Multi-portfolio Optimization: A Potential Game Approach

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Abstract. Trades from separately managed accounts are usually pooled together for execution and the transaction cost for a given account may depend on the overall level of trading. Multi-portfolio optimization is a technique for combing multiple accounts at the same time, considering their joint effects while adhering to account-specific constraints. In this paper, we model multi-portfolio optimization as a game problem and adopt as a desirable objective the concept of Nash Equilibrium (NE). By formulating the game problem as a potential game, we are able to provide a complete characterization of NE and derive iterative algorithms with a distributed nature and satisfactory convergence property.

1 Introduction

In a couple of ground-breaking articles [1,2] laying down the foundations of modern portfolio theory, Markowitz introduced half a century ago a fundamental framework for solving the canonical problem of how an individual account allocates wealth across a portfolio of risky assets by optimizing the associated risk-return tradeoff. Since then, numerous generalizations, such as limitations on transaction costs and other portfolio characteristics, have been proposed in order to effectively model realistic operating conditions underlying the practice of mean-variance optimization of single portfolio.

In a practical framework, trades of diverse clients are usually pooled and executed simultaneously for the sake of efficiency. But trading one account raises the marginal transaction costs for other accounts, so a particularly relevant problem is that of realistically modeling the trading costs incurred when rebalancing multiple accounts, and more specifically their market impact cost. Indeed, in the multi-portfolio rebalancing problem, the market impact cost associated with a given account depends on the overall level of trading of all accounts and not just on its specific trading requirements. As a consequence, the actual market impact cost of trading multiple accounts is typically larger than the sum of the estimated market impact costs of trading each account separately.

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To the best of our knowledge, O'Cinneide and his collaborators [5] are the first to introduce the simultaneous rebalancing of multiple accounts into a multi-portfolio optimization problem. Their approach is based on the maximization of the total welfare over all accounts, namely the Pareto optimal solution to the well-known "social planner" problem. Another methodology frequently used is known as the Nash Equilibrium (NE) approach [4,7]. In NE approach, rather than colluding to maximize total welfare, each account optimizes its own welfare, under the assumption that the trade decisions of other accounts have been made and are fixed. A NE is achieved when no account has an incentive to unilaterally deviate from it. Although this has been done in [7], the authors do not provide any characterizations of the NE. Besides, each account is only subject to short-selling constraint, which is hardly enough to model various complicated regulations in practice.

One common shortcoming of [5,7] is that an all-at-once approach is used to generate optimal trades for all accounts simultaneously. Compared with centralized approach, distributed algorithms are more suitable for implementation, especially when the number of accounts is large. During each iteration, accounts are rebalanced independently but taking into account the market impact of the desired trades of other accounts.

In this paper, the multi-portfolio optimization problem is modeled as a game problem and NE is adopted as the desirable objective. We first consider a Nash Equilibrium Problem where one player's feasible strategy set is independent of other players. We then generalize the problem by incorporating global constraints imposed on all accounts, which may arise due to practical considerations such as transaction size constraint over multiple accounts. In both cases, we give a complete characterizations of the NE and derive iterative algorithms that can be implemented in a distributed manner.

2 Problem Formulation

Treating the market-impact in a single account optimization as if it is the only account being traded underestimates the true trading cost of rebalancing each account. Instead, the market-impact caused by all accounts being optimized simultaneously should be considered. Under this consideration, the utility function for account n is defined as [7]

$$u_n(\mathbf{w}_n, \mathbf{w}_{-n}) = \boldsymbol{\alpha}_n^T \mathbf{w}_n - \frac{1}{2}\rho \cdot \mathbf{w}_n^T \mathbf{Q} \mathbf{w}_n - \frac{1}{2}\delta \cdot \mathbf{w}_n^T \boldsymbol{\Omega} \left(\sum_{m=1}^{N} \mathbf{w}_m \right), \tag{1}$$

where for account n, $\boldsymbol{\alpha}_n^T \mathbf{w}_n$ is the expected return, $\rho \mathbf{w}_n^T \mathbf{Q} \mathbf{w}_n$ represents the penalization for risk, $\delta \cdot \mathbf{w}_n^T \boldsymbol{\Omega} \left(\sum_{m=1}^{N} \mathbf{w}_m \right)$ is a nonlinear market-impact function while aggregate effects generated by other accounts are taken into account.

The maximization of account n's utility function (1), however, is subject to one or various kinds of constraints due to practical considerations. Some of them include short-selling constraint, holding constraint, budget constraint and cardinality constraint. Since our discussion does not depend on the type of constraints, we use a simple notation $\mathcal{K}_n$ to denote the set of account n's feasible strategies. We further assume that $\mathcal{K}_n$ is a non-empty, closed and convex set, and it is independent of the other accounts' strategies.

Given the utility fuction (1) and strategy set $\mathcal{K}_n$, we formulate the system design as a Nash Equilibrium Problem (NEP) using as desirable criterion the concept of Nash Equilibrium (NE). Specifically, we consider a strategic noncooperative game in which the players are the accounts. Each player n competes against the others by choosing a strategy $\mathbf{w}_n$ that maximizes his own utility function. In other words, given the strategy of other players, player n solves the following optimization problem:

$$\left.\begin{array}{l} \underset{\mathbf{w}_n}{\text{maximize}}\ u_n(\mathbf{w}_n, \mathbf{w}_{-n}) \\ \text{subject to}\ \mathbf{w}_n \in \mathcal{K}_n \end{array}\right\} \forall n. \tag{2}$$

A NE is achieved when no player has an incentive to deviate unilaterally, which is formally defined as follows:

Definition 1. *A (pure) strategy profile* $\mathbf{w}^\star = (\mathbf{w}_n^\star)_{n=1}^N$ *is a NE of NEP (2) if*

$$u_n(\mathbf{w}_n^\star, \mathbf{w}_{-n}^\star) \geq u_n(\mathbf{w}_n, \mathbf{w}_{-n}^\star), \forall \mathbf{w}_n \in \mathcal{K}_n, \forall n,$$

with $\mathbf{w}_{-n} = (\mathbf{w}_m)_{m=1, m\neq n}^N$.

3 Potential Game and Its Characterizations

To deal with NEP (2), we use in this paper a framework given by potential game theory [3,8], which allows us to infer the properties of NEP by solving a single optimization problem.

To begin with, a potential game is formally defined below.

Definition 2. *A strategic game (2) is called an exact potential game if there exists a function* $\mathcal{P} : \mathcal{K} \to \mathbb{R}$ *such that for all* n *and* $(\mathbf{w}_n^1, \mathbf{w}_{-n}), (\mathbf{w}_n^2, \mathbf{w}_{-n}) \in \mathcal{K}$*:*

$$u_n(\mathbf{w}_n^1, \mathbf{w}_{-n}) - u_n(\mathbf{w}_n^2, \mathbf{w}_{-n}) = \mathcal{P}(\mathbf{w}_n^1, \mathbf{w}_{-n}) - \mathcal{P}(\mathbf{w}_n^2, \mathbf{w}_{-n}). \tag{3}$$

With the definition of potential game in (3), it is easy to see that the set of NE for NEP (2) remains the same when all utility functions in (2) are replaced with $\mathcal{P}(\mathbf{w})$. This implies that we can study the properties of NEs using a single function that does not depend on the particular player. Furthermore, it is natural to ask what is the relationship between NEs of NEP (2) and the maxima of $\mathcal{P}$, which can be obtained by solving:

$$\begin{array}{l} \underset{\mathbf{w}}{\text{maximize}}\ \mathcal{P}(\mathbf{w}) \\ \text{subject to}\ \mathbf{w} \in \mathcal{K}. \end{array} \tag{4}$$

Lemma 1. *[8] Let NEP (2) be a potential game with potential function $\mathcal{P}$. If $\mathbf{w}^\star$ is a maximum of $\mathcal{P}$, then it is a NE of NEP (2). If $\mathcal{K}$ is a convex set with a Cartesian structure, i.e., $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \ldots \times \mathcal{K}_N$, and $\mathbf{w}^\star$ is a NE of NEP (2), then $\mathbf{w}^\star$ is a maximum of $\mathcal{P}$.*

Recall that in Section 2, we have assumed that $\mathcal{K}_n$ is independent of other players. Then Lemma 1 indicates that $\mathbf{w}^\star$ is a NE of NEP (2) if and only if it is a maximum of NLP (4). This provides us with new intuition to study the potential game, which is the framework of standard optimization theory applied to the potential function.

To make use of Lemma 1, we first define a function $\theta_1(\mathbf{w})$ as

$$\theta_1(\mathbf{w}) \triangleq \frac{1}{2}\mathbf{w}^T\mathbf{M}_1\mathbf{w} - \boldsymbol{\alpha}, \tag{5}$$

where $\mathbf{M}_1 \triangleq \mathbf{I} \otimes (\rho\mathbf{Q} + \delta\boldsymbol{\Omega}) + \frac{\delta}{2} \cdot \mathbf{S} \otimes \boldsymbol{\Omega}$, $\boldsymbol{\alpha} \triangleq (\boldsymbol{\alpha}_n)_{n=1}^N$, $\mathbf{w} \triangleq (\mathbf{w}_n)_{n=1}^N$, and

$$\mathbf{S} \triangleq \begin{bmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{bmatrix}. \tag{6}$$

Using the definition of potential function, we can readily show that NEP (2) is a potential game, as stated in the following theorem.

Theorem 1. *[10] Suppose each player's strategy set $\mathcal{K}_n$ is independent of other players' strategies and $\mathcal{K}$ has a Cartesian structure, NEP (2) is equivalent to the following optimization problem:*

$$\begin{aligned} &\underset{\mathbf{w}}{\text{maximize}}\ \theta_1(\mathbf{w}) \\ &\text{subject to}\ \mathbf{w} \in \mathcal{K}. \end{aligned} \tag{7}$$

The equivalence of NEP (2) and NLP (7) enables us to explore the existence and uniqueness of NE of NEP (2) by considering NLP (7). The result is stated in the following theorem.

Theorem 2. *[10] Suppose each player's strategy set $\mathcal{K}_n$ is independent of other players' strategies and $\mathcal{K}$ has a Cartesian structure, the Nash Equilibrium Problem (2) always has a unique NE.*

We mention that the pareto-optimal solution of NEP (2), i.e., the optimal solution to the sum-utility maximization problem, can be interpreted as the NE of a NEP with a modified objective function. Interested readers are referred to [10] for details.

Given the existence and uniqueness of NE, a natural question is that is there any algorithm that can be implmented in a distributed manner and has satisfactory convergence behavior? We consider best-response based iterative algorithms with both sequential (Gauss-Seidel) and simultaneous (Jacobi) update, as described in Algorithm 1.

Algorithm 1. Iterative Best Response Algorithm

Data : Choose any $\mathbf{w}_n^0 \in \mathcal{K}_n$ for all $n = 1, 2, \ldots, N$, and set $q = 0$.
Step 1: If $\mathbf{w}^q$ satisfies a suitable termination criterion: STOP.
Step 2: Sequentially or Simultaneously for $n = 1, 2, \ldots, N$, update $\mathbf{w}_n^{q+1}$ as follows:

Sequential (Gauss-Seidel) Update:
$\mathbf{w}_n^{q+1} \triangleq \arg\min_{\mathbf{w}_n \in \mathcal{K}_n} u_n(\mathbf{w}_1^{q+1}, \ldots, \mathbf{w}_{n-1}^{q+1}, \mathbf{w}_n, \mathbf{w}_{n+1}^q, \ldots, \mathbf{w}_N^q)$.
Simultaneous (Jacobi) Update:
$\mathbf{w}_n^{q+1} \triangleq (1 - \frac{1}{N})\mathbf{w}_n^q + \frac{1}{N} \cdot \arg\min_{\mathbf{w}_n \in \mathcal{K}_n} u_i(\mathbf{w}_1^q, \ldots, \mathbf{w}_{n-1}^q, \mathbf{w}_n, \mathbf{w}_{n+1}^q, \ldots, \mathbf{w}_N^q)$.

Step 3: Set $q \leftarrow q + 1$; and go to **Step 1**.

Theorem 3. *[10] Any sequence* $\{\mathbf{w}^q\}_{q=0}^{\infty}$ *generated by the sequential and simultaneous update of iterative best-response algorithm in Algorithm 1 converges to the unique NE of NEP (2), for any given updating order of the users.*

4 Generalized Nash Equilibrium Problem

In all previous developments we have assumed that one player's strategy set is independent of the rival players' actions, but this is not always the case. There are many applications of interest where the feasible sets naturally depend on the variables of the player's rivals. In this section, we consider the NEP (2) with global constraints as in (8). This results in a generalized Nash Equilibrium Problem (GNEP), which are formally described as follows:

$$\left.\begin{array}{ll}\underset{\mathbf{w}_n}{\text{maximize}} & \boldsymbol{\alpha}_n^T \mathbf{w}_n - \frac{1}{2}\rho \mathbf{w}_n^T \mathbf{Q} \mathbf{w}_n \\ \text{subject to} & \sum_{m=1}^N \left| w_{n,i} \right| \leq C_i, \forall i = 1, \ldots, I \\ & \sum_{m=1}^N \sum_{j \in \mathcal{J}_l} \left| w_{m,j} \right| \leq U_l, \forall l = 1, \ldots, L \\ & \left(\sum_{m=1}^N \mathbf{w}_m\right)^T \delta \boldsymbol{\Omega} \left(\sum_{m=1}^N \mathbf{w}_m\right) \leq T \\ & \mathbf{w}_n \in \mathcal{K}_n \end{array}\right\} \forall n, \tag{8}$$

Note that we have preserved $\mathcal{K}_n$ to exclusively denote one player's individual constraints such as budget constraint. The first and second global constraint in (8) represents the transaction size constraint over multiple accounts and limitations on the amount invested over groups of assets with related characteristics, respectively. In this formulation, we remove the market-impact function from the objective and incorporate it as the third global constraint in (8).

We call a game problem with coupled constraint sets as defined in (8) a Generalized Nash-Equilibrium Problem (GNEP). To analyze the GNEP (8), we can follow a similar approach as that in Section 3. After some elementary algebra, it can be shown that GNEP (8) is a potential game with the following constrained optimization problem:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{maximize}} & \boldsymbol{\alpha}^T \mathbf{w} - \frac{1}{2}\mathbf{w}^T \left(\mathbf{I} \otimes \rho\mathbf{Q}\right)\mathbf{w} \\ \text{subject to} & \mathbf{w} \in \mathcal{K}_1 \times \ldots \times \mathcal{K}_N \\ & \mathbf{g}(\mathbf{w}) \leq \mathbf{0}, \end{array} \tag{9}$$

where

$$\mathbf{g}(\mathbf{w}) \triangleq \begin{bmatrix} \left(\sum_{n=1}^{N} |w_{n,i}| - C_i\right)_{i=1}^{I} \\ \left(\sum_{n=1}^{N} \sum_{j \in \mathcal{J}_l} |w_{n,j}| - U_l\right)_{l=1}^{L} \\ \left(\sum_{n=1}^{N} \mathbf{w}_n\right)^T \delta\boldsymbol{\Omega} \left(\sum_{n=1}^{N} \mathbf{w}_n\right) - T, \end{bmatrix}. \tag{10}$$

We denote the feasible set of (9) as $\mathcal{K} \triangleq \{\mathbf{w} : \mathbf{w} \in \mathcal{K}_1 \times \ldots \times \mathcal{K}_N, \mathbf{g}(\mathbf{w}) \leq \mathbf{0}\}$. It is easy to see that NLP (9) is a strongly convex optimization problem, and it has a unique maximum.

Note that in NLP (9), $\mathcal{K}$ does not have a Cartesian structure. The equivalence between GNEP (8) and NLP (9) as indicated by Lemma 1 may not hold any more. As shown in [9,6,8], a NE of GNEP (8) is not necessarily a maximum of NLP (9). Nash Equilibriums of the GNEP (8) that are also maxima of NLP (9) are termed as Variational Equilibriums (VE). In another word, GNEP (8) and NLP (9) are equivalent in the sense of VE. From now on, we will focus on the VE of the GNEP (8) and give detailed analysis on its existence, uniqueness and algorithms.

Theorem 4. *[10] There always exists a unique variational equilibrium of GNEP (8).*

The potential game formulation of GNEP (8), i.e., NLP (9), not only serves as a direct way to characterize the VE, but also provides us with some intuition to devise distributed algorithms achieving the VE. First we derive a result that is valid for all potential games.

Theorem 5. *[10] Consider a Nash Equilibrium game where each players solves the following convex optimization problem*

$$\left.\begin{array}{ll} \underset{\mathbf{w}_n}{\text{maximize}} & f_n(\mathbf{w}_n, \mathbf{w}_{-n}) \\ \text{subject to} & \mathbf{w}_n \in \mathcal{K}_n, \end{array}\right\} \forall n \tag{11}$$

with a concave potential function $\mathcal{P}(\mathbf{w})$. Now suppose a global convex constraint $\mathbf{g}(\mathbf{w}) \leq \mathbf{0}$ is imposed on all players, i.e., each player solves the following optimization problem

$$\left.\begin{array}{ll} \underset{\mathbf{w}_n}{\text{maximize}} & f_n(\mathbf{w}_n, \mathbf{w}_{-n}) \\ \text{subject to} & \mathbf{w}_n \in \mathcal{K}_n \\ & \mathbf{g}(\mathbf{w}) \leq \mathbf{0}. \end{array}\right\} \forall n. \tag{12}$$

Then $\mathbf{w}^\star$ is a Variational Equilibrium of GNEP (12) if and only if it is a Nash Equilibrium of the following NEP

$$\left.\begin{array}{ll}\underset{\mathbf{w}_n}{\text{maximize}} & f_n(\mathbf{w}_n, \mathbf{w}_{-n}) - \boldsymbol{\lambda}^T \cdot \mathbf{g}(\mathbf{w}) \\ \text{subject to} & \mathbf{w}_n \in \mathcal{K}_n\end{array}\right\} \forall n \tag{13}$$

with $\boldsymbol{\lambda}$ *such that* $\mathbf{0} \leq \boldsymbol{\lambda} \perp \mathbf{g}(\mathbf{w}^\star) \leq \mathbf{0}$.

Thanks to Theorem 5, we have transformed GNEP (12) with coupled strategy set into NEP (13) with no coupling in strategy set. The transformation is beneficial because we can solve GNEP (12) in a distributed manner. Specifically, we can design a double-loop algorithm, where in the outer loop, the price tuple $\boldsymbol{\lambda}$ is updated by sub-gradient method, and in the inner loop, NEP (13) is solved using Algorithm 1. We summarize this double-loop algorithm in Algorithm 2.

Algorithm 2. Sub-gradient Algorithm

Data: Choose any $\boldsymbol{\lambda}^{(0)} \geq \mathbf{0}$, and set $q = 0$.
Step 1: If $\boldsymbol{\lambda}^{(q)}$ satisfies a suitable termination criterion: STOP.
Step 2: Compute the unique NE $\mathbf{w}^\star\left(\boldsymbol{\lambda}^{(q)}\right)$ of NEP (13) using Algorithm 1.
Step 3: $\boldsymbol{\lambda}^{(q+1)} = \left[\boldsymbol{\lambda}^{(q)} - \gamma^{(q)} \cdot \mathbf{g}\left(\mathbf{w}^\star\left(\boldsymbol{\lambda}^{(q)}\right)\right)\right]$, where $\gamma^{(q)}$ is the q-th stepsize.
Step 4: $q \leftarrow q + 1$; go to **Step 1**.

A common criterion for choosing the stepsize $\gamma^{(q)}$ in Algorithm is that $\gamma^{(q)}$ must be square summable, but not absolute summable. The convergence property of Algorithm 2 is given by the following theorem.

Theorem 6. *[10] Algorithm 2 solving GNEP (12) converges as long as Algorithm 1 solving NEP (11) converges.*

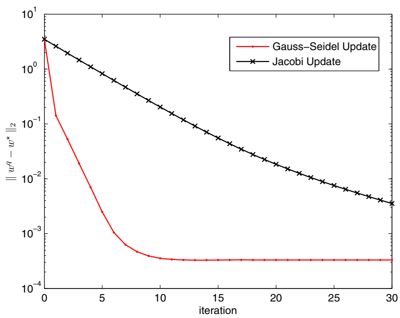
Theorem 6 indicates that the introduction of global convex constraints does not require stricter convergence conditions. For GNEP (8), as we have already proved in Theorem 3 that Algorithm 1 always converges, we can therefore conclude that Algorithm 2 can surely converge to the unique VE of GNEP (8).

5 Discussions and Conclusions

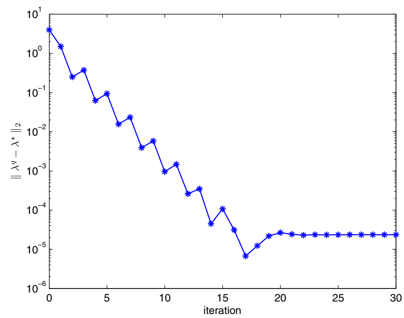
In Figure 1, we show the convergence of Algorithm 1 and Algorithm 2. From Figure 1(a), the sequential update of best-response iterative algorithm converges to the unique NE very fast. On the other hand, the convergence speed of the simultaneous update of best-response iterative algorithm depends on N, the number of accounts. When there are a large number of accounts, its convergence speed is typically small.

Figure 1(b) shows that the outer-loop price tuple $\boldsymbol{\lambda}$ converges with a satisfactory convergence speed. As we have pointed out before, the convergence of inner-loop best-response iterative algorithm guarantees the convergence of Algorithm 2.

In conclusion, we have modeled the multi-portfolio optimization problem as a Nash Equilibrium problem and analyze it under the framework of potential game.



(a) Algorithm 1



(b) Algorithm 2

Fig. 1. Convergence Behavior of Algorithm 1 and Algorithm 2

Specifically, we consider both NEP with uncoupled strategy set and generalized NEP with global constraints imposed on all players. We then give a complete characterizations of NE of NEP and VE of GNEP. We further derive iterative algorithms that can be implemented in a distributed manner and has satisfactory convergence behavior.

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