cost towards each provider i ($i = 1, 2, ..., n$). It is defined as $c_k(i) = p_i - \beta_k q_i$ where p_i and q_i are the access price and QoS level for provider i respectively, while $\beta_k \in [0; 1]$ captures source k's sensitivity to the QoS.

Besides, we make the assumption that all source have a same maximum opportunity cost $c_{\max} > 0$, above which they will refuse to buy access. The need to introduce a maximum admissible opportunity cost results from the following observation: a source will refuse to buy access or delay the subscription process either if the access price is too high, or if the QoS is not good enough.

We make the fundamental assumption that the source chooses the provider having the smallest opportunity cost or refuses to subscribe if this latter is larger than $c_{\max}$.

n_i measures provider i's market share. Provider i's utility is the difference between the revenue generated by the source subscriptions and his fixed cost I_i. Let n_i be the pourcentage of sources subcribing to provider i

$$\pi_i = n_i N p_i - I_i \tag{1}$$

In a non-cooperative setting, each provider aims at maximizing his own utility by determining the optimal access price. The two-level game between the providers can be described as follows

(1) Providers determine simultaneously and independently their access prices.

(2) Depending on the perceived opportunity costs, source k chooses the provider having the smallest opportunity cost or refuses to connect.

To cope with the uncertainty on the sources' preferences, we assume that the QoS sensitivity parameter is distributed according to the uniform density on the interval $[0; 1]$ i.e., $\beta_k \sim \mathcal{U}[0; 1]$. Besides, the QoS levels being fixed a priori, we make the assumption that $0 < q_n < q_{n-1} < q_{n-2} < ... < q_1 < +\infty$.

2.1 Relationship between Two Providers

We focus on two providers i, j. Without loosing generality, we assume that $i < j$. As already stated, the opportunity cost associated by source k to provider i is

$$c_k(i) = p_i - \beta_k q_i \tag{2}$$

and for provider j, we get

$$c_k(j) = p_j - \beta_k q_j \tag{3}$$

According to the assumptions introduced previously in this section, we know that the QoS are initially fixed so that $0 < q_j < q_i < +\infty$.

Lemma 1. *Source k prefers provider i over provider j if, and only if,* $0 < p_i < p_j$.

Proof of Lemma 1. Suppose that $0 < p_i < p_j$. It implies that $c_k(i) < c_k(j)$ by definition of both providers' opportunity costs as described in Equations (2) and (3). But, it means that provider j would not have any client i.e., his market share, as defined in Section 1, would vanish ($n_j = 0$) since by definition, the sources subscribe to the provider having the smallest opportunity cost. □

An immediate consequence of Lemma 1 is that it is mandatory to impose $0 < p_j < p_i$ to guarantee that both providers i and j might have non-negative market shares.

We now aim at determining the values of the indifference bounds when studying the interactions between two providers only.

Lemma 2. *For any provider i, source k prefers provider i to no subscription if, and only if, $B_i < \beta_k$ where $B_i \equiv \frac{p_i - c_{\max}}{q_i}$.*

Proof of Lemma 2. Source k prefers provider i to no subscription when $c_k(i) < c_{\max}$ (by definition of opportunity cost). So, we have

$$c_k(i) < c_{\max} \Leftrightarrow p_i - \beta_k q_i < c_{\max}.$$

This last inequality can be re-ordered to give a lower bound for β_k

$$p_i - \beta_k q_i < c_{\max} \Leftrightarrow \frac{p_i - c_{\max}}{q_i} < \beta_k$$

□

Note that if $B_i = \beta_k$, source k is indifferent between provider i and no subscription.

Lemma 3. *For any providers $i < j$, source k prefers provider i to provider j if, and only if, $B_{i,j} < \beta_k$ where $B_{i,j} \equiv \frac{p_i - p_j}{q_i - q_j}$.*

Proof of Lemma 3. Source k prefers provider i to provider j if, and only if, $c_k(i) < c_k(j)$. The last inequality can be rewritted as follow:$p_i - \beta_k q_i < p_j - \beta_k q_j$. So, this last inequality gives a lower bound for β_k.

$$\begin{aligned} p_i - \beta_k q_i < p_j - \beta_k q_j &\Leftrightarrow p_i - p_j < \beta_k (q_i - q_j) \\ &\Leftrightarrow \frac{p_i - p_j}{q_i - q_j} < \beta_k. \end{aligned}$$

□

Presently, we aim at ordering these bound values on the interval $[0;1]$ which will enable us to determine the analytical expressions of both providers' market shares. For the remaining of this paper, we will introduce some notations : for any providers i, j, we define $B_i \equiv \frac{p_i - c_{\max}}{q_i}$. and $B_{i,j} \equiv \frac{p_i - p_j}{q_i - q_j}$. Moreover, an additional proposition is required:

Proposition 1. *Let two providers i, j such that $i < j$.*

$$B_j < B_{i,j} \Leftrightarrow B_j < B_i < B_{i,j}$$

Due to the lack of place, the proof of this proposition is in [2].

2.2 Case of Two Providers

In this section, we assume that provider 1 and 2 only, propose access services to N independent sources. We want to determine the complete ordering of the bounds B_1, B_2, $B_{1,2}$. Using Proposition 1, two cases might arise.

Case (1): $B_2 < B_{1,2} \Leftrightarrow B_2 < B_1 < B_{1,2}$.
Case (2) : $B_{1,2} < B_2 \Leftrightarrow B_{1,2} < B_1 < B_2$.

We detail these two cases below.

Case (1): $B_2 < B_{1,2} \Leftrightarrow B_2 < B_1 < B_{1,2}$.

Figure 2 represents the source preferences ordered according to their β_k values. We have placed the indifference bounds on the β_k-axis. Using Lemmas 2 and 3, we infer the source prefences as functions of the β_k values. If $\beta_k \in [0; B_2[$, then the source prefers to not subscribe; if $\beta_k \in [B_2; B_{1,2}[$, the source prefers provider 2; if $\beta_k \in [B_{1,2}; 1]$, the source prefers provider 1.

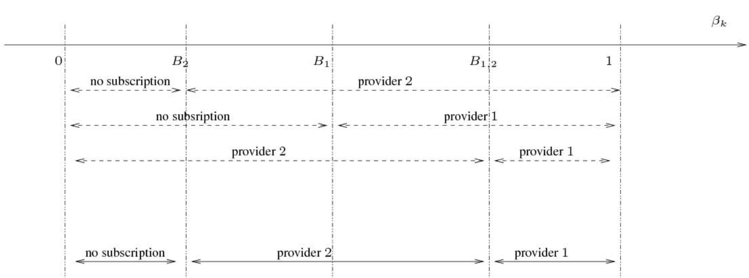

Fig. 2. Case where $B_2 < B_1 < B_{1,2}$

Now, we will focus on Case (2) : $B_{1,2} < B_2 \Leftrightarrow B_{1,2} < B_1 < B_2$. Figure 3 represents the source preferences ordered according to their β. We have placed the indifference bounds on the β_k-axis. Using the same way as Case (1) (applying Lemmas 2 and 3), we infer the source prefences as functions of the β_k values. If $\beta_k \in [0; B_1[$, then the source prefers to not subscribe; if $\beta_k \in [B_1; 1]$, the source prefers provider 1.

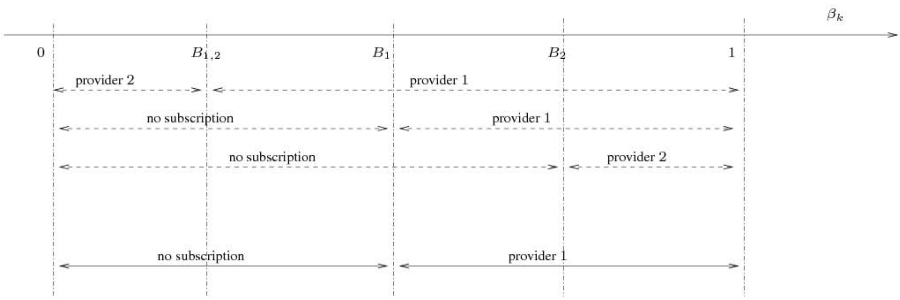

Fig. 3. Case where $B_{1,2} < B_1 < B_2$

Note that in Case (1), both providers might have non-negative market shares. Under such assumptions, both providers would have the opportunity to survive and to contract with sources. As a consequence, a duopoly might emerge [7], [11]. Recall that a duopoly is the simplest case of oligopoly. In economics, duopoly models are shared between cooperative equilibrium models like cartel and non-cooperative equilibrium models like Cournot, Stackelberg or Bowley duopoly models [7]. On the contrary, in Case (2), solely provider 1 can survive on the market leading to a monopol [7], [11]. This remark is summarized in the following corollary.

Corollary 1. *In case of two providers, a duopoly might emerge if, and only if* $B_2 < B_1 < B_{1,2}$.

We want to determine conditions on the game parameters $q_1,\ q_2$ and the access prices upper bound denoted by $p_{\max}$, so that the game belongs to Case (1). We let $\alpha_{1,2} = p_1 - p_2$ be providers 1 and 2's price margin and suppose that there exists a real $\varepsilon > 0$ such that $\varepsilon < \alpha_{1,2}$.

Proposition 2. *If* ε *and* $p_{\max}$ *are chosen so that* $\frac{p_{\max}}{\varepsilon} < \frac{q_2}{q_1 - q_2}$, *then a duopoly might emerge.*

Proof of Proposition 2. The game belongs to Case (1) if, and only if

$$B_{1,2} > B_2 \Leftrightarrow p_1 q_2 - p_2 q_1 > \underbrace{-c_{\max}(q_1 - q_2)}_{<0}$$

A sufficient condition to guarantee the emergence of a duopoly is to have

$$\begin{aligned} p_1 q_2 - p_2 q_1 > 0 &\Leftrightarrow \frac{\alpha_{1,2}}{p_2} > \frac{q_1}{q_2} - 1 \\ &\Leftrightarrow p_2 < \alpha_{1,2} \frac{q_2}{q_1 - q_2} \end{aligned}$$

If $p_{\max} < \varepsilon \frac{q_2}{q_1 - q_2}$, the last inequality is automatically satisfied. This condition is equivalent with $\frac{p_{\max}}{\varepsilon} < \frac{q_2}{q_1 - q_2}$. □

As already stated, the sources' QoS sensitivity being distributed according to the uniform density on the interval $[0;1]$, it is easy to infer both providers' market shares using Figure 2:

$$n_1 = 1 - B_{1,2} = 1 - \frac{p_1 - p_2}{q_1 - q_2},$$
$$n_2 = B_{1,2} - B_2 = \frac{p_1 - p_2}{q_1 - q_2} - \frac{p_2 - c_{\max}}{q_2}.$$

This short computation enables us to obtain the market segmentation as a function of the game parameters.

In a non-cooperative setting, each provider determines selfishly the access price maximizing his utility using the market segmentation. This is summarized in the following proposition.

Proposition 3. *In a non-cooperative game, the prices maximizing the providers' utilities are* $p_1^* = (q_1 - q_2)\left[\frac{1}{2} + \frac{1}{4q_1 - q_2}(\frac{q_2}{2} + c_{\max})\right]$ *and* $p_2^* = 2\frac{q_1 - q_2}{4q_1 - q_2}\left[\frac{q_2}{2} + c_{\max}\right]$.

Proof of Proposition 3. Provider i ($i = 1, 2$)'s utility has been introduced in Equation 1: $\pi_i = n_i N p_i - I_i$ where I_i is provider i's fixed cost. By differentiation of π_1 with respect to p_1, we obtain p_1^* as a linear function of p_2^* i.e., $p_1^* = \frac{(q_1 - q_2) + p_2^*}{2}$, provider 1's cost ($I_1$) being fixed. Identically, by differentiation of π_2 with respect to p_2, we obtain p_2^* as a linear function of p_1^* i.e., $p_2^* = \frac{q_2}{2q_1}p_1^* + \frac{c_{\max}(q_1 - q_2)}{2q_1}$. Note that π_1 and π_2 are second order polynomial equations in p_1 and p_2 respectively whose highest order coefficient is negative; hence the extremum obtained by differentiation of their utilities coincides with a global maximum. Solving a linear system of two equations with two unknown variables, we obtain $p_1^* = (q_1 - q_2)\left[\frac{1}{2} + \frac{1}{4q_1 - q_2}(\frac{q_2}{2} + c_{\max})\right]$ and $p_2^* = 2\frac{q_1 - q_2}{4q_1 - q_2}\left[\frac{q_2}{2} + c_{\max}\right]$. □

To end this two provider game section, we compare the profit resulting from cooperation for the providers against the selfish maximization of their utilities and try to answer the following question: in case of a duopoly, do the providers have incentives to cooperate?

We suppose that when the access providers become allied, they share the alliance revenue according to the Shapley value. There exists many other sharing mechanisms like the nucleolus, proportional allocation, supply chain contract mechanisms, etc. [11], [13]. But, their study is out of the scope of the present article.

We prove that in this case, the providers always prefer to form an alliance than to maximize their utility independently. This result is summarized in the following lemma.

Lemma 4. *In the case of a duopoly, the providers always prefer to become allied than to selfishly maximize their own utility provided the alliance revenue is shared according to the Shapley value.*

Note that Lemma 4 is simply a by-product of the Shapley value definition [11].

2.3 Case of Three Providers

In this section, we focus on the case of three interacting providers. As previously, we assume that the QoS levels are ordered so that $0 < q_3 < q_2 < q_1 < +\infty$. Generalizing Propostion 2 to the case of three interacting providers, we obtain the following price ordering: $0 < p_3 < p_2 < p_1$.

Applying Proposition 2 to all the coalitions containing two providers and assuming that there exists a real $\varepsilon > 0$ such that $\varepsilon < \min\{\alpha_{i,j} | \forall (i,j) \in \{1,2,3\}^2, i \neq j\}$, where $\alpha_{i,j}$ is the difference between provider i's price and provider j's price (i.e : $\alpha_{i,j} = p_i - p_j$). We obtain three relations on the bound ordering:

- If $\frac{p_{\max}}{\varepsilon} < \frac{q_2}{q_1 - q_2}$ then $B_2 < B_1 < B_{1,2}$.
- If $\frac{p_{\max}}{\varepsilon} < \frac{q_3}{q_1 - q_3}$ then $B_3 < B_1 < B_{1,3}$.
- If $\frac{p_{\max}}{\varepsilon} < \frac{q_3}{q_2 - q_3}$ then $B_3 < B_2 < B_{2,3}$.

If these three relations are simultaneously satisfied then $B_3 < B_2 < B_1$.

Presently, we want to determine the total ordering of $B_{2,3},\ B_{1,3},\ B_{1,2}$ on the interval $[0; 1]$.

Lemma 5.

$$B_1 < B_{1,2} \Leftrightarrow B_1 < B_{2,3} < B_{1,2}$$

Due to the lack of place, the proof of Lemma 5 is in [2].

As in the case with two providers, we want to determine conditions on the game parameters $q_1\ , q_2,\ q_3,\ c_{\max},\ p_{\max}$ guaranteeing that competition is possible on the market i.e., that the three providers might have non-negative market shares. Note that in this case, the competition would be total (i.e : all providers are in competition).

Proposition 4. *If ε and $p_{\max},\ c_{\max}$ are chosen so that $\frac{p_{\max} - c_{\max}}{\varepsilon} < \frac{q_1}{q_1 - q_2}$, then the competition would be total.*

Proof of Proposition 4.

$$\begin{aligned} B_1 < B_{1,2} &\Leftrightarrow \frac{p_1 - c_{\max}}{q_1} < \frac{p_1 - p_2}{q_1 - q_2} \\ &\Leftrightarrow p_1 - c_{\max} < \alpha_{1,2} \frac{q_1}{q_1 - q_2} \end{aligned}$$

A sufficient condition to satisfy this last inequality is to have $p_{\max} - c_{\max} < \varepsilon \frac{q_1}{q_1 - q_2}$. □

Under the conditions introduced in Propositions 5 and 4, we obtain the following indifference bound ordering in case of three providers in competition: $0 < B_3 < B_{2,3} < B_{1,2} < 1$ as depicted in Figure 4.

This reinforces the economic intuition behind the problem. Indeed, the market should enable competition between providers which might improve the sources' welfare. Competition is possible if each provider's market share remains positive. Besides, it is quite logical to note that on the β_k-axis, the provider with the smallest QoS (q_3) captures the sources having small β_k values; the provider with intermediate QoS (q_2) captures the sources having intermediate β_k values; the provider with the highest QoS (q_1) captures the sources having hight sensitivites in the QoS.

Note that according to Lemma 5, if $B_1 > B_{1,2}$ then $B_1 > B_{2,3} > B_{1,2}$ and provider 2 market share equals zero since provider 2 is always dominated by the other provider in the source preferences. This point is summarized in the corollary below.

Corollary 2. *In case of three providers, the competition is total if, and only if* $B_1 < B_{2,3} < B_{1,2}$.

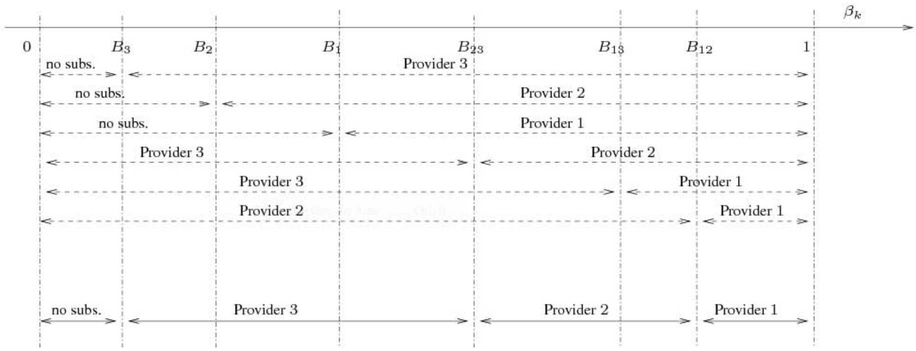

Fig. 4. Indifference bound ordering in case of three providers in competition

3 Can Cooperation Emerge in Case of n Providers?

In this section, we aim at generalizing the results obtained in Section 2 to the case of n interacting providers.

3.1 Generalization of the Game Resolution to n Providers

The indifference bound ordering derived in Sections 2.1 and 2.3, can be generalized recursively, to give the following ordering for n providers.

Theorem 1. *In case of* $n \geq 2$ *providers, the competition is total if, and only if,* $0 < B_n < B_{j-1,j} < B_{j-2,j-1} < 1$ *for any integer* j *such that* $3 \leq j \leq n$*. Moreover, in this case each source* k *prefers* j *if* $B_{j,j+1} \leq \beta_k \leq B_{j-1,j}$ *where* $2 \leq j \leq n-1$, 1 *if* $B_{1,2} \leq \beta_k \leq 1$*, and* n *if* $B_n \leq \beta_k \leq B_{n-1,n}$.

Proof of Theorem 1. We proceed by recurrence on the number n, of interacting providers. For $n = 3$, the result has been proved in Corollary 2.

Suppose that at rank n, $B_n < B_{n-1,n} < B_{n-2,n-1} < ... < B_{1,2}$ and each source k prefers j if $B_{j,j+1} \leq \beta_k \leq B_{j-1,j}$ where $2 \leq j \leq n-1$, 1 if $B_{1,2} \leq \beta_k \leq B_1$, and n if $B_n \leq \beta_k \leq B_{n-1,n}$.

At rank $n+1$, a new provider enters the market. Using the assumption introduced in Section 2, provider $n+1$ chooses his QoS so that $0 < q_{n+1} < q_n < q_j$ for any integer j such that $3 \leq j < n$. Using Lemma 1, we obtain the following ordering on the providers' prices: $p_{n+1} < p_n < p_j$ for any integer j such that $3 \leq j < n$.

Now, we want to order B_n, $B_{n,n+1}$, B_{n+1} and $B_{n-1,n}$. We need to compare $B_{n,n+1}$ to B_n. Two cases are possible: $B_{n,n+1} < B_n$ and $B_{n,n+1} > B_n$.

If $B_{n,n+1} < B_n$ then from Proposition 1, we get $B_{n,n+1} < B_{n+1} < B_n$. But, from Lemma 3, source k prefers provider $n+1$ to provider n if $B_{n,n+1} > \beta_k$. Moreover, from Lemma 2 source k prefers no subcription to provider n if $B_{n,n+1} < \beta_k < B_n$. And by recurrence hypothesis, each source k prefers another provider to provider n if $\beta_k > B_n$. So, in this case, provider n market share would vanish since each source prefers $n+1$ to n. Therefore, the competition cannot be total.

If $B_n < B_{n,n+1}$ then from Proposition 1, we infer that $B_{n,n+1} > B_n > B_{n+1}$. In this case, we need to compare $B_{n-1,n}$ to $B_{n,n+1}$.

By absurd reasoning, assume that $B_{n-1,n} < B_{n,n+1}$. From Lemma 2 source k prefers no subcription to provider n if $\beta_k < B_n$. Moreover, source k prefers provider $n+1$ to provider n if $B_n < \beta_k < B_{n,n+1}$ and also if $B_n < \beta_k < B_{n-1,n}$. Moreover, by recurrence hypothesis, each source k prefers another provider to provider n if $B_{n,n+1} < \beta_k$. So, on any sub-interval on $[0;1]$, provider n would be dominated by an other provider meaning that none of the sources would agree to subscribe to provider n's service. Thus, provider n market share would vanish and the competition cannot be total.

Finaly, we focus on the fact that $B_{n+1} < B_{n,n+1} < B_{n-1,n}$. By recurrence hypothesis, we have each source k prefers j if $B_{j,j+1} \leq \beta_k \leq B_{j-1,j}$ where $2 \leq j \leq n-1$, 1 if $B_{1,2} \leq \beta_k \leq 1$. Moreover each source k prefers n to provider ℓ if $B_n \leq \beta_k \leq B_{n-1,n}$ and if $1 \leq \ell < n$. It remains to focus on interval $[B_{n+1}, B_{n,n+1}]$. From Lemma 3, each source k prefers $n+1$ to provider n if $\beta_k \leq B_{n,n+1}$ and n to provider $n+1$ otherwise. So, it this case, the competition is total is total and the reccurence hypothesis holds.

Hence, the competition is total if, and only if $B_{n-1,n} > B_{n,n+1} > B_n > B_{n+1}$.

Note that all the other indifference bounds ($B_{j-1,j}$ forall integer j such that $2 \leq j < n$) remain identical to the n provider case. □

Using the same principles as in Sections 2.1 and 2.3, the sources are shared between the providers according to the following rule.

For provider $k = 2, ..., n-1$, the market share is $n_k = B_{k-1,k} - B_{k,k+1}$ (from Theorem 1). On the boundaries, provider 1's market share takes the form $n_1 = 1 - B_{1,2}$ while provider n's market share is $n_n = B_{n-1,n} - B_n$. The market segmentation is perfectly determined as a function of the game parameters i.e., $c_{\max}, q_1, q_2, ..., q_n$.

To determine the prices maximizing each provider's utility, we substitute the analytical expressions of the providers' market shares in their utility as defined in Equation (1) and derive the providers' utilities with respect to the prices. It gives us a linear system on n equations in the n unknown prices

$$
\begin{aligned}
p_1^* &= \frac{1}{2} p_2^* + \frac{1}{2}(q_1 - q_2), \\
p_k^* &= \frac{1}{2} \frac{q_k - q_{k+1}}{q_{k-1} - q_{k+1}} p_{k-1}^* + \frac{1}{2} \frac{q_{k-1} - q_k}{q_{k-1} - q_{k+1}} p_{k+1}^*, \text{ for } k = 2, ..., n-1, \\
p_n^* &= \frac{1}{2} \frac{q_n}{q_{n-1}} p_{n-1}^* + \frac{1}{2} \frac{q_{n-1} - q_n}{q_{n-1}}.
\end{aligned}
$$

The problem can be written under a matricial form. To simplify the expressions, we let

$$A_n = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{1}{2}\frac{q_2-q_1}{q_1-q_3} & 0 & \frac{1}{2}\frac{q_1-q_2}{q_1-q_3} & 0 & \dots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & \dots & \frac{1}{2}\frac{q_k-q_{k+1}}{q_{k-1}-q_{k+1}} & 0 & \frac{1}{2}\frac{q_{k-1}-q_k}{q_{k-1}-q_{k+1}} & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{2}\frac{q_{n-1}-q_n}{q_{n-2}-q_n} & 0 & \frac{1}{2}\frac{q_{n-2}-q_{n-1}}{q_{n-2}-q_n} \\ 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{2}\frac{q_n}{q_{n-1}} & 0 \end{pmatrix}.$$

Note that A_n is a tri-diagonal matrix. Using this notation, the linear system of equations in the prices can be arranged to give

$$\begin{pmatrix} p_1^* \\ p_2^* \\ \vdots \\ p_n^* \end{pmatrix} \underbrace{(I - A_n)}_{B_n} = \begin{pmatrix} \frac{q_1-q_2}{2} \\ \vdots \\ 0 \\ \vdots \\ \frac{c_{\max}(q_{n-1}-q_n)}{2q_{n-1}} \end{pmatrix} \tag{4}$$

Lemma 6. *The prices maximizing the providers' utilities are uniquely defined as solutions of the matricial Equation (4).*

Proof of Lemma 6. For the sake of simplicity, we let $B_n = I - A_n$. The prices maximizing the providers' utilities are uniquely defined if, and only if, B_n is invertible i.e., its determinant does not vanish. We proceed by recursion.

At rank $n = 1$, we have $B_1 = 1$ and hence det $B_1 = 1 > 0$.

At rank $n = 2$, det $B_2 = 1 - \frac{1}{4}\frac{q_2-q_3}{q_2-q_3}$. We check that det $B_2 > \frac{3}{4}$. A fortiori, we get: det $B_2 - \frac{1}{4}$det $B_1 > 0$.

At rank $n-1$, we make the recursive hypothesis that: det $B_{n-1} - \frac{1}{4}$det $B_{n-2} > 0$.

At rank n, B_n being a tri-diagonal matrix, we have the following well-known relation between its determinant and the minor determinants: det B_n = det $B_{n-1} - \frac{1}{4}\frac{q_n}{q_{n-1}}\frac{q_{n-2}-q_{n-1}}{q_{n-2}-q_n}$det B_{n-2}. We check easily that: $-\frac{1}{4}\frac{q_n}{q_{n-1}}\frac{q_{n-2}-q_{n-1}}{q_{n-2}-q_{n-1}} > -\frac{1}{4}$. Then, det $B_n - \frac{1}{4}$det $B_{n-1} >$ det $B_{n-1} - \frac{2}{4}$det $B_{n-2} > 0$ using rank $n-1$ recursive hypothesis. □

3.2 Stability of the Shapley Value as a Revenue Sharing Mechanism

Suppose that $\mathcal{S}$ is the set of all the possible coalitions of providers. It is well-known that the cardinal of $\mathcal{S}$ equals $2^n - 1$ provided n providers are interacting on the market. The providers cooperate in order to maximize their joint utility by increasing the alliance total market share. Formally, it can be described as follows: for any coalition $s \in \mathcal{S}$, π_s is the utility of coalition s, and we have

$$\max_{n_s} \pi_s = \sum_{k \in s} \pi_k$$
$$\max_{n_{\mathcal{S}-s}} \pi_{\mathcal{S}-s} = \sum_{k \in \mathcal{S}-s} \pi_k \tag{5}$$

where n_s (resp. $n_{\mathcal{S}-s}$) contains coalition s (resp. $\mathcal{S}-s$) total market share. Going back to the analytical expressions of the providers' market shares, we note that coalition s market share relies solely on the prices proposed by the providers having the highest and the worst QoS levels respectively, in the coalition.

Proposition 5. *The utility $\pi(.)$ is submodular in the n providers cooperative game described by Equations (5).*

Proof of Proposition 5. Consider two coalitions s, s' such that $s \subset s' \subset \{1,2,...,n\}$ and a provider $j \in \{1,2,...,n\} - s'$. We want to show that $\pi_{s\cup j} - \pi_s \geq \pi_{s'\cup j} - \pi'_s \Leftrightarrow \pi_{s\cup j} - \pi_{s'\cup j} \geq \pi_s - \pi'_s$.

We observe that
$[\pi_{s\cup j} - \pi_{s'\cup j}] - [\pi_s - \pi'_s] = (p'_s - p_{s',j})n_{s'} - (p_s - p_{s,j})n_s + \underbrace{(p_{s,j} - p_{s',j})n_j}_{\geq 0}$.

Now, we note that $(p'_s - p_{s',j})n_{s'} - (p_s - p_{s,j})n_s \geq (p'_s - p_{s,j})n_{s'} - (p_s - p_{s,j})n_s$ since $s \subset s'$ implies that $p_s \geq p'_s$ and in turn that $p_{s,j} \geq p_{s',j}$.

We let l be the provider belonging to coalition s which proposes the smallest QoS in coalition s i.e., $q_l = \min_{i\in s} q_i$ and k be the provider belonging to coalition s' which proposes the smallest QoS in coalition s' i.e., $q_k = \min_{i\in s'} q_i$.

Then $n_s = B - \frac{p_k - p_l}{q_k - q_l}$ and $n_{s'} = B - \frac{p_j - p_k}{q_j - q_k}$ with $B_{j,k} \leq B_{k,l}$ and B the upper indifference bound delimiting coalitions s and s' market shares. By definition and using the inclusion property $(p_{s'} - p_{s,j})(B_{j,k} - B) \geq (p_s - p_{s,j})(B_{k,l} - B)$. Multiplying each term of the inequality by -1 we obtain the proposition result. □

The Shapley value is then the center of gravity of the core of the n provider cooperative game as described in Equations (5). Therefore, it is still a fair and stable revenue sharing mechanism.

4 Conclusion

We have proved that the Shapley value is the center of gravity of the n provider cooperative game taking into account the sources' individual preferences. Therefore, it is always a stable mechanism to share the grand coalition total revenue, meaning that none of the providers has incentives to deviate from it or leave the grand coalition. It is therefore most likely that tacit alliances emerge, to the disadvantage of the sources.

However, in practice, alliances do not emerge on every market since for instance, collusion may be forbidden due to competition policy. Indeed, courts punish explicit accords whose objectives are clearly to decrease the competition. Heavy sanctions have been applied to the international accords on the vitamin market (855 millions of euros), on lysine and citric acid (200 millions of dollars and emprisonment years) [**?**] and more recently, on the memory chip market (331 millions of euros).

In the present article, we have assumed that the providers' QoS levels were fixed. Extensions should be envisaged by adding another level in the game description. The resulting three level game resolution might be tackled using numerical approaches. In practice i.e., in the telecommunication business area, when information about the providers' QoS is known publicly, it is possible for some providers to set a price that kick other providers out of the market. However, QoS is difficult to estimate with accuracy and performance measures are usually very costly to perform. Consequently, it might be interesting to assume that in the game with the other providers, each provider ignores his rivals' true QoS levels but try to infer them using side observations.

Finally, hierarchical relations between the providers should be taken into account. Indeed, some of them might lack the infrastructure and buy their QoS to some others, owning a network. The QoS fixation/negotiation market will add another level of complexity to the game and the revenue sharing mechanism might be more complex to design [10]. Generally speaking, it might be possible to design contract mechanisms which would force the providers to cooperate. Analogies with the supply chain theory is possible [3] but it would require to evaluate complex power relations between all the involved providers. This task is not straightforward compared to the Shapley value which provides a direct measure of the providers' inter-relations.

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