Bottleneck Routing Games on Grids

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Abstract. We consider routing games on grid network topologies. The social cost is the worst congestion in any of the network edges (bottleneck congestion). Each player's objective is to find a path that minimizes the bottleneck congestion in its path. We show that the price of anarchy in bottleneck games in grids is proportional to the number of bends β that the paths are allowed to take in the grids' space. We present games where the price of anarchy is $\widetilde{O}(\beta)$. We also give a respective lower bound of $\Omega(\beta)$ which shows that our upper bound is within only a poly-log factor from the best achievable price of anarchy. A significant impact of our analysis is that there exist bottleneck routing games with small number of bends which give a poly-log approximation to the optimal coordinated solution that may use an arbitrary number of bends. To our knowledge, this is the first tight analysis of bottleneck games on grids.

Keywords: algorithmic game theory, bottleneck games, routing games, price of anarchy, grid networks.

1 Introduction

We consider non-cooperative routing games with n players, where each player's *pure strategy set* consists of a set of paths in the network. A player selfishly selects a strategy (a single path) that maximizes the player's utility cost function. Such games are also known as *atomic* or *unsplittable-flow* games. We focus on *bottleneck routing games* where the objective for the social outcome is to minimize the bottleneck congestion C, the maximum congestion on any edge. Each player's objective is also to select a path with the smallest bottleneck congestion along the selected path's edges. Typically, the congestion on a edge is a non-decreasing function on the number of paths that use the edge; here, we consider the congestion to be simply the number of players that use the edge.

Bottleneck routing games have been studied in the literature [1,3,2]. In [1] the authors observe that bottleneck games are important in networks for various practical reasons. In wireless networks, the maximum congested link is related to the lifetime of the network since the nodes adjacent to high congestion links transmit large number of packets which results to higher energy depletion. High congestion links also result to congestion hot-spots which may slow-down the network throughput. Hot spots also increase the vulnerability of the network to malicious attacks which aim to increase the congestion of links in the hope to

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bring down the network. Bottleneck games are also important from a theoretical point of view since the bottleneck congestion is immediately related to optimal packet scheduling. In a seminal result, Leighton *et al.* [9] showed that there exist packet scheduling algorithms that deliver the packets along their chosen paths in time very close to $C+D$, where D is the maximum chosen path length. When $C \gg D$, the congestion becomes the dominant factor in the packet scheduling performance. Thus, smaller bottleneck congestion C immediately implies faster packet delivery time.

A natural problem that arises in games concerns the effect of the players' selfishness on the welfare of the whole system measured with the *social cost* C. We examine the consequence of the selfish behavior in pure *Nash equilibria* which are stable states of the game in which no player can unilaterally improve her situation. We quantify the effect of selfishness with the *price of anarchy* (PoA) [8,13], which expresses how much larger is the worst social cost in a Nash equilibrium compared to the social cost in the optimal coordinated solution in the strategy space. The price of anarchy provides a measure for estimating how closely do Nash equilibria of bottleneck congestion games approximate the optimal C^* of the respective coordinated optimization problem in the player's strategy set.

Ideally, the price of anarchy should be small. However, the current literature results have only provided weak bounds for bottleneck games. In [1] it is shown that if the resource congestion delay function is bounded by some polynomial with degree k then $PoA = O(|E|^k)$, where E is the set of edges in the graph. In [3] it is shown that if $k = 1$ there are game instances with $PoA = \Omega(|E|)$. A natural question that we explore here is the circumstances under which there are bottleneck games with alternative and better price of anarchy bounds.

1.1 Contributions

We consider grid network topologies in which the nodes are placed in a d-dimensional array and each node connects with edges to at most $2d$ neighbors. The number of nodes is $n^d = N$. Grid networks have been used as interconnection networks in parallel multiprocessor computer architectures [10]. In wireless networks 2-dimensional grids provide a framework for formulating and analyzing wireless communication problems. In other communication networks routing and scheduling algorithms are typically first tested and analyzed on grids and then extended to arbitrary network topologies [4].

We explore games where the price of anarchy is expressed in terms of the numbers of bends that the paths use in the grid. A *bend* is a node in a path which connects two path segments in different dimensions. We explore games where the strategies of the players consists of paths whose bends are bounded by β, where β can be any number of nodes up to N. We first examine basic bottleneck games on grids with at most β bends for the paths. We show that there are instances in the 2-dimensional grid with $\beta = O(1)$ and price of anarchy $\Omega(\sqrt{N})$. However, this is not satisfactory. In order to obtain price of anarchy bounded by β, we explore two alternative games.

In the first game we utilize *channels*, where path segments on straight lines are routed in different channels according to their lengths. An edge accommodates $\alpha = \log n$ channels (logarithms are base 2), such that channel j is used by path segments of length in range $[2^j, 2^{j+1} - 1]$. Channels do not interfere with each other so that congestion can be created only by path segments in the same channel. Channels can be implemented with different frequencies in the physical communication medium, or with time division multiplexing, or with other means of signal multiplexing. The use of channels enables us to control the price of anarchy. We show that in *channel bottleneck games* if paths are allowed to use at most β bends, the price of anarchy is $PoA = O((\beta/d)\log N)$. We also provide a lower bound $PoA = \Omega(\beta)$. Thus, for constant d, the upper bound is tight within a $\log n$ factor.

We then explore games which use only one channel. Now, in order to control the price of anarchy we split the path segments into different grid lines according to the lengths of the segments. Odd lines with index $2i+1$ are used to route path segments of length in range $[2^{i \bmod \alpha}, 2^{(i \bmod \alpha)+1} - 1]$, where $\alpha = \log n$ (logarithms are base 2). Even index lines are used to route paths segments with length at most $2\alpha - 1$. Even index lines are uses to route paths close to the source and destination and when path segments switch to different lengths. This gives $\alpha + 1$ different types of lines. Thus, path segments are separated in space, and a single channel suffices. Note that we can still perform routing from every node to any other node without significantly increasing the number of bends, compared to a routing mode without space separated path segments. We show that in the respective *split bottleneck games* if paths are allowed to use at most β bends, the price of anarchy is $PoA = O((\beta/d^2)\log^2 N)$. We also provide a lower bound $PoA = \Omega(\beta)$. Thus, for constant d, the upper bound is tight within a $\log^2 n$ factor.

1.2 Impact of Games with Small Number of Bends

We demonstrate that Nash equilibria of bottleneck games with small number of bends can approximate efficiently the best coordinated solution that uses an arbitrary number of bends. Assuming that every path in the network can be used, there exist oblivious routing algorithms in grids which find paths with $O(d \log N)$ bends and achieve $O(d \log N)$ approximation to the optimal solution that uses an arbitrary number of bends [4]. Let $\widehat{C}$ denote the solution returned by the oblivious algorithm and $\widehat{C}^*$ denote the global optimal solution with an arbitrary number of bends. Clearly, $\widehat{C}/\widehat{C}^* = O(d \log N)$.

Consider now channel bottleneck games where the strategy of each player contains all possible paths in the grid with $\beta = O(d \log N)$ bends. Let C^* denote the smallest social cost. Clearly, $C^* \le \widehat{C}$. Let C be any Nash equilibrium of the game. Since $C/C^* \le PoA$, and $PoA = O((\beta/d)\log N) = O(\log^2 N)$, we obtain $C/\widehat{C}^* = O(d \log^3 N)$. Therefore, Nash equilibria of channel bottleneck games with small number of bends provide good approximations to the optimal coordinated solution with arbitrary number of bends.

We can obtain a similar result for split bottleneck games. Note that any solution of an oblivious routing algorithm with congestion C' and x bends is translated to a solution with congestion $C' \cdot \log n$ and $O(x)$ bends in the split grid, since some of the path segments have to be rerouted to nearby lines that accommodate their length. Since $PoA = O((\beta/d^2)\log^2 N) = O((1/d) \cdot \log^3 N)$, we obtain: $C/\widehat{C}^* = O((1/d)\log^5 N)$.

1.3 Related Work

Congestion games were introduced and studied in [12,14]. In [14], Rosenthal proves that congestion games have always pure Nash equilibria. Koutsoupias and Papadimitriou [8] introduced the notion of price of anarchy in the specific *parallel link networks* model in which they provide the bound $PoA = 3/2$. Roughgarden and Tardos [17] provided the first result for splittable flows in general networks in which they showed that $PoA \leq 4/3$ for a player cost which reflects to the sum of congestions of the resources of a path. Pure equilibria with atomic flow have been studied in [3,5,11,19] (our work fits into this category), and with splittable flow in [15,16,17,18]. Most of the work in the literature uses a player cost metric related to the aggregate sum of congestions on all the edges of the player's path; and the social cost metric is also an aggregate expression of all the edge congestions [5,16,17,18,19].

Bottleneck routing games have been studied in [1], where the authors consider the maximum congestion metric in general networks with splittable and atomic flow. They prove the existence and non-uniqueness of equilibria in both the splittable and atomic flow models. They show that finding the best Nash equilibrium that minimizes the social cost is a NP-hard problem. Further, they show that the price of anarchy may be unbounded for specific resource congestion functions. In [3], the authors consider bottleneck routing games in general networks where they prove that $\ell \leq PoA \leq c(\ell^2 + log^2|V|)$, where ℓ is the size of the largest edge-simple cycle in the graph and c is a constant. In [2] the authors consider bottleneck games with the $C + D$ metric. In [6], the authors prove the existence of strong Nash equilibria (which concern coalitions of players) for games with the lexicographic improvement property; such games include the bottleneck routing games that we consider here. In [7], the authors provide games with the bottleneck social cost which achieve low price of anarchy when the players use a cost function which is an aggregate exponential expression of the congestions of the edges in their selected paths.

Outline of Paper: In Section 2 we give basic definitions. In Section 3 we present a basic bottleneck routing game with high price of anarchy. In Sections 4 and 5 we present the channel and split bottleneck games, respectively, which achieve price of anarchy bounded by the number of bends β. We finish with providing lower bounds in Section 6.

2 Definitions

Grids: The *d-dimensional grid* $G = (V, E)$ consists $N = |V| = n^d$ nodes arranged in a grid of d dimensions with side length n in each dimension. There is an edge connecting a node with each of its $2d$ neighbors (except for the nodes at the boundaries of the grid). Each node has a coordinate $(a_1, a_2, \ldots, a_d)$, where $a_i \in [0, n-1]$ denotes the position in the ith dimension. An example of a 2-dimensional grid is shown in Figure 1. A line segment with x edges in the kth dimension is a sequence of nodes $(a_1, \ldots, a_k, \ldots, a_d), \ldots, (a_1, \ldots, a_k + x, \ldots, a_d)$.

Routings: Let $\Pi = \{\pi_1, \ldots, \pi_\kappa\}$ be a set of players such that each π_i corresponds to a path request from a source u_i and destination v_i. A *routing* $\mathbf{p} = [p_1, p_2, \cdots, p_\kappa]$ is a collection of paths, where p_i is a path for player π_i from u_i to v_i. For any routing $\mathbf{p}$ and any edge $e \in E$, the *edge-congestion* $C_e(\mathbf{p})$ is the number of paths in $\mathbf{p}$ that use edge e. For any path q, the *path-congestion* $C_q(\mathbf{p})$ is the maximum edge congestion over all edges in q, namely, $C_q(\mathbf{p}) = \max_{e \in q} C_e(\mathbf{p})$. Player's π_i congestion is denoted as $C_{\pi_i}(\mathbf{p}) = C_{p_i}(\mathbf{p})$. The *network (bottleneck) congestion* $C(\mathbf{p})$ is the maximum edge-congestion over all edges in E, that is, $C(\mathbf{p}) = \max_{e \in E} C_e(\mathbf{p})$.

We denote the length (number of edges) of any path p as $|p|$. For a grid G, the path p consists of a sequence path segments which change dimensions. A *bend* of a path is a node that connects two consecutive path segments in different dimensions. By default, we take the source and destination nodes to be bends.

Routing Games: A routing game in graph G is a tuple $\mathcal{R} = (G, \Pi, \mathcal{P})$, where $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\kappa\}$ is the set of players such that each player π_i has a source node u_i and destination v_i. The set $\mathcal{P}$ is the strategy state space of the game, $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_\kappa$, where $\mathcal{P}_i$ is the *strategy set* of player π_i which is a collection of available paths in G for player i from u_i to v_i. Any path $p \in \mathcal{P}_i$ is a *pure strategy* available to player π_i. A *pure strategy profile (or game state)* is any routing $\mathbf{p} = [p_1, p_2, \cdots, p_\kappa] \in \mathcal{P}$.

For game $\mathcal{R}$ and routing $\mathbf{p}$, the *social cost* (or *global cost*) is a function of routing $\mathbf{p}$, and it is denoted $SC(\mathbf{p})$. The *player or local cost* is also a function on $\mathbf{p}$ denoted $pc_i(\mathbf{p})$. We use the standard notation $\mathbf{p}_{-i}$ to refer to the collection of paths $\{p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_\kappa\}$, and $(p_i; \mathbf{p}_{-i})$ as an alternative notation for $\mathbf{p}$ which emphasizes the dependence on p_i. A *greedy move* is available to player π_i if the player can obtain lower cost by changing the current path from p_i to p'_i. Specifically, the greedy move takes the original routing $\mathbf{p} = (p_i; p_{-i})$ to $\mathbf{p}' = (p'_i; p_{-i})$ (in which path p_i is replaced by p'_i) such that $pc_i(\mathbf{p}') < pc_i(\mathbf{p})$.

Player i is *locally optimal* (or *stable*) in routing $\mathbf{p}$ if $pc_i(\mathbf{p}) \leq pc_i(p'_i; \mathbf{p}_{-i})$ for all paths $p'_i \in \mathcal{P}_i$. In other words, no greedy move is available for a locally optimal player. A routing $\mathbf{p}$ is in a Nash Equilibrium if every player is locally optimal. Nash Equilibria quantify the notion of a stable selfish outcome. A routing $\mathbf{p}^* \in \mathcal{P}$ is an *optimal pure strategy profile* if it has minimum attainable social cost: for any other pure strategy profile $\mathbf{p} \in \mathcal{P}$, $SC(\mathbf{p}^*) \leq SC(\mathbf{p})$.

We quantify the quality of the Nash Equilibria with the *price of anarchy* (PoA) (sometimes referred to as the coordination ratio) and the *price of stability* (PoS). Let $\mathbf{Q}$ denote the set of distinct Nash Equilibria, and let SC^* denote the social cost of the optimal routing $\mathbf{p}^* \in \mathcal{P}$. Then,

$$PoA = \sup_{\mathbf{p} \in \mathbf{Q}} \frac{SC(\mathbf{p})}{SC^*}, \qquad PoS = \inf_{\mathbf{p} \in \mathbf{Q}} \frac{SC(\mathbf{p})}{SC^*}.$$

3 Basic Game

Consider a routing game $\mathcal{R} = (G, \Pi, \mathcal{P})$ in a d-dimensional grid $G = (V, E)$, where each path in $\mathcal{P}_i$ is allowed to have at most β bends. For any routing $\mathbf{p} = [p_1, p_2, \cdots, p_\kappa] \in \mathcal{P}$, the social cost function is the bottleneck congestion, $SC(\mathbf{p}) = C(\mathbf{p})$, and the player cost function is the bottleneck congestion of its path, $pc_i(\mathbf{p}) = C_{\pi_i}(\mathbf{p}) = C_{p_i}(\mathbf{p})$.

We first show that such (basic) games have always Nash equilibria and the price of stability is 1. However, there are game instances where the price of anarchy is very large compared to the number of bends β. For this reason we explore alternative games with low price of anarchy in Sections 4 and 5.

The stability of the above basic game follows from techniques in [3,6] related to potential functions based on lexicographic ordering. We give the details here for completeness. For routing $\mathbf{p}$, the *congestion vector* $M(\mathbf{p}) = [m_0(\mathbf{p}), m_1(\mathbf{p}), \ldots, m_\kappa(\mathbf{p})]$ is defined such that each component $m_j(\mathbf{p})$ is the number of edges with congestion j. Note that $\sum_j m_j(\mathbf{p}) = |E|$. The network congestion $C(\mathbf{p})$ is the maximum index j for which $m_j > 0$. We define a lexicographic total order on routings according to their congestion vectors. Let $\mathbf{p}$ and $\mathbf{p}'$ be two routings, with $M(\mathbf{p}) = [m_0, m_1, \ldots, m_\kappa(\mathbf{p})]$, and $M(\mathbf{p}') = [m'_0, m'_1, \ldots, m'_\kappa(\mathbf{p})]$. Two routings are equal, written $\mathbf{p} = \mathbf{p}'$, if and only if $m_j = m'_j$ for all $j \geq 0$. Routing $\mathbf{p}$ is smaller than $\mathbf{p}'$, written $p < p'$, if and only if there is some $j \in [0, \kappa]$ such that $m_j < m'_j$ and $\forall j' > j, m_{j'} = m'_{j'}$.

It is easy to verify that for any greedy move of a player from a routing $\mathbf{p}$ to routing $\mathbf{p}'$ it holds that $\mathbf{p}' < \mathbf{p}$, since a lower index vector position increases in $M(\mathbf{p}')$ and a higher index vector position decreases in $M(\mathbf{p}')$ with respect to $M(\mathbf{p})$. Let $\mathbf{p}^* \in \mathcal{P}$ be the minimum routing (according to the total lexicographic order) in the available game state set. Routing $\mathbf{p}^*$ is a Nash equilibrium since no player can perform a greedy move to improve its cost. Further, $\mathbf{p}^*$ has optimal social cost, since if there was another state with smaller social cost then $\mathbf{p}^*$ wouldn't be minimum. Therefore, we obtain:

Theorem 1. *Any basic bottleneck game instance $\mathcal{R}$ has at least one Nash Equilibrium and $PoS(\mathcal{R}) = 1$.*

Next, we show that there are instances of the basic bottleneck game with large price of anarchy even when β is small.

Theorem 2. *There is a basic bottleneck game instance $\mathcal{R}$ in the 2-dimensional grid, with $\beta = 0(1)$ bends, such that $PoA(\mathcal{R}) = \Omega(\sqrt{N})$.*

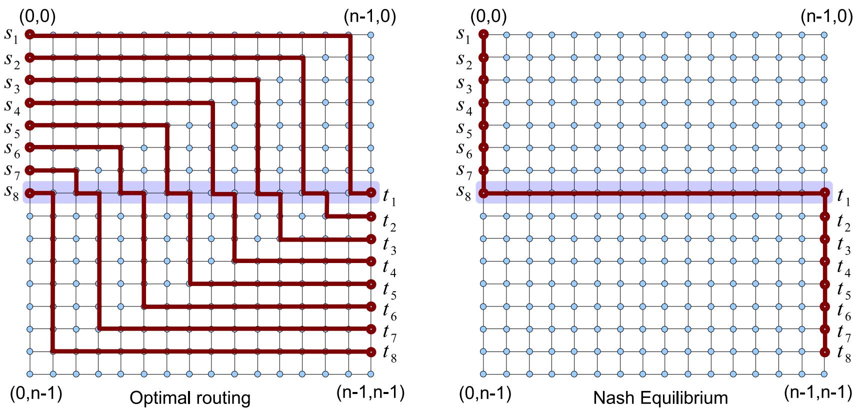

Fig. 1. A game with large price of anarchy and small number of bends

Proof. Consider an $n \times n$ grid. In the game there are $\kappa = n/2$ players, where each player π_i has source s_i in node $(0, i-1)$ of the column 0, and destination t_i in node $(n-1, i+n/2-2)$ of column $n-1$ (see Figure 1). The strategy set of player π_i consists of two paths $\mathcal{P}_i = \{p_i^1, p_i^2\}$. Both of the paths cross row $r = n/2 - 1$ (the row is highlighted in Figure 1). Path p_i^1 uses one "dedicated" edge in row r, so that the dedicated edges of different players do have any common nodes (see left of Figure 1). The remaining path segments of p_i^1 are used to connect the source and destination so that the first strategy paths of the players are disjoint. Note that path p_i^1 consists of at most five path segments (6 bends). Path p_i^2 uses all the edges of row r, and it consists of at most three path segments (4 bends), one in column 0, one in row r, and one in column $n-1$ (see right of Figure 1).

The routing with the first path choices $\mathbf{p}^1 = [p_1^1, p_2^1, \ldots, p_\kappa^1]$ is optimal, since the congestion is $C(\mathbf{p}^1) = 1$. The routing with the second path choices $\mathbf{p}^2 = [p_1^2, p_2^2, \ldots, p_\kappa^2]$ has congestion $C(\mathbf{p}^2) = \kappa$ and every player has cost $pc_i(\mathbf{p}^2) = \kappa$, due to the path segments in row r. Routing $\mathbf{p}^2$ is a Nash equilibrium, since if any player π_i switches to path p_i^1, then its cost remains κ because it still uses the dedicated edge in row r. Therefore: $PoA(\mathcal{R}) \geq C(\mathbf{p}^2)/C(\mathbf{p}^1) = \kappa = n/2 = \Omega(\sqrt{N})$.

4 Channel Game

Let $G = (V, E)$ be a d-dimensional grid, with $n^d = N$ nodes. We consider bottleneck routing games where each path is allowed to have at most β bends, and achieve price of anarchy bounded by β. In order to get this price of anarchy we use $\log n$ *channels*, as we describe below.

We can write any path p as a sequence of path segments $p = (q_1, q_2, \ldots, q_k)$, where each q_l is in a line which is in a different dimension than q_{l+1}, where $1 \leq l < k$. The number of nodes (bends) in the path p is bounded by $k + 1 \leq \beta$; thus, the number of path segments is $k \leq \beta - 1$.

Let $\alpha = \log n$. Each edge e accommodates $\alpha = \log n$ distinct *channels* $A_0, A_1, \ldots, A_{\alpha-1}$. The purpose of the channels is to route path segments of different lengths separately. A path segment q whose length is in range $|q| \in [2^j, 2^{j+1}-1]$ uses channel A_j; we also say that the channel of q is $A(q) = A_j$. Note that a path may use multiple channels according to the lengths of its constituent segments.

Consider a routing $\mathbf{p}$. For any edge e denote by $C_e^{A_j}(\mathbf{p})$ the congestion caused by the path segments of channel A_j, which is equal to the number of path segments in channel A_j that use edge e. The congestion of a path segment q is $C_q(\mathbf{p}) = \max_{e \in q} C_e^{A(q)}(\mathbf{p})$, which is the maximum edge congestion along the path segment and its respective channel. Given a path $p = (q_1, q_2, \ldots, q_k)$, we denote the congestion of the path as the maximum congestion along any of its path segments, namely, $C_p(\mathbf{p}) = \max_{q_i \in p} C_{q_i}(\mathbf{p})$. Using this notion of path congestion all the congestion definitions in Section 2 can be extended in a grid with channels.

We are now ready to define the *channel bottleneck game* $\mathcal{R} = (G, \Pi, \mathcal{P})$. As in the basic bottleneck game, there is limit β on the allowed number of bends in a selected path. The social and player cost functions are also similar, $SC(\mathbf{p}) = C(\mathbf{p})$, and $pc_i(\mathbf{p}) = C_{\pi_i}(\mathbf{p}) = C_{p_i}(\mathbf{p})$, where all congestions are calculated using the channel model of the grid. Similar to the basic congestion game we obtain:

Theorem 3. *Any channel bottleneck game instance $\mathcal{R}$ has at least one Nash Equilibrium and $PoS(\mathcal{R}) = 1$.*

4.1 Price of Anarchy Analysis for Channel Game

Consider a Nash equilibrium $\mathbf{p} \in \mathcal{P}$. Let $\mathbf{p}^* = [p_1^*, p_2^*, \ldots, p_\kappa^*] \in \mathcal{P}$ be an optimal routing with lowest congestion $C^* = C(\mathbf{p}^*)$. Consider a set of players $\Pi' \subseteq \Pi$ such that the smallest congestion of any player of Π' in routing $\mathbf{p}$ is at least C'. Since $\mathbf{p}$ is an equilibrium, each player $\pi_i \in \Pi'$ has congestion at least $C'-1$ in its optimal path p_i^*, namely, $C_{p_i^*}(\mathbf{p}) \geq C'-1$. The $C'-1$ congestion in p_i^* is due to some path segment $q_i^* \in p_i^*$ with congestion at least $C'-1$, namely, $C_{p_i^*}(\mathbf{p}) \geq C_{q_i^*}(\mathbf{p}) \geq C'-1$. Thus, there is an edge $e \in q_i^*$ such that $C_e^{A(q_i^*)} \geq C'-1$. We call e the *special edge* of player π_i and the respective channel $A(q_i^*)$ the *special channel* of player π_i. Note that a player could have multiple special edges and respective special channels, in which case we choose one of them arbitrarily.

We say that two edges e_1 and e_2 are *far-apart with respect to channel A_j* if the edges are in different dimensions, or if the edges are in the same dimension and in different lines, or if the edges are in the same line and the shortest path length that connects any of their adjacent nodes is at least $2^{j-1}-1$. If two edges are not far-apart with respect to channel A_j, then we say that they are *close* with respect to channel A_j.

Let $\widehat{A}(\Pi')$ denote the channel which is special for the majority of the players in Π'. Let $B(\Pi')$ be the subset of players in Π' with special channel $\widehat{A}(\Pi')$. Clearly, since there are α channels, $|B(\Pi')| \geq |\Pi'|/\alpha$. Let $\Gamma(\Pi')$ denote the set of special edges for the players in $B(\Pi')$. Let $\Delta(\Pi')$ denote a maximum set of edges such that $\Delta(\Pi') \subseteq \Gamma(\Pi')$, and each pair of edges in $\Delta(\Pi')$ is far-apart with respect to channel $\widehat{A}(\Pi')$. Let $\Phi(\Pi')$ denote the set of players which in routing $\mathbf{p}$ use an edge in $\Delta(\Pi')$ such that the path segment that crosses the edge belongs to channel $\widehat{A}(\Pi')$. Each player $\pi_i \in B(\Pi')$ has either (i) its special edge $e \in \Delta(\Pi')$, or (ii) there is an edge $e' \in \Delta(\Pi')$ such that e' is close to e with respect to channel $\widehat{A}$. In either case, we say that player π_i is *assigned* to respective edge e or e' of $\Delta(\Pi')$.

Lemma 1. *For any set of players $\Pi' \subseteq \Pi$, each edge in $\Delta(\Pi')$ has assigned to it at most $5C^*$ players of Π' in routing $\mathbf{p}$.*

Proof. Suppose that the channel $\Delta(\Pi')$ is in dimension x. Assume that there is an edge $e \in \Delta(\Pi')$ such that there are at least $z \geq 5C^* + 1$ players assigned to it. Let X be the set of players in $B(\Pi')$ which are assigned to e because e is their special edge (case (i) above). Let Y be the number of players in $B(\Pi')$ which are assigned to e because e is near their special edge (case (ii) above). We have that $z = |X| + |Y|$. If $|X| > C^*$, then the edge e is used in the optimal path of at least $C^* + 1$ players, which is impossible since the optimal congestion is C^*. Therefore, $|X| \leq C^*$, and hence $|Y| \geq 4C^* + 1$.

For ease of presentation, assume without loss of generality that x is the horizontal dimension. For any player $\pi_i \in Y$ we say that its special edge is in the *first (second)* part of its optimal path segment if it is positioned in the left (right) half of its optimal path segment (if the special edge is positioned exactly in the middle of the path segment then it is simultaneously in the first and second parts). Let Y_l and Y_r denote the players whose special edges appear on the left and right of e, respectively. Without loss of generality, assume that $|Y_l| \geq |Y|/2$. Without loss of generality, assume also that at least half of the special edges in Y_l are in the first half of their respective optimal segments. Denote by Y_l' these players. We have that $|Y_l'| \geq |Y|/4$. By the positions of the special edges of Y_l' all their optimal path segments intersect, which implies that there is an edge on same line with e which in the optimal routing $\mathbf{p}^*$ has congestion at least $|Y_l'| \geq |Y|/4 \geq (4C^* + 1)/4 > C^*$. This is a contradiction.

From Lemma 1, each edge in $\Delta(\Pi')$ is assigned at most $5C^*$ players of $B(\Pi')$. Since $|B(\Pi')| \geq |\Pi'|/\alpha$, we have:

Corollary 1. *For any set of players $\Pi' \subseteq \Pi$, $|\Delta(\Pi')| \geq |\Pi'|/(5\alpha C^*)$.*

Lemma 2. *For any set of players $\Pi' \subseteq \Pi$ with congestion at least C', $|\Phi(\Pi')| \geq (C' - 1)|\Pi'|/(20\alpha\beta C^*)$.*

Proof. Each edge in $\Delta(\Pi')$ is special for some player in $B(\Pi')$. Without loss of generality, let $\widehat{A}(\Pi') = A_j$. Then, $2^{j+1} - 1$ is the maximum path segment of any path that uses channel $\widehat{A}(\Pi')$. By the definition of the special edges $\Delta(\Pi')$, each path segment of channel $\widehat{A}(\Pi')$ can have at most four special edges. Since each player in $\Phi(\Pi')$ has at most β path segments each using at most four special edges in $\Delta(\Pi')$, and each special edge in $\Delta(\Pi')$ is used by at least $C'-1$ players in $\Phi(\Pi')$ (since the edge e has congestion $C'-1$ in channel $\widehat{A}(\Pi')$), from Corollary 1 we obtain: $|\Phi(\Pi')| \geq (C'-1)|\Delta(\Pi')|/(4\beta) \geq (C'-1)|\Pi'|/(20\alpha\beta C^*)$

Theorem 4. *$C(\mathbf{p}) \leq 40\alpha\beta C^* + \log(5\alpha d n^d C^*)$.*

Proof. Suppose that $C(\mathbf{p}) > 40\alpha\beta C^* + \log(5\alpha d n^d C^*)$. There is a player $\pi_i \in \Pi$ with congestion $C_{\pi_i}(\mathbf{p}) = C(\mathbf{p})$. We define recursively a sequence of player sets $\Pi_0, \Pi_1, \ldots, \Pi_k$, where $k = \log(5\alpha d n^d C^*)$ as follows. We define $\Pi_0 = \{\pi_i\}$. Suppose we have defined the set Π_t, where $t \geq 1$; we define $\Pi_{t+1} = \Phi(\Pi_t)$. From the above definition of Π_t, we have that for each $\pi_j \in \Pi_t$, $C_{\pi_j}(\mathbf{p}) \geq C(\mathbf{p}) - t \geq C(\mathbf{p}) - k \geq 40\alpha\beta C^* + 1$. From Lemma 2, $|\Pi_{t+1}| \geq 2|\Pi_t|$. Therefore, $|\Pi_k| \geq 2^k \geq 5\alpha d n^d C^*$. Consequently, from Corollary 1, $|\Delta(\Pi_k)| \geq |\Pi_k|/(5\alpha C^*) \geq dn^d$. However, we have a contradiction, since $|\Delta(\Pi_k)| \leq |E| < dn^d$.

From Theorem 4, since $\alpha = O(\log n)$ and $N = n^d$, we obtain the following corollary:

Corollary 2. *For any channel bottleneck game $\mathcal{R}$ in the d-dimensional grid which allows paths with at most β bends, $PoA(\mathcal{R}) = O((\beta/d)\log N)$.*

5 Split Game

We describe a way to split the path segments of a path in different lines according to their lengths. In this way we only need to use a single channel that all players can share. For ease of presentation, we first describe the respective game in the 2-dimensional grid, and then explain below how it can be extended to higher dimensions.

Let $G = (V, E)$ be a 2-dimensional $n \times n$ grid. Let $\alpha = \log n$. For convenience take n to be a multiple of $2\log n$. The odd index rows (columns) $1, 3, \ldots, n-1$ are used to route horizontal (vertical) path segments of lengths ranging from 2 to $n-1$. In particular, row $(2i+1) \bmod \alpha$ (column $2i \bmod \alpha$), where $i \in [0, n/2-1]$, is used for horizontal (vertical) path segments whose length is in range $[2^{i \bmod \alpha}, 2^{(i \bmod \alpha)+1} - 1]$. The even rows (columns) $0, 2, \ldots, n-2$ are reserved to route horizontal (vertical) path segments whose length is in range $[1, 2\alpha-1]$. Note that path segments in range $[2, 2\alpha-1]$ have a chance to be routed either in even or odd rows and columns. We say that an odd row (column) $2i+1$ $(2i)$ is of *type-($i \bmod \log n$)*, while any even row (column) is of the *local*-type. Note that there are $\alpha+1$ types in total. Any edge $e \in E$ has the same type of the row or column that it belongs to. Note that with splitting the path segments into different rows we achieved to have a single channel that all players can share.

We are now ready to define the *split bottleneck game* $\mathcal{R} = (G, \Pi, \mathcal{P})$. As in the basic bottleneck game, there is limit β on the number of bends of a path. Each path has to follow the rules for using the appropriate rows and columns for its segments as described above. The social and player cost functions are similar, $SC(\mathbf{p}) = C(\mathbf{p})$, and $pc_i(\mathbf{p}) = C_{\pi_i}(\mathbf{p}) = C_{p_i}(\mathbf{p})$. Similar to the basic congestion game we obtain:

Theorem 5. *Any split bottleneck game instance $\mathcal{R}$ has at least one Nash Equilibrium and $PoS(\mathcal{R}) = 1$.*

5.1 Price of Anarchy Analysis for Split Game

Consider a Nash equilibrium $\mathbf{p} \in \mathcal{P}$. Consider a set of players $\Pi' \subseteq \Pi$. We can define the special edge and special type for a player in the same way as we did for channel bottleneck games. The only difference is that instead of the notion of the channel we use the notion of the type. Let $\tau(\Pi')$ be the type which is special for the majority of the players in Π'. Using $\tau(\Pi')$ we can define the sets: $B(\Pi')$, $\Gamma(\Pi')$, $\Delta(\Pi')$, and $\Phi(\Pi')$, as we did in Section 4, where $\tau(\Pi')$ plays the role of $\widehat{A}(\Pi')$. We have that $|B(\Pi')| \geq |\Pi'|/(\alpha+1)$, since there are $\alpha+1$ types.

Lemma 3. *For any set of players $\Pi' \subseteq \Pi$, each edge $e \in \Delta(\Pi')$ has assigned to it at most $c_1\alpha C^*$ players of Π' in routing $\mathbf{p}$, for some constant c_1.*

From Lemma 3, each edge in $\Delta(\Pi')$ is assigned at most $c_1\alpha C^*$ players of $B(\Pi)$. Since $|B(\Pi')| \geq |\Pi'|/(\alpha+1)$, we have $|\Delta(\Pi')| \geq |\Pi'|/((\alpha+1)\cdot(c_1\alpha C^*))$. Therefore,

Corollary 3. *For any set of players $\Pi' \subseteq \Pi$, $|\Delta(\Pi')| \geq |\Pi'|/(c_2\alpha^2 C^*)$, for some constant c_2.*

Lemma 4. *For any set of players $\Pi' \subseteq \Pi$ with congestion at least C', $|\Phi(\Pi')| \geq (C'-1)|\Pi'|/(c_3\alpha^2\beta C^*)$, for some constant c_3.*

Theorem 6. $C(\mathbf{p}) \leq 2c_3\alpha^2\beta C^* + \log(2c_2\alpha^2 n^2 C^*)$.

From Theorem 6, since $\alpha = \log n$ and $N = n^2$, we obtain the following corollary:

Corollary 4. *For any split bottleneck game $\mathcal{R}$ in the 2-dimensional grid which allows paths with at most β bends, $PoA(\mathcal{R}) = O(\beta \log^2 N)$.*

5.2 Split Game in the d-Dimensional Grid

We can extend the split games to a grid with d dimensions. The first dimension takes the role of the horizontal dimension, and the second dimension takes the role of the vertical dimension. Any other dimension (third and above) uses the first dimension to split the path segments. For example, in the 3-dimensional grid, a path segment q in the third dimension is a sequence of nodes with coordinates $q = (x, y, z), \ldots, (x, y, z+k)$. This path segment is placed in an appropriate odd first coordinate $x = 2^i + 1$ if $k \in [2^{i \bmod \alpha}, 2^{(i \bmod \alpha)+1} - 1]$, and if $k \leq 2\alpha - 1$ then it could use an even first coordinate $x = 2^i$. In this way we can characterize q as *type-i*, or local type, respectively. The total number of types for the d-dimensional grid remains $\alpha + 1$.

The main difference in the price of anarchy analysis is that Theorem 6 now returns $C(\mathbf{p}) \leq 2c_3\alpha^2\beta C^* + \log(c_2\alpha^2 d n^d C^*)$. Since $\alpha = O(\log n)$ and $N = n^d$, Corollary 4 now becomes:

Corollary 5. *For any split bottleneck game $\mathcal{R}$ in the d-dimensional grid which allows paths with at most β bends, $PoA(\mathcal{R}) = O((\beta/d^2)\log^2 N)$.*

6 Lower Bounds

Here, we give lower bounds in terms of bends for the price of anarchy for the channel and split games.

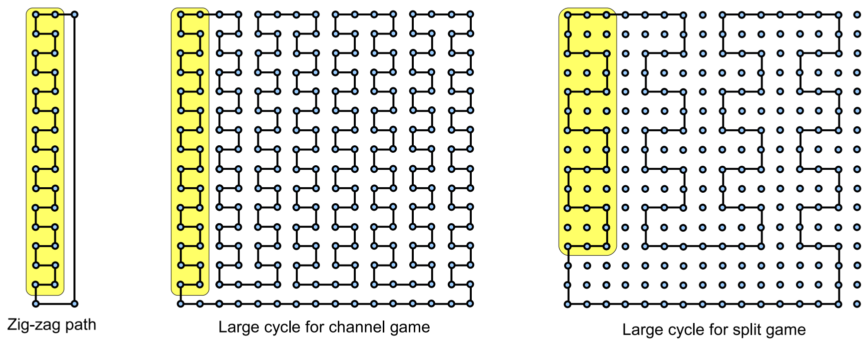

Fig. 2. Zig-zag path and cycles

Theorem 7. *In the d-dimensional grid with N nodes, given any $\beta \leq c'N$, for a specific constant c', there is a channel bottleneck game instance R with at most β bends, such that $PoA(R) = \Omega(\beta)$.*

Proof. We present the result for the 2-dimensional $n \times n$ grid G, and it can be extended to the d-dimensional grid. We define a game along a cycle c of the grid. The main building block of the cycle is the *zig-zag* path which is formed in two consecutive columns, by alternating edges between the columns and rows, as shown highlighted in the left of Figure 2. A x-zig-zag path contains x horizontal edges and $x-1$ vertical edges, giving $2x-2$ bends (without counting the end nodes). Given an x-zig-zag path we can build a cycle by closing the end points with 4 additional bends, giving a cycle with total $2x+2$ bends. Since $x < n-1$ (last row is reserved to close the cycle), the maximum number of bends that a single zig-zag path can provide is bounded by $2(n-1)+2 = 2n$.

In order to obtain a cycle with larger number of bends, we combine multiple zig-zag paths, as shown in the middle of Figure 2. The largest cycle is formed by using $n/2$ instances of $(n-1)$-zig-zag paths by combining their original version and their horizontal mirrors, and connecting them with bridge edges in rows 0 and $n-2$ and closing the loop with a path in row $n-1$ and bridge edges in the bottoms of columns 0 and $n-1$. This construction gives a cycle with total $\ell = (2(n-1)+2) \cdot n/2 + 4 = n^2 + n + 4$ bends. Using the above construction and adjusting appropriately the sizes of the zig-zag paths it is possible to obtain a cycle with any number of bends β up to ℓ. Clearly, the total number of edges in the cycle is $|c| = \Theta(\beta)$.

We define now a channel bottleneck game $\mathcal{R} = (G, \Pi, \mathcal{P})$. Let Z denote the set of edges in the zig-zag paths, excluding the edges adjacent to the end nodes of each zig-zag path. The game has $\kappa = |Z|$ players $\Pi = \{\pi_1, \ldots, \pi_\kappa\}$. Player π_i has two strategy sets: $\mathcal{P}_i = \{p_i^1, p_i^2\}$, where p_i^1 consists only of edge $e_i = (u_i, v_i) \in Z$ in a zig-zag path, and path p_i^2 consists of the alternate path in the cycle from v_i to u_i that traverses all the edges of c except e_i. The edges $e_i \in Z$ are chosen so that different players use different edges. Note that the first path has 2 bends, while the second path has β bends.

The optimal routing $\mathbf{p}^* \in \mathcal{P}$ is the one where each player π_i uses strategy p_i^1, namely, $\mathbf{p}^* = [p_1^1, p_2^1, \ldots, p_\kappa^1]$. The congestion of $\mathbf{p}^*$ is $C(\mathbf{p}^*) = 1$, since edge is used by at most one player. Consider now routing $\mathbf{p} = [p_1^2, p_2^2, \ldots, p_\kappa^2]$, consisting of the second strategy of each player. Routing $\mathbf{p}$ has congestion $C(\mathbf{p}) = \kappa - 1$, since all players except π_i use edge $e_i \in Z$ and all the path segments that use e_i belong to the same channel A_0 for unit length segments. The routing $\mathbf{p}$ is a Nash equilibrium, since if any user π_i attempts to switch to alternate strategy p_i^1, the congestion of the becomes $\kappa + 1 > C(\mathbf{p})$. Therefore we have that: $PoA \geq C(\mathbf{p})/C(\mathbf{p}^*) = \kappa - 1 = |Z| - 1 = \Omega(\beta)$.

Using similar zig-zag paths for the split model by adjusting appropriately the bend distances (see right of Figure 2) we can obtain the following lower bound:

Theorem 8. *In the d-dimensional grid with N nodes, given any $\beta \leq c''N$, for a specific constant c'', there is a split bottleneck game instance R with at most β bends, such that $PoA(R) = \Omega(\beta)$.*

7 Conclusions

We presented new bottleneck games on multidimensional grids whose price of anarchy is analyzed in terms of the number of bends that the paths are allowed to follow. We found that the price of anarchy is proportional to the number of bends. We also provided game instances that show that the price of anarchy results are tight within poly-log factors. A natural question that remains open is whether we can obtain tighter bounds by removing the poly-log factors. Another interesting problem is to study other network topologies and examine how the notion of bends is generalized in them.

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