The Reconstruction Conjecture

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Abstract. The Reconstruction Conjecture is one of the most engaging problems under the domain of Graph Theory. The conjecture proposes that every graph with at least three vertices can be uniquely reconstructed given the multiset of subgraphs produced by deleting each vertex of the original graph one by one. This conjecture has been proven true for several infinite classes of graphs, but the general case remains unsolved. In this paper we will outline the problem and give a practical method for reconstructing a graph from its node-deleted.

Keywords: Multiset, Card, Deck, Matching, Perfect matching, K-matching, Matching polynomial, Tree-Decomposition.

1 Introduction

The Reconstruction Conjecture, formulated by Kelly and Ulam in 1942 [11, 16], asserts that every finite, simple, undirected graph on at least three vertices is determined uniquely (up to isomorphism) by its collection of 1-vertex deleted subgraphs.

Harary [7] formulated the Edge-Reconstruction- Conjecture, which states that a finite simple graph with at least four edges can be reconstructed from its collection of one edge deleted sub graphs.

The Reconstruction Conjecture is interesting not only from a mathematical or historical point of view but also due to its applicability in diverse fields. Archaeologists may try to assemble broken fragments of pottery to find the shape and pattern of an ancient vase. Chemists may infer the structure of an organic molecule from knowledge of its decomposition products. In bioinformatics the Multiple Sequence- Alignment problem [1] is to reconstruct a sequence with minimum gap insertion and maximum number of matching symbols, given a list of protein or DNA

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sequences. In computer networking, a reconstruction problem can appear in the following scenario: given a collection of sketches depicting partial network connection in a city from different locations, reconstruct the network of the entire city.

In this paper we will give a solution to this problem, using Matching Polynomial of the graph. In general, the reconstruction of graph polynomials can shed some light on the Reconstruction Conjecture itself. For example, reconstruction was investigated by several authors. In particular, Gutman and Cvetkovic [6] investigated the reconstruction of the characteristic polynomial. The existence of a reconstruction for the characteristic polynomial was eventually established by Tutte [15]. Tutte also established the reconstructibility of the rank polynomial and the chromatic polynomial.

Although the reconstruction of several graph polynomials has been established, no practical means of reconstruction exists for any of them. Farrell and Wahid [4] investigated the reconstructibility of matching polynomial and gave a practical method for reconstruction.

Although Farrell and Wahid gave a method for reconstruction of the matching polynomial they did not provide a practical method for generate the matching polynomial for a graph.

In this paper we will first give an algorithm that can generate the matching polynomial of a graph in polynomial time. Then we will give an algorithm for reconstructing a graph from its node deleted sub graphs.

2 Preliminaries

2.1 Some Definitions

Multiset. In mathematics, a Multiset (or bag) is a generalization of a set. While each member of a set has only one membership, a member of a multiset can have more than one membership (meaning that there may be multiple instances of a member in a multiset).

Card. A Card of G is an unlabelled graph formed by deleting 1 vertex and all edges attached to it.

Deck. The Deck of G, D (G), is the collection of all G‘s cards. Note this is in general a multiset.

Graph Isomorphism. Let V(G) be the vertex set of a simple graph and E(G) its edge set. Then a graph isomorphism from a simple graph G to a simple graph H is a bijection f: $V(G) \rightarrow V(H)$ such that $u\,v \in E(G)$ iff $f(u)\,f(v) \in E(H)$ (West 2000, p. 7). If there is a graph isomorphism for G to H, then G is said to be isomorphic to H, written $G \approx H$.

2.2 Notations

Our alphabet set is Σ={0, 1}. We use {., ., ., } to denote sets and [:; : : : ; :] to denote multiset. We use U to denote set union as well as multiset union. We consider only finite, undirected graphs with no self-loops. Given a graph G, let V(G) denote its vertex set and let E(G) denote its edge set. For notational convenience, we sometimes represent a graph G by (V; E), where V=V(G) and E=E(G). By the order of a graph G we mean | V(G) |, i.e., the cardinality of its vertex set.

3 Matching Polynomial and Reconstruction Conjecture

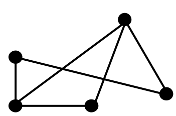
3.1 Matching Polynomial [4, 5]

Matching. A matching cover (or simply a matching) in a graph G is taken to be a subgraph of G consisting of disjoint (independent) edges of G, together with the remaining nodes of G as (isolated) components.

K–matching. A matching is called a K–matching if it contains exactly K edges.

Matching Polynomial. If G contains P nodes, and if a matching contains K edges, then it will have P–2K component nodes. Now assign weights W_1 and W_2 to each node and edge of G, respectively. Take the weight of a matching to be the product of the weights of all its components. Then the weight of a K–matching will be $W_1^{P-2K}W_2^{K}$. The matching polynomial of G, denoted by m(G), is the sum of the weights of all the matchings in G. The matching polynomial of G has been defined as $m(G)=\sum ak W_1^{P-2K} W_2^{K}$. ak is the number of matchings in G with k edges.

Example.



0 – matching = $1\, W_1^{5} W_2^{0}$
1 – matching = $6\, W_1^{3} W_2^{1}$
2 – matching = $6\, W_1^{1} W_2^{2}$
No 3–matching.

$$m(G) = W_1^{5} + 6\, W_1^{3} W_2 + 6\, W_1 W_2$$

Perfect Matching. A perfect matching is a matching of a graph containing n/2 edges, the largest possible. Perfect matchings are therefore only possible on graphs with an even number of vertices. We denote the number of perfect matchings in G by δ(G). Clearly δ(G) is the coefficient of the term independent of W_1 in m(G).

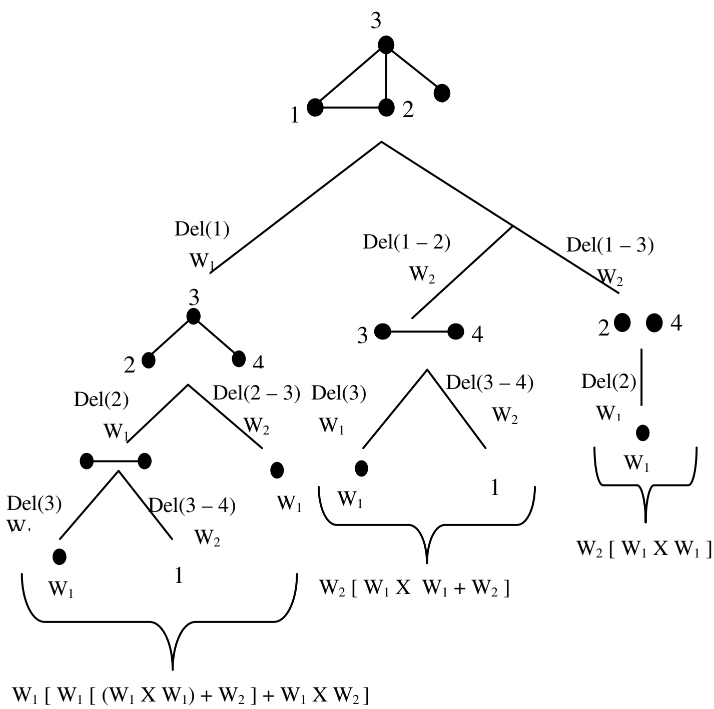
3.2 Matching Polynomial Generation Using Tree Decomposition

Algorithm: gen_mpoly(Graph G)

Input: A simple connected graph G.
Output: The Matching Polynomial m(G) of this graph.

Steps:
1. if | V(G) | = 1 then
   return (create_mpoly("W_1"));
2. else if | V(G) | = 0 then
   return (create_mpoly("1"));
3. else
   return (create_mpoly(W_1 + gen_mpoly(G – V_i) + $W_2 X \sum_{e : V_i V_j \in E(G)}$ gen_mpoly(G – e)));
   end if.
4. End

Example.



$$m(G) = W_1[W_1[(W_1XW_1)+W_2]+W_1XW_2]+W_2[W_1XW_1+W_2]+W_2[W_1XW_1]$$
$$= W_1^{4}+4W_1^{2}W_2+W_2^{2}$$

3.3 Reconstruction of Matching Polynomial [4, 6, 7]

In order to establish our main result, we will need the following lemma.

LEMMA 1 : Let G be a graph with n nodes. Then

$$\frac{d}{d W_1} m(G) = \sum_{i=1}^{n} m(G - V_i)$$

Proof.

We establish (1) by showing that the two polynomials $A = \frac{d}{d W_1} m(G)$ and $B = \sum_{i=1}^{n} m(G - V_i)$ have precisely the same terms with equal coefficients.

Let $W_1^j W_2^k$ be a term of A. Then m(G) has a term in $W_1^{j+1} W_2^k$. It follows that G has a matching S with (j + 1) nodes and k edges. Let $V_r \in V(G)$. Then $G - V_r$ will contain the matching $S - V_r$. Hence B also contains a term in $W_1^j W_2^k$.

Conversely, if B contains a term in $W_1^j W_2^k$, then there exists a node V_r such that $G - V_r$ has a matching with nodes and k edges. Therefore G has a matching with (j+1) nodes and k edges. It follows that m(G) has a term in $W_1^{j+1} W_2^k$. Hence A has a term in $W_1^j W_2^k$. We conclude that A and B have the same kinds of terms.

We will show that the coefficients of like terms are equal. Let $a_k W_1^j W_2^k$ be a term in m(G). Then the corresponding term in A will be $ja_k W_1^{j-1} W_2^k$. Hence the coefficient of $W_1^j W_2^k$ in A will be ja_k. Now, for each matching in G with j nodes, there will be exactly corresponding matchings in the graphs $G - V_r$ with j–1 nodes. Since G contains a_k such matchings, then B will contain the term $W_1^{j-1} W_2^k$ with coefficient ja_k. Therefore the coefficients of like terms are equal. Hence the result follows.

Theorem 1:

$$m(G) = \sum_{i=1}^{n} \int m(G - V_i)\, dW_1 + \delta(G)$$

Proof. This is straightforward from the lemma(1), by integrating with respect to W_1.

4 Node Reconstruction of a Graph

4.1 Algorithm

Input: Deck D=[H_1;H_2;H_3;···;H_k] (multiset of subgraphs produced by deleting each vertex of the original graph).

Output: S={G1,G2,G3,···,Gn} where n≥1 and G1≈G2≈G3≈··.≈Gn [≈ : Isomorphic].

Steps:

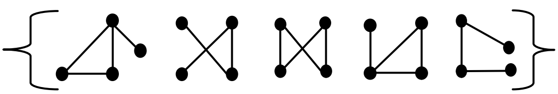
1. Reconstruct the matching polynomial [m1(G)] of the graph from the given deck D by using Theorem 1.
2. Select any card [vertex deleted sub graph of the original graph] H_i from the deck D.

3. Add a new vertex V_K to H_i and connect it with any vertex of H_i by a tentative edge, obtaining a graph $G_{tentative}$ on K vertices.
4. for i←1 to k – 1 do
   label the tentative edge between i and V_K as X_{iK}.
5. Generate the 2nd matching polynomial[m2(G)] with variable coefficient using the *Tree Decomposition* method [m2(G)=tree_decomposition($G_{tentative}$)].
6. Compare coefficients of m1(G) and m2(G) and generate one or more solutions. Store the solutions in solution vectors $Soln_1$, $Soln_2$, ·········· $Soln_n$. In each solution vector a variable coefficient X_{ij} can have only two values 0 or 1. 0 means the tentative edge will be deleted from $G_{tentative}$, and 1 means the tentative edge will be converted in to a permanent edge. Also calculate the Perfect Matching by substituting the solutions into m2(G).
7. for every solution [$Soln_{1\ to\ n}$] generate a new graph G_i [$i \leq n$] from the $G_{tentative}$ according to step6 and add the generated graph in to the output set S [S=SU{G_i}].

All of the graphs in the set S are isomorphic. So, select any one of them and this is the reconstructed graph isomorphic to the original graph.

4.2 Example

Input:

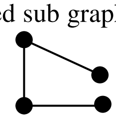
Deck **D** = 

Step 1:[Reconstruct the matching polynomial [m1(G)] of the graph from the given deck D by using Theorem 1.]

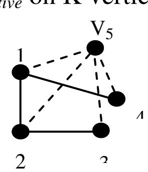
$m(H_1) = W_1^4 + 4 W_1^2 W_2 + W_2^2 \quad \int m(H_1) = W_1^5/5 + 4/3 W_1^3 W_2 + W_1 W_2^2$
$m(H_2) = W_1^4 + 3 W_1^2 W_2 + W_2^2 \quad \int m(H_2) = W_1^5/5 + 3/3 W_1^3 W_2 + W_1 W_2^2$
$m(H_3) = W_1^4 + 4 W_1^2 W_2 + 2W_2^2 \quad \int m(H_3) = W_1^5/5 + 4/3 W_1^3 W_2 + 2 W_1 W_2^2$
$m(H_4) = W_1^4 + 4 W_1^2 W_2 + W_2^2 \quad \int m(H_4) = W_1^5/5 + 4/3 W_1^3 W_2 + W_1 W_2^2$
$m(H_5) = W_1^4 + 3 W_1^2 W_2 + W_2^2 \quad \int m(H_5) = W_1^5/5 + 3/3 W_1^3 W_2 + W_1 W_2^2$

$m1(G) = W_1^5 + 6 W_1^3 W_2 + 6 W_1 W_2^2 + \delta(G)$ [$\delta(G)$: Perfect Matching]

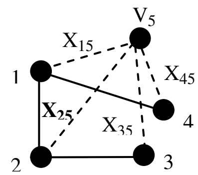
Step 2: [Select any card (vertex deleted sub graph of the original graph) H_i from the deck D.]

Here we select H_5 

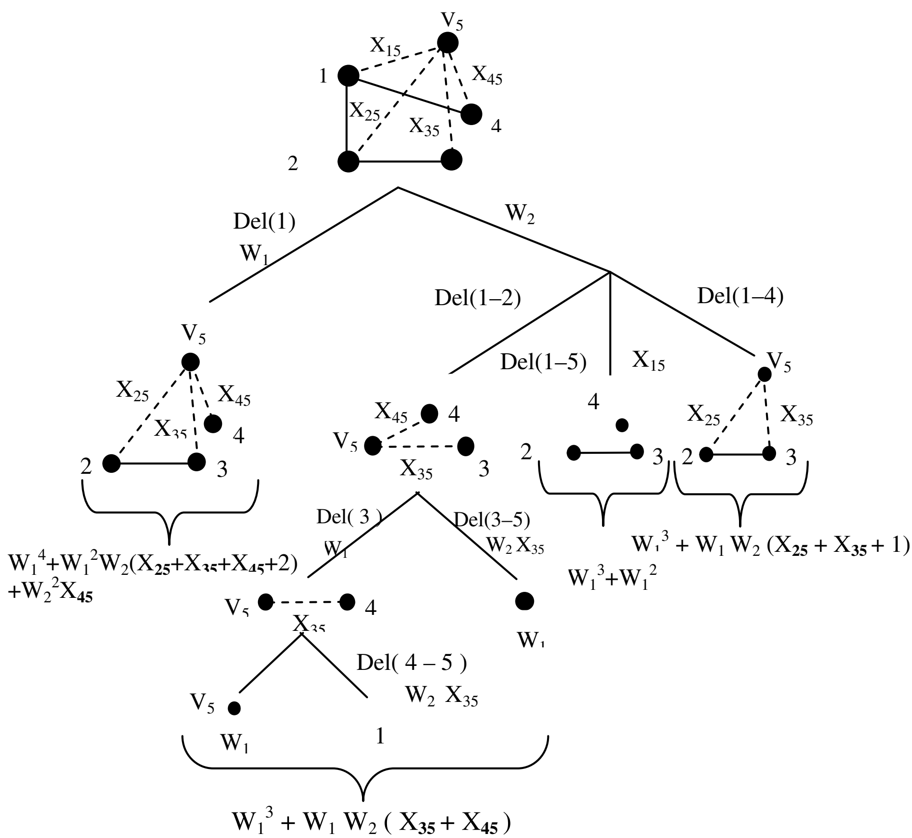
Step 3: [Add a new vertex V_K to H_i and connect it with any vertex of H_i by a tentative edge, obtaining a graph $G_{tentative}$ on K vertices.]



Step 4: [for i ←1 to k – 1 do
label the tentative edge between i and V_K as X_{iK} .]



Step 5: [Generate the 2nd matching polynomial [m2(G)] with variable coefficient using the *Tree Decomposition* method [m2(G) = tree_decomposition($G_{tentative}$)].]



$$m2(G) = W_1^5+W_1^3W_2(X_{15}+X_{25}+X_{35}+X_{45}+3)+W_1W_2^2(X_{15}+X_{25}+2X_{35}+2X_{45}+1)$$

Step 6 : [Compare coefficients of m1(G) and m2(G) and generate one or more solutions. Store the solutions in solution vectors $Soln_1, Soln_2, \cdots, Soln_n$.]
by comparing coefficients of m1(G) and m2(G) we get

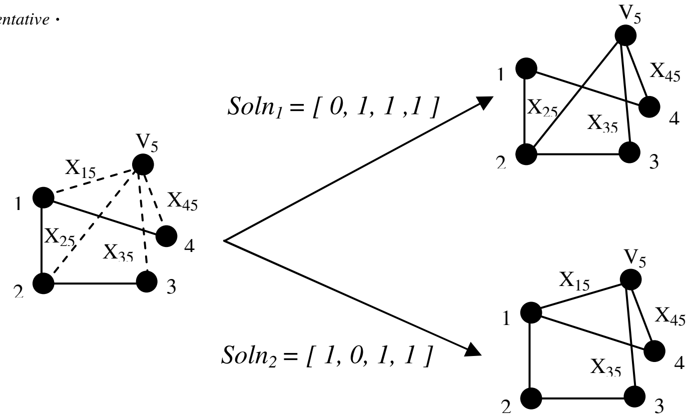
$$X_{15} + X_{25} + X_{35} + X_{45} = 3 \qquad X_{15} + X_{25} + X_{35} + X_{45} = 5$$

It is clear that the only solutions to these equations are

$Soln_1$ = [0, 1, 1 ,1] and $Soln_2$ = [1, 0, 1, 1]

by substituting into m2(G) we get in both cases δ(G) = 0.

Step 7: [for every solution [$Soln_{1\ to\ n}$] generate a new graph $G_i[i \leq n]$ from the $G_{tentative}$ according to step6 and add the generated graph in to the output set S [$S=SU\{G_i\}$]].
We can now use $Soln_1$ to constructed a graph G_1 and $\mathbf{Soln_2}$ to constructed a graph G_2 from $G_{tentative}$.



It can be easily confirmed that the mapping defined by φ:G1 → G2 such that φ(1) ≡ 3; φ(2) ≡ 5; φ(3) ≡ 4; φ(4) ≡ 2 and φ(5) ≡ 1 is an isomorphism. Hence GI≈G2.

4.3 Analysis

On arbitrary graphs or even planar graphs, computing the matching polynomial is #P–Complete(Jerrum 1987)[10].

But using Tree Decomposition we can compute the matching polynomials for all the subgraphs for the given deck in polynomial time.

Tree Decomposition contains recursive calls to itself, its running time can often be described by a recurrence. The recurrence for Tree Decomposition –

$$T(n) = \begin{cases} O(1) & \text{if } n \leq 1 \\ T(n-V_i) + \sum_{V_iV_j \in E(G)} T(n-V_iV_j) & \text{otherwise [for } i \leftarrow 1 \text{ to n-1]} \end{cases}$$

$T(n) = O(n^2)$

Step 1 requires $O(n^2)$ time. Step 2 & 3 require O(1) time. Step 4 requires O(n) times. Step 5 (Tree Decomposition) requires $O(n^2)$ time. Step 6 requires exponential time $O(2^n)$, and Step 7 requires O(n) times.

So, if we can find a polynomial time algorithm for solving this Underdetermined system in Step7 then we can easily reconstruct the original graph in polynomial time.

5 Conclusions

As presented in this paper, if G is a simple undirected graph with at least three vertices then we can use the proposed algorithm for reconstruct G from its vertex deleted subgraphs. Although the proposed algorithm is not tested for all classes of graph, but we can conclude that suppose that a graph G is characterized by a particular matching polynomial and suppose that the matching polynomial is reconstructible, then G is reconstructible.

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