# Hierarchical Auctions for Network Resource Allocation

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**Abstract.** Motivated by allocation of cloud computing services, bandwidth and wireless spectrum in secondary network markets, we introduce a hierarchical auction model for network resource allocation. The Tier 1 provider owns all of the resource, who holds an auction in which the Tier 2 providers participate. Each of the Tier 2 providers then holds an auction to allocate the acquired resource among the Tier 3 users. The Tier 2 providers play the role of middlemen, since their utility for the resource depends entirely on the payment that they receive by selling it. We first consider the case of indivisible resource. We study a class of mechanisms where each sub-mechanism is either a first-price or a second-price auction, and show that incentive compatibility and efficiency cannot be simultaneously achieved. We then consider the resource to be divisible and propose the hierarchical network second-price mechanism in which there exists an efficient Nash equilibrium with endogenous strong budget balance.

**Keywords:** Network economics, mechanism design, auctions, hierarchical models.

## 1 Introduction

As networks have become increasingly complex, so has the ownership structure. This means that the traditional models and resource allocation mechanisms that are used for resource exchange between the primary owners and the end-users are no longer always relevant. Increasingly, there are middlemen, operators who buy network resources from the primary owners and then sell them to the end-users. This potentially causes inefficiencies in network resource allocation.

Consider the case of bandwidth allocation. The network bandwidth is primarily owned by a Tier 1 ISP (Internet Service Provider) or carrier, who then sells it to various Tier 2 ISPs. The Tier 2 ISPs then sell it further either to corporate customers or to the Tier 3 ISPs, who provide service directly to consumers. The presence of the Tier 2 ISPs can potentially skew the network resource allocation, and cause it to be inefficient from a social welfare point of view. Another case

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in point, is the emerging market of cloud computing services. Providers such as IBM, Google, Amazon and others are providing cloud computing services which end-users (e.g., enterprises having small computational or data center needs) can buy. Of course, the distribution channel for these services is likely to involve middlemen. This raises the key question what hierarchical mechanisms can be used in the presence of middlemen that are incentive compatible and/or efficient.

Auctions as mechanisms for network resource allocation have received lots of attention recently. Following-up on the network utility model proposed by Kelly [12], Johari and Tsitsiklis showed that the Kelly mechanism can have up to 25% efficiency loss [10]. This led to a flurry of activity in designing efficient network resource allocation mechanisms, including the work of Maheshwaran and Basar [15], Johari and Tsitsiklis [11], Yang and Hajek [20], Jain and Walrand [5], Jia and Caines [8] among others [2,16]. Most of the work focused on one-sided auctions for divisible resources, and is related to the approach of Lazar and Semret [13]. Double-sided network auctions for divisible resources were developed in [5]. The only work to focus on indivisible network resources is Jain and Varaiya [6] which proposed a Nash implementation combinatorial double auction. This is also the only work known so far that presents incomplete information analysis of combinatorial market mechanisms [7].

All those mechanisms either involve network resource allocation by an auctioneer among multiple buyers, or network resource exchange among multiple buyers and sellers. Most of the proposed mechanisms are Nash implementations, i.e., truth telling is a Nash equilibrium but not necessarily a dominant strategy equilibrium, and have either unique Nash equilibria which are efficient, or at least one that is. In reality, however, markets for network resources often have middlemen operators, and efficiency can be rather hard, if not impossible to achieve with their presence. Unfortunately, models with middlemen have not been studied at all, primarily due to the difficulty of designing appropriate mechanisms. Even in the economic and game theory literature, the closest related auction models are those that involve a resale after an auction. That is, there is only a single tier auction, and the winners can then resell the resources acquired in the auction [3].

There is indeed some game-theoretic work on network pricing in a more general topology. Johari, Mannor and Tsitsiklis [9] studied a network game where the nodes of the network wish to form a graph to route traffic between themselves, and they characterized connected link stable equilibria. Shakkottai and Srikant [18] examined how transit and customer prices and quality of service are set in a network consisting of multiple ISPs, where a 3-tier hierarchical model is proposed. However, such work only focused on the pricing equilibrium, and issues like mechanism design and auctions were never studied.

In this paper, we consider a multi-tier setting. We consider a homogeneous network resource. This could be bandwidth, wireless spectrum or cloud computing service, all owned by a single entity, the Tier 1 provider. He conducts an auction to allocate the resource among the Tier 2 operators. The Tier 2 operators then further allocate the network resource they have acquired in the auction among the Tier 3 entities, who may be the end-users. Each Tier 3 user has a valuation for the resource, which is strictly increasing and concave with respect to the capacity. On the other hand, the Tier 2 entities are more like middlemen. They do not have any intrinsic valuation for the network resource but a quasi-valuation which depends on the revenue that they will acquire by selling it off in an auction. Our goal is to design a *hierarchical auction mechanism* that specifies one sub-mechanism for each tier.

We develop a general hierarchical mechanism design framework and consider the setting where all auctions are conducted simultaneously. Admittedly, this does not fully meet the reality (where auctions in different tiers may take place one after another), but provides insights into the problem from a theoretical point of view. We first consider the resource to be indivisible. We investigate a class of mechanisms where each sub-mechanism is either a first-price or a second-price auction. We show that the all-tier second-price auction mechanism is incentive compatible but not efficient, i.e., social-welfare maximizing. This is a surprising observation and the only known instance of its type involving the VCG/second-price mechanism [19]. We then show that the hierarchical mechanism with a first-price or a second-price sub-mechanism at Tier 1, and first-price sub-mechanisms at all other tiers is indeed efficient but not incentive compatible.

When the resource is divisible, it is impossible for bidders to report their arbitrary real-valued valuation functions. They are thus asked to report a two-dimensional bid signal, i.e., a per-unit bid price and the maximum quantity that they want to buy/sell. We note that while the Tier 1 sub-mechanism is a single-sided auction, the sub-mechanisms at all lower tiers are double-sided auctions. In this framework, we propose a hierarchical mechanism with a VCG-type auction at each tier. We show that in this hierarchical mechanism, there exists an efficient Nash equilibrium that is strongly budget-balanced at all tiers except the top tier, where a single-sided auction is conducted.

## 2 Model and Problem Statement

### 2.1 The Hierarchical Model

Consider a Tier 1 provider who owns a homogeneous network resource, say bandwidth. Such an entity could be either a carrier (e.g., AT&T), the FCC, or a cloud service provider such as IBM. The Tier 1 provider auctions $C$ units of the resource among Tier 2 operators via a single-sided auction. We will refer to this as the *Tier 1 auction*. Each of the Tier 2 operators then auctions off the resource acquired in the Tier 1 auction to the Tier 3 operators. These will be referred to as the *Tier 2 auctions*, and in general at Tier $k$ as the *Tier $k$ auctions*. We will assume that there are $K$ tiers. The only Tier 1 provider will be considered as the social planner, and the Tier $K$ entities are the end-users or the consumers, while operators at all other tiers will be considered as middlemen. Tier $k$ ($1 < k \leq K$) operators can participate in and acquire the resource only from their Tier $k-1$ parent. That is, each middleman has exclusive access to his children, and there is only one seller in each auction. This simplifies the network topology. Extension (e.g., allowing competition among sellers) is possible but would be much more complicated.

An example of the network topology is shown in Fig. 1. We now introduce the notation to ease further discussion. Let the nodes in the tree network be numbered $i = 0, 1, \ldots, N$ with $M$ terminal nodes, where node 0 at the root of the tree is the social planner. Let $\mathcal{T}(i)$ be a function that specifies the tier to which node $i$ belongs. The tier of node 0 is considered as Tier 1. By $\texttt{parent}(i)$, we shall denote the parent of node $i$ in the tree network, and by $\texttt{children}(i)$, we shall denote the set of the children of node $i$. Each node represents a player. To avoid cumbersomeness, we shall use the redundant notation $P_i^{(k)}$ for node $i$ that is at Tier $k$. Let the capacity acquired by $P_i^{(k)}$ denoted by $x_i^{(k)}$ (in the Tier $k-1$ auction), and the capacity that node $i$ offers denoted by $y_i^{(k)}$ (in the Tier $k$ auction). Note that $\sum_{j\in\texttt{children}(i)} x_j^{(k+1)} \le y_i^{(k)} \le x_i^{(k)}$, for $k = 1, ..., K-1$.

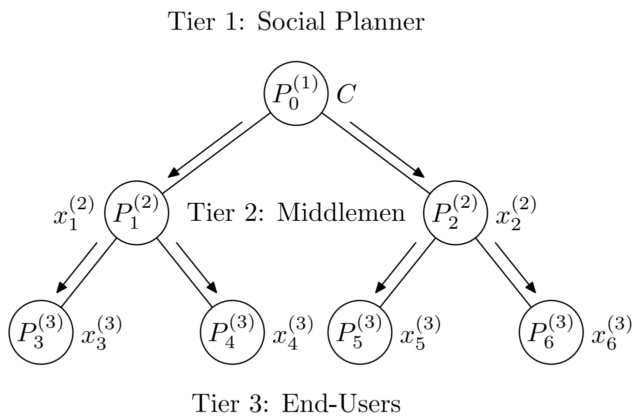


**Fig. 1.** An example of a 3-tier network

### 2.2 The Mechanism Design Framework

We now describe the mechanism design framework. We assume that each player $P_i^{(k)}$ has a quasi-linear utility function $u_i^{(k)}(x, \overline{w}_i) = v_i^{(k)}(x) - \overline{w}_i$, where $v_i^{(k)}(x)$ is the valuation of player $P_i^{(k)}$ when he is allocated a capacity $x$, and $\overline{w}_i$ is the payment made to his parent. Typically, for the middlemen $P_i^{(k)}$ $(k = 2, \ldots, K-1)$, $v_i^{(k)}(x) = \underline{w}_i - c_i^{(k)}(x)$, where $\underline{w}_i$ is the revenue from reselling and $c_i^{(k)}(x)$ is the cost function, since they do not derive any utility from the allocation but may incur a transaction cost.

We define the *social welfare* to be the total utility derived by the end-users minus the total cost incurred by the middlemen, i.e.,

$$S(\mathbf{x}) = \sum_{i:\mathcal{T}(i)=K} v_i^{(K)}(x_i^{(K)}) - \sum_{2\le k\le K-1} \sum_{j:\mathcal{T}(j)=k} c_j^{(k)}(x_j^{(k)})$$

where $\mathbf{x} = (x_1, \ldots, x_N)$. The social planner's objective is to realize an *(allocatively) efficient* allocation $\mathbf{x}^{**}$ that maximizes the social welfare, and solves

$$\begin{aligned}
\textbf{HN-OPT}: \quad & \max \quad S(\mathbf{x}) && && (1)\\
\text{s.t.} \quad & \sum_{i \in \texttt{children}(0)} x_i^{(2)} && \leq C, \\
& \sum_{i \in \texttt{children}(j)} x_i^{(k+1)} && \leq x_j^{(k)}, \quad \forall (j,k): 2 \leq k \leq K-1, \\
& x_i^{(k)} && \geq 0, \quad \forall (i,k): 2 \leq k \leq K.
\end{aligned}$$

The first constraint follows because in the Tier 1 auction, the auctioneer (player 0) allocates the total capacity $C$ among the Tier 2 players. The second constraint follows from the fact that the total allocation among the buyers in the Tier $k$ auction cannot be greater than the allocation received by $P_j^{(k)}$ from the Tier $k-1$ auction. The third constraint is required to ensure non-negative allocations. Furthermore, we could consider the resource to be indivisible and let the $x_i$'s to be integral, or consider it to be divisible and allow the $x_i$'s to be real.

The social planner cannot achieve the objective (i.e., social-welfare maximizing) by himself as he does not know the valuation and cost functions of the end-users and the middlemen respectively. Thus, a decentralized implementation is necessary. However, the strategic players are selfish and may misreport their information. Furthermore, in the hierarchical model, the mechanism is distributed with multiple auctions at each tier. This makes the achievement of the social welfare maximization even more difficult.

Our goal thus is an incentive *mechanism* $\Gamma$ that is composed of various sub-mechanisms $(\Gamma_i^{(k)}, i = 0, 1, \ldots, N-M, k = \mathcal{T}(i))$. Each sub-mechanism (i.e., auction) $\Gamma_i^{(k)}$ is conducted at each node $i$ of the tree, except the $M$ leaf (terminal) nodes. Note that the auction $\Gamma_i^{(k)}$ involves player $P_i^{(k)}$ as a seller, and the players $\texttt{children}(i)$ as the buyers. Note that node 0 acts only as an seller and the terminal nodes only act as buyers, whereas the middlemen $P_i^{(k)}$ $(2 \leq k \leq K-1)$ act as buyers in the Tier $k-1$ auction, and as sellers in the Tier $k$ auction. Generally, each $\Gamma_i^{(k)}$ can be different, though we consider the setting for which $\Gamma_i^{(k)} = \Gamma^{(k)}$, i.e., a common sub-mechanism is used at each Tier $k$. Still, this is a simplification but subject to extension.

Since the middlemen have no intrinsic valuation for the resource itself, we define the notion of quasi-valuation functions for the middlemen. Let $\mathbb{X}$ denote the allocation space, which is $\mathbb{Z}_+$ when the resource is indivisible and $\mathbb{R}_+$ when the resource is divisible.

**Definition 1.** *A quasi-valuation function of player $P_i^{(k)}$ is a function $\bar{v}_i^{(k)}$ : $\mathbb{X} \to \mathbb{R}_+$ that specifies the revenue he receives in the auction $\Gamma_i^{(k)}$ from his children for each possible allocation, when all the players* $\texttt{children}(i)$ *report their valuation functions (for end-users) or quasi-valuation functions (for middlemen) truthfully.*

Note the backward-recursiveness in the definition of quasi-valuation functions. They can be easily computed by the players in complete information settings. We now see the role of such functions in defining hierarchical incentive compatible and Nash implementation mechanisms.

**Definition 2.** *The (direct) hierarchical mechanism $\Gamma = (\Gamma^{(1)}, \ldots, \Gamma^{(K-1)})$ is incentive compatible (or strategy-proof) if there is a dominant strategy equilibrium wherein all the end-users report their valuation functions and all the middlemen report their quasi-valuation functions, truthfully.*

Such equilibrium strategies will be referred to as "truth telling" as a counterpart to standard notions of "truth-telling" in non-hierarchical mechanisms [17]. We now define the notion of efficiency in hierarchical mechanisms.

**Definition 3.** *The (direct) hierarchical mechanism $\Gamma = (\Gamma^{(1)}, \ldots, \Gamma^{(K-1)})$ is* efficient *if there is an equilibrium that maximizes the social welfare in the optimization problem* ***HN-OPT*** *(1).*

We study *simultaneous hierarchical mechanisms*, in which all sub-mechanism auctions take place simultaneously (which are modeled as a normal form game). Thus, usual notions of *Nash equilibrium* shall be studied [1,4].

## 3 Hierarchical Auctions for Indivisible Resources

When the resource is indivisible, we present a class of hierarchical mechanisms $\Gamma = (\Gamma^{(1)}, \ldots, \Gamma^{(K-1)})$ wherein a common sub-mechanism is used at each tier, and each such sub-mechanism is either a first-price auction (denoted by $\mathcal{F}$) or a second-price auction (denoted by $\mathcal{S}$), i.e., $\Gamma^{(k)} \in \{\mathcal{F}, \mathcal{S}\}$. We investigate the efficiency and incentive compatibility of such hierarchical mechanism designs.

We first consider the case where there is only a single unit to be allocated, i.e., $C = 1$. Here, we assume that the middlemen have no transaction costs, i.e., $c_i^{(k)}(x) = 0$. We note that the introduction of transaction costs would be trivial in the case of indivisible resources, and it can be easily extended if desired.

Let $b_i^{(k)}$ denote the buy-bid of player $i$ who is at Tier $k$, and $x_i^{(k)}$ the unit he acquires (in the Tier $k-1$ auction as defined). Recall that there are $N - M + 1$ auctions that are conducted simultaneously, though some auction outcomes cannot be fulfilled since there is only a single indivisible unit. This, however, is not unreasonable since there is really a single winner among the end-users. The middlemen that connect this end-user to the root will also be purported to be winners.

**Theorem 1.** *Assume each player except the end-users has at least two children. Suppose a single indivisible unit is to be allocated through a hierarchical auction mechanism $\hat{\Gamma}$ such that $\hat{\Gamma}^{(1)} \in \{\mathcal{F}, \mathcal{S}\}, \hat{\Gamma}^{(2)} = \cdots = \hat{\Gamma}^{(K-1)} = \mathcal{F}$. Then, there exists an $\epsilon$-Nash equilibrium which is efficient.*

*Proof.* We prove by construction. Consider a Tier $K-1$ auction $\hat{\Gamma}_i^{(K-1)}$. Find the player that has the highest valuation in that auction, i.e., $j^* \in \arg\max_{j \in \texttt{children}(i)} v_j^{(K)}$. Define the bids of the player as follows

$$b_{j^*}^{(K)} = v_{j^*}^{(K)},$$

$$b_j^{(K)} = v_{j^*}^{(K)} - \epsilon, \forall j \in \texttt{children}(i), j \neq j^*,$$

i.e., player $j^*$ bids truthfully, while all others in that auction bid just a bit below.

Consider a Tier $k$ auction $\Gamma_i^{(k)}$ $(1 < k < K-1)$. As before, find a player $j^* \in \arg\max_{j \in \texttt{children}(i)} \bar{v}_j^{(k+1)}$, and define the bids of the players in this auction as

$$b_{j^*}^{(k+1)} = \bar{v}_{j^*}^{(k+1)},$$

$$b_j^{(k+1)} = \bar{v}_{j^*}^{(k+1)} - \epsilon, \forall j \in \texttt{children}(i), j \neq j^*.$$

Now, consider the Tier 1 auction $\hat{\Gamma}^{(1)}$.Find a player $j^* \in \arg\max_{j \in \texttt{children}(0)} \bar{v}_j^{(2)}$. If $\hat{\Gamma}^{(1)} = \mathcal{F}$, define the bids of players in this auction as

$$b_{j^*}^{(2)} = \bar{v}_{j^*}^{(2)},$$

$$b_j^{(2)} = \bar{v}_{j^*}^{(2)} - \epsilon, \forall j \in \texttt{children}(0), j \neq j^*.$$

Otherwise, if $\hat{\Gamma}^{(1)} = \mathcal{S}$, define the bids of players in this auction as

$$b_j^{(2)} = \bar{v}_j^{(2)}, \forall j \in \texttt{children}(0).$$

It is obvious that such bids induce the efficient allocation. We argue that these bids constitute an $\epsilon$-Nash equilibrium. Note that every player gets a non-negative payoff in such a bid profile.

Consider a player $P_i^{(K)}$. If he is a winner, he has no incentive to increase his bid since $b_i^{(K)} = v_i^{(K)}$, and he has no incentive to decrease his bid since there exists a player $P_{i'}^{(K)}$ with $\texttt{parent}(i) = \texttt{parent}(i')$ whose bid is $b_i^{(K)} - \epsilon$. If he is a loser, we have $b_i^{(K)} > v_i^{(K)} - \epsilon$. Clearly, he has no incentive to either increase or decrease his bid.

Consider a player $P_i^{(k)}$ $(2 < k < K)$. If he is a winner, he has no incentive to increase his bid since $b_i^{(k)} = \bar{v}_i^{(k)}$, and he has no incentive to decrease his bid since there exists a player $P_{i'}^{(k)}$ with $\texttt{parent}(i) = \texttt{parent}(i')$ whose bid is $b_i^{(k)} - \epsilon$. If he is a loser, we have $b_i^{(k)} > \bar{v}_i^{(k)} - \epsilon$. Clearly, he has no incentive to either increase or decrease his bid.

It is also easy to verify that such bids are the best responses of the Tier 2 players for $\hat{\Gamma}^{(1)} = \mathcal{F}$ and $\hat{\Gamma}^{(1)} = \mathcal{S}$ respectively. This proves the claim.

The following example shows that the above mechanism $\hat{\Gamma}$ achieves efficiency but is not incentive compatible.

**Proposition 1.** *The hierarchical mechanism $\hat{\Gamma}$ is efficient but not incentive compatible.*

*Proof.* We prove by providing a counter example. Assume the network topology is as in Fig. 1, i.e., there are two Tier 2 players $P_1^{(2)}$ (with his Tier 3 children $P_3^{(3)}$, $P_4^{(3)}$) and $P_2^{(2)}$ (with his Tier 3 children $P_5^{(3)}$, $P_6^{(3)}$). Let the valuation functions of the Tier 3 players be

$$v_3^{(3)} = 2, v_4^{(3)} = 3, v_5^{(3)} = 1, v_6^{(3)} = 4.$$

Since $\hat{\Gamma}^{(2)} = \mathcal{F}$, the quasi-valuation functions of the Tier 2 players can be easily computed to be

$$\bar{v}_1^{(2)} = 3, \bar{v}_2^{(2)} = 4.$$

However, truth telling is not an equilibrium in this auction. Rather, it is easy to verify that an $\epsilon$-Nash equilibrium is

$$(b_1^{(2)}, b_2^{(2)}) = (4 - \epsilon, 4),$$

$$(b_3^{(3)}, b_4^{(3)}, b_5^{(3)}, b_6^{(3)}) = (3 - \epsilon, 3, 4 - \epsilon, 4).$$

The corresponding equilibrium allocation is

$$(x_1^{(2)}, x_2^{(2)}) = (0, 1),$$

$$(x_3^{(3)}, x_4^{(3)}, x_5^{(3)}, x_6^{(3)}) = (0, 0, 0, 1),$$

which is exactly the efficient allocation.

Thus, this mechanism is efficient but not incentive compatible.

We now introduce a natural hierarchical extension of the second-price or VCG auction mechanism.

**Theorem 2.** *Suppose multiple units of an indivisible resource are to be allocated through a hierarchical auction mechanism $\tilde{\Gamma}$ such that $\tilde{\Gamma}^{(1)} = \cdots = \tilde{\Gamma}^{(K-1)} = \mathcal{S}$ (which we shall call the second-price hierarchical auction). Then, the mechanism is incentive-compatible.*

*Proof.* We argue by backward induction that truth telling is a dominant strategy equilibrium. Consider the Tier $K-1$ auction, which is a second-price sub-mechanism. The end-user $P_i^{(K)}$ ($\mathcal{T}(i) = K$) will bid truthfully, no matter how the other players bid and what capacity his parent is allocated, since that is his dominant strategy in a second-price auction. This is the fundamental property of the VCG mechanism.

Given that all the $P_i^{(K)}$'s report truthfully, the quasi-valuation functions of the players $P_i^{(K-1)}$'s are true. Furthermore, the Tier $K-2$ auction is again a VCG mechanism in which truth-telling is a dominant strategy equilibrium.

Now, we can argue by backward induction. Assuming the Tier $k+1$ players in the Tier $k$ auctions have true quasi-valuation functions, they will bid truthfully. So, the quasi-valuation functions of the Tier $k$ players will be true as well. Since this is true for $k = K-1$, it is true for all $k = K-1, \ldots, 1$. Hence, all the players bid truthfully, and the hierarchical mechanism is incentive-compatible.

The second-price hierarchical auction mechanism, as can be expected, has truth telling by each player as a dominant strategy equilibrium. The surprise is that unlike non-hierarchical settings, efficiency may not be achieved.

**Proposition 2.** *The second-price hierarchical mechanism $\tilde{\Gamma}$ is not necessarily efficient.*

*Proof.* We prove by providing a counter example. Let $C = 5$ be allocated by the second-price hierarchical mechanism in a 3-tier network as in Fig. 1. Let the valuation functions of the Tier 3 players be

$$(v_3^{(3)}(x), x = 1, 2, 3, 4, 5) = (10, 18, 24, 28, 30),$$

$$(v_4^{(3)}(x), x = 1, 2, 3, 4, 5) = (20, 25, 29, 32, 34),$$

$$(v_5^{(3)}(x), x = 1, 2, 3, 4, 5) = (15, 24, 32, 39, 45),$$

$$(v_6^{(3)}(x), x = 1, 2, 3, 4, 5) = (16, 20, 24, 27, 29).$$

According to (1), the efficient allocation is

$$(x_1^{(2)**}, x_2^{(2)**}) = (2, 3),$$

$$(x_3^{(3)**}, x_4^{(3)**}, x_5^{(3)**}, x_6^{(3)**}) = (1, 1, 2, 1).$$

Since $\tilde{\Gamma}^{(2)} = \mathcal{S}$, the quasi-valuation functions of the Tier 2 players can be easily computed to be

$$(\bar{v}_1^{(2)}(x), x = 1, 2, 3, 4, 5) = (10, 13, 15, 16, 15),$$

$$(\bar{v}_2^{(2)}(x), x = 1, 2, 3, 4, 5) = (15, 13, 16, 18, 19).$$

In the mechanism $\tilde{\Gamma}$, truth telling is a Nash equilibrium as we have already proved. Thus, the corresponding equilibrium allocation is

$$(x_1^{(2)}, x_2^{(2)}) = (4, 1),$$

$$(x_3^{(3)}, x_4^{(3)}, x_5^{(3)}, x_6^{(3)}) = (3, 1, 0, 1),$$

which however, is different from the efficient allocation.

Thus, in the case of multiple units of an indivisible resource, this hierarchical mechanism is incentive-compatible but not efficient.

An even greater surprise is the following impossibility result if we restrict our attention to first-price and second-price sub-mechanisms.

**Theorem 3 (Hierarchical Impossibility).** *Suppose we allocate a single unit of the indivisible resource through a hierarchical auction mechanism $\Gamma$ such that $\Gamma^{(k)} \in \{\mathcal{F}, \mathcal{S}\}$ (for all $k = 1, \ldots, K-1$ and $K \geq 3$). Then, there exists no such hierarchical mechanism which is both incentive-compatible and efficient.*

*Proof.* As we have already seen in Proposition 1 that incentive compatibility is not guaranteed if there exists a $k$ such that $\Gamma^{(k)} = \mathcal{F}$. We have also seen in Proposition 2 that efficiency is not guaranteed if there exists a $k$ such that $\Gamma^{(k)} = \mathcal{S}$. Thus, if the choices of the $\Gamma^{(k)}$'s are restricted to the two alternatives ($\mathcal{F}$ or $\mathcal{S}$), incentive compatibility and efficiency cannot be simultaneously achieved.

Our conjecture is that this "limited" impossibility theorem foretells a more general impossibility result for hierarchical mechanism design.

## 4 Hierarchical Auctions for Divisible Resources

We now consider the resource to be divisible, and propose a hierarchical auction mechanism. We will now consider the setting where the middlemen have transaction costs as well. While the Tier 1 auction will remain single-sided, Tier 2 through Tier $K-1$ auctions will be double-sided, i.e., in such auctions buyers will make buy-bids, and sellers will make sell-bids.

For simplicity of exposition, we will only consider a 3-tier network as in Fig. 1. Also, we drop the superscripts and adopt a more concise notation here, i.e., denote the $i$th Tier 2 player by $P_i$ and the $j$th child of $P_i$ by $P_{ij}$ (Tier 3 player). The notations of valuation functions, bids, etc. are changed correspondingly.

We will assume that the valuation functions of the end-users, $v_{ij}(x_{ij})$ are strictly increasing and concave, and smooth, with $v_{ij}(0) = 0$. The cost functions of the middlemen, $c_i(x_i)$ are assumed to be strictly increasing and convex, and smooth, with $c_i(0) = 0$.

The end-user's payoff is $u_{ij} = v_{ij}(x_{ij}) - \overline{w}_{ij}$, where $\overline{w}_{ij}$ is the payment made by player $P_{ij}$ to player $P_i$. A middleman $P_i$ has a utility $u_i = \underline{w}_i - \overline{w}_i - c_i(x_i)$, where $\underline{w}_i$ is $P_i$'s revenue from reselling and $\overline{w}_i$ is the payment made by $P_i$ to player 0.

In this setting the social welfare optimization problem is as following

$$\begin{aligned} \textbf{DIV-OPT}: \quad \max \quad & \sum_{i,j} v_{ij}(x_{ij}) - \sum_i c_i(x_i) \\ \text{s.t.} \quad & \sum_i x_i \leq C, \quad [\lambda_0] \\ & \sum_j x_{ij} \leq x_i, \quad \forall i, \quad [\lambda_i] \\ & x_i \geq 0, \quad \forall i, \\ & x_{ij} \geq 0, \quad \forall (i,j). \end{aligned} \tag{2}$$

Here, $\lambda_0$ and $\lambda_i$'s are the Lagrange multipliers of the corresponding constraints above. The above is a convex optimization problem, and a solution exists, which is characterized by the KKT conditions [14]

$$\begin{aligned} (c_i'(x_i) + \lambda_0 - \lambda_i)x_i &= 0, \quad \forall i, \\ c_i'(x_i) + \lambda_0 - \lambda_i &\geq 0, \quad \forall i, \\ (v_{ij}'(x_{ij}) - \lambda_i)x_{ij} &= 0, \quad \forall (i,j), \\ v_{ij}'(x_{ij}) - \lambda_i &\leq 0, \quad \forall (i,j), \\ \lambda_0(\sum_i x_i - C) &= 0, \\ \lambda_i(\sum_j x_{ij} - x_i) &= 0, \quad \forall i. \end{aligned} \tag{3}$$

Our objective is to design a hierarchical mechanism to allocate the divisible resource that achieves the social welfare optimum despite the strategic behavior of the players. An important issue in the context of divisible resources is that it is impossible for a bidder to communicate a complete arbitrary real-valued valuation function. Thus, the bidders must communicate an approximation to it from a finite-dimensional bid space.

**Hierarchical Network Second-Price Mechanism**

We now propose the hierarchical network second-price (**HNSP**) mechanism $\bar{\Gamma}$ that can be used to allocate a divisible resource in a multi-tier network. We take a 3-tier network as an example. The mechanism $\bar{\Gamma} = (\bar{\Gamma}^{(1)}, \bar{\Gamma}^{(2)})$ is composed of two sub-mechanisms $\bar{\Gamma}^{(1)}$ and $\bar{\Gamma}^{(2)}$. The sub-mechanism $\bar{\Gamma}^{(1)}$ employed at Tier 1 is a single-sided VCG-type auction mechanism in which Tier 2 players (the middlemen) report bids $b_i = (\beta_i, d_i)$ where $\beta_i$ is interpreted to be the per-unit bid price, and $d_i$ as the maximum quantity wanted. The sub-mechanism $\bar{\Gamma}^{(2)}$ employed at Tier 2 is a double-sided VCG-type auction mechanism where Tier 2 players report sell-bids $a_i = (\alpha_i, q_i)$ where $\alpha_i$ is the per-unit sell-bid price and $q_i$ is the maximum quantity offered for sale, while the Tier 3 players (end-users) report buy-bids $b_{ij} = (\beta_{ij}, d_{ij})$ where $\beta_{ij}$ is the per-unit buy-bid price and $d_{ij}$ is the maximum quantity wanted.

Once the bids are received in all the auctions, the auction outcomes are determined as follows. In the Tier 1 auction $\bar{\Gamma}^{(1)}$, the allocation $\hat{\mathbf{x}}$ is a solution of the optimization problem

$$\begin{aligned} \textbf{HNSP-1}: \quad \max \quad & \textstyle\sum_i \beta_i x_i \\ \text{s.t.} \quad \sum_i x_i \quad &\leq \quad C, \quad [\lambda_0'] \\ x_i \quad &\leq \quad d_i, \quad \forall i, \quad [\mu_i] \\ x_i \quad &\geq \quad 0, \quad \forall i. \end{aligned} \tag{4}$$

Let $\hat{\mathbf{x}}^{-i}$ denote the allocation as a solution of the above with $d_i = 0$, i.e., when the player $P_i$ (a middleman) is not present. Then, the payment made by $P_i$ is

$$\overline{w}_i = \sum_{j \neq i} \beta_j (\hat{x}_j^{-i} - \hat{x}_j), \tag{5}$$

which is the "externality" that $P_i$ imposes on all the other players (in the Tier 1 auction) by his participation. Let $\lambda_0'$ and $\mu_i$'s be the Lagrange multipliers of the corresponding constraints. Then, the solution of **HNSP-1** is characterized by the KKT conditions

$$\begin{aligned} (\beta_i - \lambda_0' - \mu_i) x_i = 0, \quad & \forall i, \\ \beta_i - \lambda_0' - \mu_i \leq 0, \quad & \forall i, \\ \lambda_0' (\sum_i x_i - C) = 0, & \\ \mu_i (x_i - d_i) = 0, \quad & \forall i. \end{aligned} \tag{6}$$

In the Tier 2 auction $\bar{\Gamma}^{(2)}$, the middleman is the seller and his children (the end-users) are the buyers. The sub-mechanism $\bar{\Gamma}^{(2)}$ is a VCG-type double-sided auction, i.e., both the seller and the buyers place bids, and the allocation $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is a solution of the optimization problem

$$\begin{aligned} \textbf{HNSP-2}: \quad \max \quad & \sum_j \beta_{ij} x_{ij} - \alpha_i y_i \\ \text{s.t.} \quad & \sum_j x_{ij} \leq y_i, \quad [\lambda_i'] \\ & x_{ij} \leq d_{ij}, \quad \forall j, \quad [\mu_{ij}] \\ & y_i \leq q_i, \quad [\nu_i] \\ & x_{ij} \geq 0, \quad \forall j, \\ & y_i \geq 0. \end{aligned} \tag{7}$$

Let $(\tilde{\mathbf{x}}^{-j}, \tilde{\mathbf{y}}^{-j})$ denote the allocation as a solution of the above with $d_{ij} = 0$. Then, the payment made by player $P_{ij}$ is

$$\overline{w}_{ij} = \sum_{k \neq j} \beta_{ik} (\tilde{x}_{ik}^{-j} - \tilde{x}_{ik}) - \alpha_i (\tilde{y}_i^{-j} - \tilde{y}_i), \tag{8}$$

and the payment received by player $P_i$ is

$$\underline{w}_i = \sum_j \beta_{ij} \tilde{x}_{ij}. \tag{9}$$

These transfers are the externalities that the players impose on the others through their participation. Let $\lambda_i'$'s, $\mu_{ij}$'s and $\nu_i$'s be the Lagrange multipliers corresponding to the constraints in the **HNSP-2** above. Then, the solution is characterized by the following KKT conditions

$$\begin{aligned}(\beta_{ij} - \lambda_i' - \mu_{ij})x_{ij} &= 0, \quad \forall j, \\ \beta_{ij} - \lambda_i' - \mu_{ij} &\leq 0, \quad \forall j, \\ (\alpha_i - \lambda_i' + \nu_i)y_i &= 0, \\ \alpha_i - \lambda_i' + \nu_i &\geq 0, \\ \lambda_i'(\sum_j x_{ij} - y_i) &= 0, \\ \mu_{ij}(x_{ij} - d_{ij}) &= 0, \quad \forall j \\ \nu_i(y_i - q_i) &= 0.\end{aligned} \tag{10}$$

This completes the definition of the **HNSP** mechanism.

We now show the existence of an efficient Nash equilibrium in the simultaneous hierarchical network second-price mechanism by construction. Moreover, we show that the Tier 2 sub-mechanism $\bar{\Gamma}^{(2)}$ achieves *endogenous strong budget balance* at this equilibrium, i.e., the payment received by each middleman equals the total payments made by his children.

**Theorem 4.** *In the **HNSP** mechanism $\bar{\Gamma}$, there exists an efficient Nash equilibrium with endogenous strong budget balance.*

*Proof.* Let $\mathbf{x}^{**}$ be an efficient allocation corresponding to the problem **DIV-OPT** in (2). Then, there exist Lagrange multipliers $\lambda_0$ and $\lambda_i$'s that satisfy the KKT conditions (3). Consider the bid profile $d_i = q_i = x_i^{**}$, $d_{ij} = x_{ij}^{**}$, $\beta_i = \sum_j v_{ij}'(x_{ij}^{**}) - c_i'(x_i^{**})$, $\alpha_i = c_i'(x_i^{**}) + \lambda_0$, and $\beta_{ij} = v_{ij}'(x_{ij}^{**})$.

First, we prove that the bid profile induces the efficient allocation. Let $\lambda_0' = \lambda_0$, $\lambda_i' = \lambda_i$, $\mu_i = \sum_j \beta_{ij} - \lambda_i$ and $\mu_{ij} = \nu_i = 0$. Then the KKT conditions (6) and (10) are equivalent to the KKT conditions (3). This implies $\mathbf{x}^{**}$ is also a solution of the problem **HNSP-1** in (4) and the problem **HNSP-2** in (7) with these bids.

Now, we prove that the strategy profile is a Nash equilibrium. Consider an end-user $P_{ij}$ with bid $(\beta_{ij}, d_{ij})$. His payoff at the efficient allocation is $u_{ij} = v_{ij}(x_{ij}^{**}) - \overline{w}_{ij} = v_{ij}(x_{ij}^{**}) - \alpha_i x_{ij}^{**}$. Then, given the bids of others, if he changes his bid to decrease his allocation $x_{ij}^{**}$ by a $\delta > 0$ (when $x_{ij}^{**} > 0$), then note that the allocations of buyers $P_{ik}$ $(k \neq j)$ do not change but seller $P_i$ sells less. His new payoff is $u_{ij}' = v_{ij}(x_{ij}^{'*}) - \alpha_i x_{ij}^{'*} = v_{ij}(x_{ij}^{**} - \delta) - \alpha_i(x_{ij}^{**} - \delta)$. Thus, his payoff changes by

$$\begin{aligned}&u_{ij}' - u_{ij} \\ &= \alpha_i \delta + v_{ij}(x_{ij}^{**} - \delta) - v_{ij}(x_{ij}^{**}) \\ &= (c_i'(x_i^{**}) + \lambda_0)\delta + v_{ij}(x_{ij}^{**} - \delta) - v_{ij}(x_{ij}^{**}) \\ &= \lambda_i \delta + v_{ij}(x_{ij}^{**} - \delta) - v_{ij}(x_{ij}^{**}) \\ &= v_{ij}'(x_{ij}^{**})\delta + v_{ij}(x_{ij}^{**} - \delta) - v_{ij}(x_{ij}^{**}) \\ &< 0.\end{aligned}$$

Thus, his payoff will decrease. Now suppose he changes his bid to increase his allocation $x_{ij}^{**}$ by a $\delta > 0$, then note that the allocation of player $P_i$ does not change but that of some players $P_{ik}$ $(k \neq j)$ decrease. His new payoff is $u'_{ij} = v_{ij}(x^{**}_{ij} + \delta) - \sum_{k \neq j} \beta_{ik}(x^{**}_{ik} - x'^{*}_{ik}) - \alpha_i x^{**}_{ij}$. Thus,

$$\begin{aligned}
&u'_{ij} - u_{ij} \\
&= -\sum_{k \neq j} \beta_{ik}(x^{**}_{ik} - x'^{*}_{ik}) + v_{ij}(x^{**}_{ij} + \delta) - v_{ij}(x^{**}_{ij}) \\
&\leq -\sum_{k \neq j} \lambda'_i(x^{**}_{ik} - x'^{*}_{ik}) + v_{ij}(x^{**}_{ij} + \delta) - v_{ij}(x^{**}_{ij}) \\
&= -\lambda'_i \delta + v_{ij}(x^{**}_{ij} + \delta) - v_{ij}(x^{**}_{ij}) \\
&= -\lambda_i \delta + v_{ij}(x^{**}_{ij} + \delta) - v_{ij}(x^{**}_{ij}) \\
&\leq -v'_{ij}(x^{**}_{ij})\delta + v_{ij}(x^{**}_{ij} + \delta) - v_{ij}(x^{**}_{ij}) \\
&< 0.
\end{aligned}$$

Thus, his payoff will decrease again. Therefore, the best response of an end-user $P_{ij}$ is to bid $(\beta_{ij}, d_{ij})$, and he has no incentive to deviate.

Consider a middleman $P_i$ with bid $(\beta_i, d_i)$ in Tier 1 auction and bid $(\alpha_i, q_i)$ in Tier 2 auction. His payoff at the efficient allocation is $u_i = \sum_j \beta_{ij} x^{**}_{ij} - c_i(x^{**}_i)$. Then, given the bids of others, if he changes his bid to increase his allocation $x^{**}_i$ by a $\delta > 0$, his payoff will be $u'_i = \sum_j \beta_{ij} x^{**}_{ij} - c_i(x^{**}_i + \delta) - \overline{w}'_i < u_i$. That is, his revenue remains the same, while his cost and his payment to player 0 increase. Thus, he has no incentive to increase his allocation.

Now, suppose he changes his bid to decrease his allocation $x^{**}_i$ by a $\delta > 0$ (when $x^{**}_i > 0$). His payment to player 0 does not change but the payment he receives changes. His new payoff is $u'_i = \sum_j \beta_{ij} x'^{*}_{ij} - c_i(x^{**}_i - \delta)$. Thus,

$$\begin{aligned}
&u'_i - u_i \\
&= \sum_j \beta_{ij} x'^{*}_{ij} - \sum_j \beta_{ij} x^{**}_{ij} - c_i(x^{**}_i - \delta) + c_i(x^{**}_i) \\
&= \sum_j \beta_{ij} x'^{*}_{ij} - \sum_j \lambda'_i x^{**}_{ij} - c_i(x^{**}_i - \delta) + c_i(x^{**}_i) \\
&\leq \sum_j \lambda'_i x'^{*}_{ij} - \sum_j \lambda'_i x^{**}_{ij} - c_i(x^{**}_i - \delta) + c_i(x^{**}_i) \\
&= \lambda'_i \sum_j (x'^{*}_{ij} - x^{**}_{ij}) - c_i(x^{**}_i - \delta) + c_i(x^{**}_i) \\
&= -\lambda'_i \delta - c_i(x^{**}_i - \delta) + c_i(x^{**}_i) \\
&= -\lambda_i \delta - c_i(x^{**}_i - \delta) + c_i(x^{**}_i) \\
&= -(c'_i(x_i) + \lambda_0)\delta - c_i(x^{**}_i - \delta) + c_i(x^{**}_i) \\
&\leq -c'_i(x_i)\delta - c_i(x^{**}_i - \delta) + c_i(x^{**}_i) \\
&< 0.
\end{aligned}$$

Thus, his payoff will decrease. Since his payoff will decrease by deviation in either direction, bidding $(\beta_i, d_i)$ and $(\alpha_i, q_i)$ is his best response. This implies that the constructed bid profile is a Nash equilibrium in the **HNSP** mechanism and yields an efficient outcome.

We now prove that there is endogenous strong budget balance at this Nash equilibrium. Note that $\overline{w}_{ij} = \alpha_i x_{ij}^{**} = \lambda_i' x_{ij}^{**} = \beta_{ij} x_{ij}^{**}$. So $\sum_j \overline{w}_{ij} = \sum_j \beta_{ij} x_{ij}^{**} = \underline{w}_i$ (for all $i$), which is what we wanted to prove.

*Remark 1.* We can easily check that each end-user and each middleman has a non-negative payoff at the Nash equilibrium constructed above.

*Remark 2.* We also note that the **HNSP** mechanism can be easily extended to the general multi-tier model wherein the Tier 1 sub-mechanism $\bar{\Gamma}^{(1)}$ is a VCG-type single-sided mechanism, while sub-mechanisms at all lower tiers, $\bar{\Gamma}^{(2)}, \ldots, \bar{\Gamma}^{(K-1)}$ are VCG-type double-sided mechanisms. Likewise, we can then establish the existence of an efficient Nash equilibrium with endogenous strong budget balance.

## 5 Conclusion

In this paper, we introduced a hierarchical auction model for network settings with multi-tier structures. We developed a general hierarchical mechanism design framework. Such a model is innovative and this paper is the first work on multi-tier auctions to our knowledge.

When the resource is indivisible, we investigated a class of mechanisms where each sub-mechanism is either a first- or a second-price auction. We showed that the hierarchical mechanism with a first- or a second-price sub-mechanism at Tier 1, and first-price sub-mechanisms at all other tiers is efficient but not incentive-compatible and surprisingly, the all-tier second-price auction mechanism is incentive-compatible but not efficient. This seems to fortell a more general impossibility of achieving incentive compatibility and efficiency at the same time in a hierarchical setting.

When the resource is divisible, we proposed the hierarchical network second-price mechanism, where the Tier 1 sub-mechanism is a single-sided VCG-type auction and the sub-mechanism at all lower tiers is a VCG-type double-sided auction. We showed that in this hierarchical mechanism, there exists an efficient Nash equilibrium with endogenous strong budget balance.

As part of future work, we intend to study more general classes of mechanisms than those where the sub-mechanisms are either first- or second-price auctions. We will also consider the *Stackelberg auction* setting, wherein the auctions at various tiers are conducted one after another. We will also consider more general network topologies wherein there may be more than one resource (e.g., bandwidth on multiple links, or bandwidth, storage and computation), and also allow for sub-mechanism auctions with multiple sellers.

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